



Oscillation criteria for second-order nonlinear neutral delay dynamic equations

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Abstract

In this paper we will establish some oscillation criteria for the second-order nonlinear neutral delay dynamic equation

$$\left(r(t) \left((y(t) + p(t)y(t - \tau))^{\Delta} \right)^{\gamma} \right)^{\Delta} + f(t, y(t - \delta)) = 0$$

on a time scale \mathbb{T} ; here $\gamma > 0$ is a quotient of odd positive integers with $r(t)$ and $p(t)$ real-valued positive functions defined on \mathbb{T} . To the best of our knowledge nothing is known regarding the qualitative behavior of these equations on time scales, so this paper initiates the study.

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1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete

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analysis (see [17]). The theory of “dynamic equations” unifies the theories of differential equations and difference equations and it also extends these classical cases to cases “in between.” Since then several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal et al. [1] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [4], summarizes and organizes much of time scale calculus, see also the book by Bohner and Peterson [5] for advances in dynamic equations on time scales.

In recent years there has been much research activity concerning the oscillation and nonoscillation of solutions of ordinary dynamic equations on time scales, we refer the reader to the papers [3,6–15,18,19]. Recently Agarwal et al. [2] have established some new oscillation criteria for second-order delay dynamic equations on time scales.

In this paper, we are concerned with oscillation properties of the second-order nonlinear neutral delay dynamic equation

$$\left(r(t)((y(t) + p(t)y(t - \tau))^{\Delta})^{\gamma}\right)^{\Delta} + f(t, y(t - \delta)) = 0 \quad (1.1)$$

on a time scale \mathbb{T} . Throughout this paper we assume that $\gamma > 0$ is a quotient of odd positive integers, τ and δ are positive constants such that the delay functions $\tau(t) = t - \tau < t$ and $\delta(t) = t - \delta < t$ satisfy $\tau(t) : \mathbb{T} \rightarrow \mathbb{T}$ and $\delta(t) : \mathbb{T} \rightarrow \mathbb{T}$ for all $t \in \mathbb{T}$, $r(t)$ and $p(t)$ are real valued positive functions defined on \mathbb{T} and

(H₁) $r(t) > 0$, $\int_{t_0}^{\infty} (1/r(t))^{1/\gamma} \Delta t = \infty$ and $0 \leq p(t) < 1$,

(H₂) $f : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function such that $uf(t, u) > 0$ for all $u \neq 0$ and there exists a nonnegative function $q(t)$ defined on \mathbb{T} such that $|f(t, u)| \geq q(t)|u^{\gamma}|$.

Recall that a solution of (1.1) is a nontrivial real function $y(t)$ such that $y(t) + p(t)y(t - \tau) \in C_{\text{rd}}^1[t_y, \infty)$, and $r(t)((y(t) + p(t)y(t - \tau))^{\Delta})^{\gamma} \in C_{\text{rd}}^1[t_y, \infty)$ for $t_y \geq t_0$ and satisfying Eq. (1.1) for $t \geq t_y$. Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_y, \infty)$ and satisfy $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The paper is organized as follows: In the next section we present some basic definitions concerning the calculus on time scales. In Section 3, first we give some oscillation criteria of (1.1) on any time scale \mathbb{T} by reducing the oscillation properties of (1.1) to the nonexistence of positive solutions of a delay dynamic inequality. Secondly, by developing a Riccati transformation technique some sufficient conditions for oscillation of all solutions of (1.1) on time scales (where all the points are right scattered) are established. Some examples are considered in Section 4 to illustrate our main results.

We note that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = 0$, $\mu(t) = 0$, $f^{\Delta}(t) = f'(t)$ and (1.1) becomes the second-order nonlinear neutral delay differential equation

$$\left(r(t)((y(t) + p(t)y(t - \tau))')^{\gamma}\right)' + f(t, y(t - \delta)) = 0, \quad t \in [t_0, \infty). \quad (1.2)$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$,

$$y^{\Delta}(t) = \Delta y(t) = y(t + 1) - y(t),$$

and (1.1) becomes the second-order neutral delay difference equation

$$\Delta \left(r(t) (\Delta (y(t) + p(t)y(t - \tau)))^\gamma \right) + f(t, y(t - \delta)) = 0, \quad t \in [t_0, \infty). \quad (1.3)$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\sigma(t) = t + h$, $\mu(t) = h$,

$$y^\Delta(t) = \Delta y_h x(t) = \frac{y(t+h) - y(t)}{h},$$

and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_h \left(r(t) (\Delta_h (y(t) + p(t)y(t - \tau)))^\gamma \right) + f(t, y(t - \delta)) = 0, \quad t \in [t_0, \infty). \quad (1.4)$$

If $\mathbb{T} = q^{\mathbb{N}} = \{t: t = q^k, k \in \mathbb{N}, q > 1\}$, then $\sigma(t) = qt$, $\mu(t) = (q - 1)t$,

$$x^\Delta(t) = \Delta_q x(t) = \frac{x(qt) - x(t)}{(q - 1)t},$$

and (1.1) becomes the second-order q -neutral delay difference equation

$$\Delta_q \left(r(t) (\Delta_q (y(t) + p(t)y(t - \tau)))^\gamma \right) + f(t, y(t - \delta)) = 0, \quad t \in [t_0, \infty). \quad (1.5)$$

If $\mathbb{T} = \mathbb{N}_0^2 = \{t^2: t \in \mathbb{N}_0\}$, then $\sigma(t) = (\sqrt{t} + 1)^2$ and $\mu(t) = 1 + 2\sqrt{t}$,

$$\Delta y_N(t) = \frac{y((\sqrt{t} + 1)^2) - y(t)}{1 + 2\sqrt{t}},$$

and (1.1) becomes the second-order neutral delay difference equation

$$\Delta_N \left(r(t) (\Delta_t (y(t) + p(t)y(t - \tau)))^\gamma \right) + f(t, y(t - \delta)) = 0, \quad t \in [t_0^2, \infty). \quad (1.6)$$

If $\mathbb{T} = \mathbb{T}_n = \{t_n: n \in \mathbb{N}_0\}$ where t_n are the so-called harmonic numbers defined by

$$t_0 = 0, \quad t_n = \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{N}_0,$$

then $\mu(t_n) = \frac{1}{n+1}$,

$$y^\Delta(t_n) = (n + 1)y(t_n),$$

and (1.1) becomes the neutral difference equation

$$\Delta_{t_n} \left(r(t_n) (\Delta_{t_n} (y(t_n) + p(t_n)y(t_n - \tau)))^\gamma \right) + f(t_n, y(t_n - \delta)) = 0, \quad t_n \in [0, \infty). \quad (1.7)$$

2. Some preliminaries on time scales

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . On any time scale \mathbb{T} we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T}: s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\}. \quad (2.1)$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be left-dense if $\rho(t) = t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$. The graininess function μ for a time scale \mathbb{T} is defined by $\mu(t) := \sigma(t) - t$.

A function $p: \mathbb{T} \rightarrow \mathbb{R}$ is called positivity regressive (we write $p \in \mathfrak{R}^+$) if it is rd-continuous function and satisfies $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$.

For a function $f: \mathbb{T} \rightarrow \mathbb{R}$ the (delta) derivative is defined by

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} \quad (2.2)$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^\Delta(t) = \lim_{s \rightarrow t} \frac{f(\sigma(t)) - f(s)}{t - s} = \lim_{t \rightarrow \infty} \frac{f(t) - f(s)}{t - s}, \quad (2.3)$$

provided this limit exists. A function $f: [a, b] \rightarrow \mathbb{R}$ is said to be right-dense continuous if it is right continuous at each right-dense point and there exists a finite left limit at all left-dense points, and f is said to be differentiable if its derivative exists. A useful formula is

$$f^\sigma = f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t). \quad (2.4)$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable function f and g :

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \quad (2.5)$$

and

$$\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \quad (2.6)$$

For $a, b \in \mathbb{T}$, and a differentiable function f , the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t) \Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)g^\Delta(t) \Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t)) \Delta t, \quad (2.7)$$

and infinite integrals are defined as

$$\int_a^\infty f(t) \Delta t = \lim_{b \rightarrow \infty} \int_a^b f(t) \Delta t.$$

3. Main results

In this section, we establish some oscillation criteria for (1.1). Since we are interested in asymptotic behavior of solutions we will suppose that the time scale \mathbb{T} under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0, \infty)$.

Recall a solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

The case $\gamma > 0$

Let $\gamma > 0$ be a quotient of odd positive integers. We now establish some sufficient conditions for the oscillation of (1.1) by reducing our study to a first-order delay dynamic inequality where we apply the results of Zhang and Deng [21].

We begin by stating the main result from [21].

Lemma 3.1. *Assume that $p(t) > 0$, $\delta(t) < t$ and $\lim_{t \rightarrow \infty} \tau(t) = \infty$. If*

$$\limsup_{t \rightarrow \infty} \sup_{\lambda > 0, -\lambda p \in \mathfrak{N}^+} \lambda e_{-\lambda p}(t, \delta(t)) < 1,$$

then all solutions of

$$z^\Delta(t) + p(t)z(\delta(t)) \leq 0$$

are oscillatory.

Now, we state and prove our main comparison theorem.

Theorem 3.1. *Assume that (H_1) – (H_2) hold. Furthermore, assume that $r^\Delta(t) \geq 0$. Then every solution of (1.1) oscillates if the inequality*

$$z^\Delta(t) + A(t)z(t - \delta) \leq 0, \quad (3.1)$$

where

$$A(t) = \frac{q(t)(1 - p(t - \delta))^\gamma}{r(t - \delta)} \left(\frac{t - \delta}{2} \right)^\gamma,$$

has no eventually positive solution.

Proof. Suppose to the contrary that (1.1) has a nonoscillatory solution y . We may assume without loss of generality that $y(t - N) > 0$ where $N = \max\{\tau, \delta\}$ for all $t \geq t_0$. Set

$$x(t) = y(t) + p(t)y(t - \tau). \quad (3.2)$$

In view of (1.1) and (H_2) we have

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + q(t)y^\gamma(t - \delta) \leq 0 \quad (3.3)$$

for all $t \geq t_0$, and so $r(t)(x^\Delta(t))^\gamma$ is an eventually decreasing function. We first show that $r(t)(x^\Delta(t))^\gamma$ is eventually nonnegative. Indeed, since $q(t)$ is a positive function, the decreasing function $r(t)(x^\Delta(t))^\gamma$ is either eventually positive or eventually negative. Suppose

there exists an integer $t_1 \geq t_0$ such that $r(t_1)(x^\Delta(t_1))^\gamma = c < 0$, then from (3.3) we have $r(t)(x^\Delta(t))^\gamma < r(t_1)(x^\Delta(t_1))^\gamma = c$ for $t \geq t_1$, and so

$$x^\Delta(t) \leq c^{1/\gamma} \left(\frac{1}{r(t)} \right)^{1/\gamma}, \quad (3.4)$$

which implies by (H_1) that

$$x(t) \leq x(t_1) + c^{1/\gamma} \int_{t_1}^t \left(\frac{1}{r(s)} \right)^{1/\gamma} \Delta s \rightarrow -\infty \quad \text{as } t \rightarrow \infty, \quad (3.5)$$

and this contradicts the fact that $x(t) > 0$ for all $t \geq t_0$. Hence $r(t)(x^\Delta(t))^\gamma$ is eventually nonnegative. Therefore, we see that there is some $t_1 \geq t_0$ such that

$$x(t) > 0, \quad x^\Delta(t) \geq 0, \quad (r(t)(x^\Delta(t))^\gamma)^\Delta < 0, \quad t \geq t_1. \quad (3.6)$$

This implies for $t \geq t_1 + \tau$ that

$$\begin{aligned} y(t) &= x(t) - p(t)y(t - \tau) = x(t) - p(t)[x(t - \tau) - p(t - \tau)y(t - 2\tau)] \\ &\geq x(t) - p(t)x(t - \tau) \geq (1 - p(t))x(t), \end{aligned}$$

and then for $t \geq t_2 \geq t_1 + \tau + \delta$ we have

$$y(t - \delta) \geq (1 - p(t - \delta))x(t - \delta).$$

From (3.3) and the last inequality we obtain

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + q(t)(1 - p(t - \delta))^\gamma x^\gamma(t - \delta) \leq 0, \quad t \geq t_2. \quad (3.7)$$

From $r^\Delta(t) \geq 0$ and (3.6) we can easily verify that $x^{\Delta\Delta}(t) \leq 0$ for $t \geq t_2$ and then $x^\Delta(t)$ is positive and nonincreasing. Using this, and fixing $t_3 \geq 2t_2$, we have for $t \in [t_3, \infty)$ that

$$x(t) = x(t_2) + \int_{t_2}^t x^\Delta(s) \Delta s \geq \int_{t_2}^t x^\Delta(t) \Delta s \geq (t - t_2)x^\Delta(t) \geq \frac{t}{2}x^\Delta(t),$$

and also for $t \geq t_3 + \delta$ we have

$$x(t - \delta) \geq \frac{t - \delta}{2}x^\Delta(t - \delta).$$

Substituting the last inequality in (3.7) we obtain for $t \geq t_3 + \delta$ that

$$(r(t)(x^\Delta(t))^\gamma)^\Delta + q(t)(1 - p(t - \delta))^\gamma \left(\frac{t - \delta}{2} \right)^\gamma (x^\Delta(t - \delta))^\gamma \leq 0. \quad (3.8)$$

Set $z(t) = r(t)(x^\Delta(t))^\gamma$. Then $z(t)$ is positive and satisfies the inequality (3.1), and this contradicts the assumption of our theorem. Thus every solution of (1.1) oscillates. The proof is complete. \square

Theorem 3.1 reduces the question of oscillation of (1.1) to the absence of eventually positive solution (the oscillatory) of the differential inequality (3.1). As a result Theorem 3.1 and Lemma 3.1 immediately imply:

Theorem 3.2. Assume that (H_1) – (H_2) hold. Furthermore, assume that $r^\Delta(t) \geq 0$. Then every solution of (1.1) oscillates if

$$\limsup_{t \rightarrow \infty} \sup_{\lambda > 0, -\lambda A \in \mathfrak{H}^+} \lambda e^{-\lambda A}(t, t - \delta) < 1,$$

where $A(t)$ is as defined in Theorem 3.1.

Theorem 3.3. Assume that (H_1) – (H_2) hold. Furthermore, assume that $r^\Delta(t) \geq 0$. Then every solution of (1.1) oscillates if

$$\limsup_{t \rightarrow \infty} \int_{t-\delta}^t \frac{q(s)(1 - p(s - \delta))^\gamma \left(\frac{s - \delta}{2}\right)^\gamma}{r(s - \delta)} \Delta s > 1. \tag{3.9}$$

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.1) and proceed as in Theorem 3.1 to get (3.1). Integrating (3.1) from $t - \delta$ to t for t sufficiently large yields

$$\begin{aligned} 0 &\geq \int_{t-\delta}^t z^\Delta(s) \Delta s + \int_{t-\delta}^t A(s)z(s - \delta) \Delta s \\ &= z(t) - z(t - \delta) + \int_{t-\delta}^t A(s)z(s - \delta) \Delta s \\ &\geq z(t) - z(t - \delta) + z(t - \delta) \int_{t-\delta}^t A(s) \Delta s \\ &= z(t) + z(t - \delta) \left(\int_{t-\delta}^t A(s) \Delta s - 1 \right) > 0, \end{aligned}$$

using (3.9). This is a contradiction. The proof is complete. \square

The case $\gamma \geq 1$

Let $\gamma \geq 1$ be a quotient of odd positive integers. We use a Riccati transformation technique to establish new oscillation criteria on time scales (where all the points are right scattered). In this case a solution y of (1.1) is said to be oscillatory on a time scale \mathbb{T} if for any $t_1 \in \mathbb{T}$ there exists a $t_2 \geq t_1$ such that $y(t_2)y(\sigma(t_2)) \leq 0$, otherwise it is nonoscillatory; it is implicitly assumed that $\sigma(t) \neq t$ for each $t \in \mathbb{T}$ here so all points of \mathbb{T} are right scattered.

Theorem 3.4. Assume that (H_1) – (H_2) hold. Furthermore, assume that $r^\Delta(t) \geq 0$ and there exists a positive rd-continuous Δ -differentiable function $\alpha(t)$ such that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[\alpha(s)Q(s) - \frac{((\alpha^\Delta(s))_+)^2 r(s - \delta)}{4\gamma \left(\frac{s - \delta}{2}\right)^{\gamma-1} \alpha(s)} \right] \Delta s = \infty, \tag{3.10}$$

where $(\alpha^\Delta(s))_+ = \max\{0, \alpha^\Delta(s)\}$ and $Q(t) = q(t)(1 - p(t - \delta))^\gamma$. Then every solution of Eq. (1.1) is oscillatory on $[t_0, \infty)$.

Proof. Suppose (1.1) has a nonoscillatory solution y . We may assume without loss of generality that $y(t - N) > 0$ where $N = \max\{\tau, \delta\}$ for all $t \geq t_0$. Proceed as in the proof of Theorem 3.1 and we get (3.6) and (3.7). Define the function w by

$$w(t) = \alpha(t) \frac{r(t)(x^\Delta(t))^\gamma}{x^\gamma(t - \delta)}, \quad t \geq t_2. \quad (3.11)$$

Then $w(t) > 0$, and using (2.5) and (2.6) we get

$$\begin{aligned} w^\Delta(t) &= r(x^\Delta)^\gamma(\sigma(t)) \left[\frac{\alpha(t)}{x^\gamma(t - \delta)} \right]^\Delta + \frac{\alpha(t)}{x^\gamma(t - \delta)} (r(t)(x^\Delta(t))^\gamma)^\Delta \\ &= \frac{\alpha(t)}{x^\gamma(t - \delta)} (r(t)(x^\Delta(t))^\gamma)^\Delta \\ &\quad + (r(x^\Delta)^\gamma)^\sigma \left[\frac{x^\gamma(t - \delta)\alpha^\Delta(t) - \alpha(t)(x^\gamma(t - \delta))^\Delta}{x^\gamma(t - \delta)x^\gamma(\sigma(t) - \delta)} \right]. \end{aligned} \quad (3.12)$$

Now (3.7) and (3.12) imply

$$w^\Delta(t) \leq -\alpha(t)Q(t) + \frac{\alpha^\Delta(t)}{\alpha^\sigma} w^\sigma - \frac{\alpha(t)(r(x^\Delta)^\gamma)^\sigma (x^\gamma(t - \delta))^\Delta}{x^\gamma(t - \delta)x^\gamma(\sigma(t) - \delta)}. \quad (3.13)$$

From (2.2) we have

$$\frac{x^\gamma(\sigma(t)) - x^\gamma(t)}{\mu(t)} = (x^\gamma(t))^\Delta.$$

Using the inequality (cf. [16])

$$x^\beta - y^\beta \geq \beta y^{\beta-1}(x - y) \quad \text{for all } x \geq y > 0 \text{ and } \beta \geq 1,$$

we have

$$\begin{aligned} (x^\gamma(t - \delta))^\Delta &= \frac{x^\gamma(\sigma(t - \delta)) - x^\gamma(t - \delta)}{\mu(t - \delta)} \\ &\geq \frac{\gamma x^{\gamma-1}(t - \delta)}{\mu(t - \delta)} (x(\sigma(t - \delta)) - x(t - \delta)) \\ &= \gamma x^{\gamma-1}(t - \delta)(x^\Delta(t - \delta)), \end{aligned} \quad (3.14)$$

so it follows from (3.13) and (3.14) that

$$w^\Delta(t) \leq -\alpha(t)Q(t) + \frac{\alpha^\Delta(t)}{\alpha^\sigma} w^\sigma - \frac{\alpha(t)(r(x^\Delta)^\gamma)^\sigma \gamma x^{\gamma-1}(t - \delta)(x^\Delta(t - \delta))}{x^\gamma(t - \delta)x^\gamma(\sigma(t) - \delta)}. \quad (3.15)$$

As in the proof of Theorem 3.1, using $r^\Delta(t) \geq 0$ and (3.6) there exists $t_3 \geq 2t_2$ so that for $t \in [t_3, \infty)$ we have

$$x(t) \geq \frac{t}{2} x^\Delta(t),$$

and so

$$\gamma x^{\gamma-1}(t) \geq \gamma \left(\frac{t}{2}\right)^{\gamma-1} (x^\Delta(t))^{\gamma-1}.$$

From (3.6), since $(r(t)(x^\gamma(t))^\Delta)^\Delta < 0$, we have

$$r(x^\Delta(t))^\gamma > (r(x^\Delta)^\gamma)^\sigma, \tag{3.16}$$

so it follows from (3.16) and (3.6) that

$$\begin{aligned} \gamma x^{\gamma-1}(t-\delta)(x^\Delta(t-\delta)) &\geq \gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} (x^\Delta(t-\delta))^\gamma \\ &\geq \gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \frac{r(\sigma(t)-\delta)}{r(t-\delta)} (x^\Delta(\sigma(t)-\delta))^\gamma \\ &\geq \gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \frac{(r(x^\Delta)^\gamma)^\sigma}{r(t-\delta)}. \end{aligned} \tag{3.17}$$

Substituting (3.17) in (3.15) and using (3.11) yields

$$w^\Delta(t) \leq -\alpha(t)Q(t) + \frac{\alpha^\Delta(t)}{\alpha^\sigma} w^\sigma - \frac{\gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \alpha(t)}{(\alpha^\sigma)^2 r(t-\delta)} (w^\sigma)^2. \tag{3.18}$$

Using the fact that $u - mu^2 \leq 1/4m$ we have

$$\begin{aligned} w^\Delta(t) &\leq -\alpha(t)Q(t) + \frac{(\alpha^\Delta(t))_+}{\alpha^\sigma} \left[w^\sigma - \frac{\gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \alpha(t)}{((\alpha^\Delta(t))_+)(\alpha^\sigma)r(t-\delta)} (w^\sigma)^2 \right] \\ &\leq -\alpha(t)Q(t) + \frac{((\alpha^\Delta(t))_+)^2 r(t-\delta)}{4\gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \alpha(t)} = - \left[\alpha(t)Q(t) - \frac{(\alpha^\Delta(t))r(t-\delta)}{4\gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \alpha(t)} \right]. \end{aligned}$$

Then

$$w^\Delta(t) \leq - \left[\alpha(t)Q(t) - \frac{((\alpha^\Delta(t))_+)^2 r(t-\delta)}{4\gamma \left(\frac{t-\delta}{2}\right)^{\gamma-1} \alpha(t)} \right].$$

Integrating the last inequality from t_3 to t we obtain

$$-w(t_3) < w(t) - w(t_3) \leq - \int_{t_3}^t \left[\alpha(s)Q(s) - \frac{((\alpha^\Delta(s))_+)^2 r(s-\delta)}{4\gamma \left(\frac{s-\delta}{2}\right)^{\gamma-1} \alpha(s)} \right] \Delta s,$$

which yields

$$\int_{t_3}^t \left[\alpha(s)Q(s) - \frac{((\alpha^\Delta(s))_+)^2 r(s-\delta)}{4\gamma \left(\frac{s-\delta}{2}\right)^{\gamma-1} \alpha(s)} \right] \Delta s < w(t_3)$$

for all large t , which contradicts (3.10). The proof is complete. \square

Remark 3.1. From Theorem 3.4 we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of $\alpha(t)$.

For example, let $\alpha(t) = t$, $t \geq t_0$. Now Theorem 3.4 yields the following result.

Corollary 3.1. Assume that (H_1) – (H_2) hold. Furthermore, assume that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[s Q(s) - \frac{r(s-\delta)}{4\gamma \left(\frac{s-\delta}{2}\right)^{\gamma-1} s} \right] \Delta s = \infty.$$

Then every solution of (1.1) is oscillatory on $[t_0, \infty)$.

Let $\alpha(t) = 1$, $t \geq t_0$. Now Theorem 3.4 yields the following well-known result (Leighton–Wintner theorem).

Corollary 3.2 (Leighton–Wintner). Assume that (H_1) – (H_2) hold. If

$$\int_{t_0}^{\infty} q(s)(1-p(s-\delta))^{\gamma} \Delta s = \infty,$$

then every solution of (1.1) is oscillatory on $[t_0, \infty)$.

The following theorem gives a Kamenev-type oscillation criteria for (1.1).

Theorem 3.5. Assume that (H_1) – (H_2) hold. Let $\alpha(t)$ be as defined in Theorem 3.4. If

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \left[\alpha(s) Q(s) - \frac{((\alpha^\Delta(s))_+)^2 r(s-\delta)}{4\gamma \left(\frac{s-\delta}{2}\right)^{\gamma-1} \alpha(s)} \right] \Delta s = \infty \quad (3.19)$$

for an odd positive integer m , then every solution of (1.1) is oscillatory on $[t_0, \infty)$.

The proof is similar to that of the proof of Theorem 3.3 in [19] using the inequality (3.18). We omit the details.

Note that when $\alpha(t) = 1$, then (3.19) reduces to

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m Q(s) \Delta s = \infty. \quad (3.20)$$

Corollary 3.3. Assume that (H_2) holds, $r(n)$ and $q(n)$ are positive sequences defined on $[t_0, \infty) \subset \mathbb{N}$ such that $\Delta r(n) \geq 0$, $0 \leq p(n) < 1$ and

$$\sum_{i=t_0}^{\infty} \left[\frac{1}{(r(i))^{1/\gamma}} \right] = \infty. \quad (3.21)$$

Furthermore, assume that there exists a positive sequence $\alpha(n)$ such that

$$\limsup_{t \rightarrow \infty} \sum_{i_0}^{t-1} \left[\alpha(i) Q(i) - \frac{r(i - \delta) ((\Delta \alpha(i))_+)^2}{4\gamma \left(\frac{i-\delta}{2}\right)^{\gamma-1} \alpha(i)} \right] = \infty, \tag{3.22}$$

where $(\Delta \alpha(i))_+ = \max\{0, \Delta \alpha(i)\}$. Then every solution of Eq. (1.3) is oscillatory.

Corollary 3.4. Assume that (H_2) holds, $r(n)$ and $q(n)$ are positive sequences defined on $[t_0, \infty) \subset h\mathbb{N}$, $h > 0$ such that $\Delta r(n) \geq 0$, $0 \leq p(n) < 1$ and

$$\sum_{i=t_0/h}^{\infty} \left[\frac{1}{(r(i))^{1/\gamma}} \right] = \infty. \tag{3.23}$$

Furthermore, assume that there exists a positive sequence $\alpha(n)$ such that

$$\limsup_{t \rightarrow \infty} \sum_{i_0/h}^{t/h-1} \left[\alpha(i) Q(i) - \frac{r(i - \delta) ((\Delta_h \alpha(i))_+)^2}{4\gamma \left(\frac{i-\delta}{2}\right)^{\gamma-1} \alpha(i)} \right] = \infty, \tag{3.24}$$

where $(\Delta_h \alpha(i))_+ = \max\{0, \Delta_h \alpha(i)\}$. Then every solution of Eq. (1.4) is oscillatory.

Corollary 3.5. Assume that (H_2) holds, $r(n)$ and $q(n)$ are positive sequences defined on $[t_0, \infty) \subset q^{\mathbb{N}}$ such that $\Delta r(n) \geq 0$, $0 \leq p(n) < 1$ and

$$\sum_{i_0}^{\infty} \mu(q^i) \left[\frac{1}{(r(q^i))^{1/\gamma}} \right] = \infty. \tag{3.25}$$

Furthermore, assume that there exists a positive sequence $\alpha(n)$ such that

$$\sum_{i_0}^{\infty} \mu(q^i) \left[\alpha(q^i) Q(q^i) - \frac{r(q^i - \delta) ((\Delta_q \alpha(q^i))_+)^2}{4\gamma \left(\frac{q^i-\delta}{2}\right)^{\gamma-1} \alpha(q^i)} \right] = \infty, \tag{3.26}$$

where $(\Delta_q \alpha(i))_+ = \max\{0, \Delta_q \alpha(i)\}$. Then every solution of Eq. (1.5) is oscillatory.

Remark 3.2. It is clear that the condition $r^\Delta(t) \geq 0$ plays an important rule in the proof of Theorem 3.4. In the following we establish some new oscillation criteria without the condition $r^\Delta(t) \geq 0$.

Theorem 3.6. Assume that (H_1) – (H_2) hold. Let $\alpha(t)$ be as defined in Theorem 3.4. If

$$\limsup_{t \rightarrow \infty} \int_{i_0}^t \left[\alpha(s) Q(s) - \frac{r(s - \delta) ((\alpha^\Delta(s))_+)^2}{2^{3-\gamma} (\mu(s - \delta))^{\gamma-1} \alpha(s)} \right] \Delta s = \infty, \tag{3.27}$$

then every solution of Eq. (1.1) is oscillatory on $[t_0, \infty)$.

Proof. As in Theorem 3.1, we assume y is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that y is an eventually positive solution of (1.1) with $y(t - N) > 0$ for all $t \geq t_0$. Proceeding as in the proof of Theorem 3.1, we see that (3.6)

and (3.7) hold. Define again the function $w(t)$ by (3.11). Then $w(t) > 0$ and satisfies (3.13). Using the inequality (cf. [20])

$$x^\gamma - y^\gamma \geq 2^{1-\gamma}(x - y)^\gamma, \quad \gamma \geq 1,$$

we have

$$\begin{aligned} (x^\gamma(t))^\Delta &= \frac{x^\gamma(\sigma(t)) - x^\gamma(t)}{\mu(t)} \geq 2^{1-\gamma} \frac{1}{\mu(t)} (x(\sigma(t)) - x(t))^\gamma \\ &= 2^{1-\gamma} (\mu(t))^{\gamma-1} \left(\frac{x(\sigma(t)) - x(t)}{\mu(t)} \right)^\gamma = 2^{1-\gamma} (\mu(t))^{\gamma-1} (x^\Delta(t))^\gamma. \end{aligned} \quad (3.28)$$

It follows from (3.6), (3.13) and (3.28) that

$$w^\Delta(t) \leq -\alpha(t)Q(t) + \frac{(\alpha^\Delta(t))_+}{\alpha^\sigma} w^\sigma - \frac{2^{1-\gamma} (\mu(t-\delta))^{\gamma-1} \alpha(t) ((r(x^\Delta)^\gamma)^\sigma)^2}{x^{2\gamma}(\sigma(t) - \delta)r(t-\delta)}. \quad (3.29)$$

Now, using (3.11) we have

$$w^\Delta(t) \leq -\alpha(t)Q(t) + \frac{(\alpha^\Delta(t))_+}{\alpha^\sigma} w^\sigma - \frac{2^{1-\gamma} (\mu(t-\delta))^{\gamma-1} \alpha(t)}{(\alpha^\sigma)^2 r(t-\delta)} (w^\sigma)^2. \quad (3.30)$$

The remainder of the proof is similar to that of the proof of Theorem 3.4 and hence is omitted. \square

Corollary 3.6. Assume that (H_1) – (H_2) hold. Furthermore, assume that

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \left[sQ(s) - \frac{r(s-\delta)}{2^{3-\gamma} (\mu(s-\delta))^{\gamma-1} s} \right] \Delta s = \infty. \quad (3.31)$$

Then, every solution of (1.1) is oscillatory on $[t_0, \infty)$.

Theorem 3.7. Assume that (H_1) – (H_2) hold. Let $\alpha(t)$ be as defined in Theorem 3.4. If

$$\limsup_{t \rightarrow \infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \left[\alpha(s)Q(s) - \frac{r(s-\delta) ((\alpha^\Delta(s))_+)^2}{2^{3-\gamma} (\mu(s-\delta))^{\gamma-1} \alpha(s)} \right] \Delta s = \infty \quad (3.32)$$

for an odd positive integer m , then every solution of (1.1) is oscillatory on $[t_0, \infty)$.

Corollary 3.7. Assume that (H_2) holds, $r(n)$ and $q(n)$ are positive sequences defined on $[t_0, \infty) \subset \mathbb{N}$ such that $0 \leq p(n) < 1$ and (3.21) holds. Furthermore, assume $0 \leq p(n) < 1$ and there exists a positive sequence $\alpha(n)$ such that

$$\limsup_{t \rightarrow \infty} \sum_{i_0}^{t-1} \left[\alpha(i)Q(i) - \frac{r(i-\delta) ((\Delta\alpha(i))_+)^2}{2^{3-\gamma} \alpha(i)} \right] = \infty,$$

where $(\Delta\alpha(i))_+ = \max\{0, \Delta\alpha(i)\}$. Then every solution of Eq. (1.3) is oscillatory.

Corollary 3.8. Assume that (H_2) holds, $r(n)$ and $q(n)$ are positive sequences defined on $[t_0, \infty) \subset h\mathbb{N}$, $h > 0$ such that $0 \leq p(n) < 1$ and (3.23) holds. Furthermore, assume $0 \leq p(n) < 1$ and there exists a positive sequence $\alpha(n)$ such that

$$\limsup_{t \rightarrow \infty} \sum_{t_0/h}^{t/h-1} h \left[\alpha(i) Q(i) - \frac{r(i-\delta)((\Delta_h \alpha(i))_+)^2}{2^{3-\gamma} h^{\gamma-2} \alpha(i)} \right] = \infty,$$

where $(\Delta_h \alpha(n))_+ = \max\{0, \Delta_h \alpha(n)\}$. Then every solution of Eq. (1.4) is oscillatory.

Corollary 3.9. Assume that (H_2) holds, $r(n)$ and $q(n)$ are positive sequences defined on $[t_0, \infty) \subset q^{\mathbb{N}}$ such that $0 \leq p(n) < 1$ and (3.25) holds. Furthermore, assume there exists a positive sequence $\alpha(n)$ such that

$$\sum_{t_0}^{\infty} (q-1)q^i \left[\alpha(q^i) Q(q^i) - \frac{r(q^i-\delta)((\Delta_q \alpha(q^i))_+)^2}{2^{1-\gamma} ((q-1)q^i-\delta)^{\gamma-1} \alpha(q^i)} \right] = \infty,$$

where $(\Delta_q \alpha(i))_+ = \max\{0, \Delta_q \alpha(i)\}$. Then every solution of Eq. (1.5) is oscillatory.

The sufficient conditions for the oscillation of (1.6) and (1.7) are left to the interested reader. One uses

$$\int_{t_0}^b f(t) \Delta t = \sum_{t \in [t_0, b)} \mu(t) f(t).$$

4. Examples

In this section, we give some examples which illustrate our main results.

Example 4.1. Consider the following second-order neutral nonlinear delay dynamic equation:

$$\left(\left(\left(y(t) + \frac{t+\delta-1}{t+\delta} y(t-\tau) \right)^\Delta \right)^\gamma \right)^\Delta + t^\alpha y^\gamma(t-\delta) = 0, \quad t \in \mathbb{T}, \tag{4.1}$$

where $\mathbb{T} = [1, \infty)$ is a time scale (where all the points are right scattered), α and γ are constants with $\gamma \geq 1$ is a quotient of odd positive integers and τ and δ are nonnegative constants. In (4.1), $r(t) \equiv 1$, $p(t) \equiv (t+\delta-1)/(t+\delta)$, and $q(t) = t^\alpha$. It is easy to see that the assumptions (H_1) and (H_2) hold. We will apply Corollary 3.1. Note

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[s Q(s) - \frac{r(s-\delta)}{4\gamma \left(\frac{s-\delta}{2}\right)^{\gamma-1} s^\gamma} \right] \Delta s \\ &= \limsup_{t \rightarrow \infty} \int_1^t \left[s^{1+\alpha-\gamma} - \frac{1}{2^{3-\gamma} (s-\delta)^{\gamma-1} s} \right] \Delta s \end{aligned}$$

$$\geq \limsup_{t \rightarrow \infty} \int_1^t \left[s^{1+\alpha-\gamma} - \frac{1}{2^{3-\gamma} s^\gamma} \right] \Delta s = \infty$$

if $\gamma > 1$ and $\alpha - \gamma \geq -2$. Therefore (4.1) with $\gamma > 1$ is oscillatory if $\alpha - \gamma \geq -2$.

Example 4.2. Consider the following dynamic equation:

$$\left((t + \delta)^{\gamma-1} \left(\left(y(t) + \frac{t + \delta - 1}{t + \delta} y(t - \tau) \right)^\Delta \right)^\gamma \right)^\Delta + \beta t^{\gamma-2} y(t - \delta) = 0, \quad t \in \mathbb{T}, \quad (4.2)$$

where $\mathbb{T} = [1, \infty)$ is a time scale (where all the points are right scattered), $\beta > 0$ and $\gamma \geq 1$ and is a quotient of odd positive integers and τ and δ are nonnegative constants. In (4.2), $r(t) \equiv (t + \delta)^{\gamma-1}$, $p(t) \equiv (t + \delta - 1)/(t + \delta)$, and $q(t) = \beta t^{\gamma-2}$. It is easy to see that the assumptions (H_1) and (H_2) hold. We will apply Corollary 3.1. Note

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \int_{t_0}^t \left[s Q(s) - \frac{r(s - \delta)}{2^{3-\gamma} \gamma (s - \delta)^{\gamma-1} s} \right] \Delta s \\ &= \limsup_{t \rightarrow \infty} \int_1^t \left[\frac{\beta}{s} - \frac{s^{\gamma-1}}{2^{3-\gamma} \gamma (s - \delta)^{\gamma-1} s} \right] \Delta s \geq \limsup_{t \rightarrow \infty} \int_1^t \left[\frac{\beta}{s} - \frac{1}{2^{3-\gamma} \gamma s} \right] \Delta s = \infty \end{aligned}$$

if $\beta > 1/(2^{3-\gamma} \gamma)$. Therefore (4.2) with $\gamma \geq 1$ is oscillatory if $\beta > 1/(2^{3-\gamma} \gamma)$.

Example 4.3. Consider the following dynamic equation:

$$\left(y(t) + \frac{t + \delta - 1}{t + \delta} y(t - \tau) \right)^{\Delta\Delta} + \beta t^{\gamma-2} y(t - \delta) = 0, \quad t \in \mathbb{T}, \quad (4.3)$$

where $\mathbb{T} = [1, \infty)$ is a time scale (where all the points are right scattered), $\beta > 0$ and $\gamma = 1$ and τ and δ are nonnegative constants. In (4.3), $r(t) \equiv 1$, $p(t) \equiv (t + \delta - 1)/(t + \delta)$, and $q(t) = \beta t^{\gamma-2}$. It is easy to see that the assumptions (H_1) and (H_2) hold. We will apply Corollary 3.6. Note

$$\limsup_{t \rightarrow \infty} \int_1^t \left[s Q(s) - \frac{r(s - \delta)}{2^{3-\gamma} (\mu(s - \delta))^{\gamma-1} s} \right] \Delta s = \limsup_{t \rightarrow \infty} \int_1^t \left[\frac{\beta}{s} - \frac{1}{4s} \right] \Delta s = \infty$$

if $\beta > 1/4$. Therefore (4.3) is oscillatory if $\beta > 1/4$.

References

- [1] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: a survey, in: R.P. Agarwal, M. Bohner, D. O'Regan (Eds.), Special Issue on "Dynamic Equations on Time Scales", J. Comput. Appl. Math. 141 (1–2) (2002) 1–26.

- [2] R.P. Agarwal, M. Bohner, S.H. Saker, Oscillation of second order delay dynamic equations, *Canad. Appl. Math. Quart.*, in press.
- [3] E.A. Bohner, J. Hoffacker, Oscillation properties of an Emden–Fowler type equations on discrete time scales, *J. Difference Equations Appl.* 9 (2003) 603–612.
- [4] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.
- [5] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
- [6] M. Bohner, S.H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, *Rocky Mountain J. Math.* 34 (2004), in press.
- [7] E.A. Bohner, M. Bohner, S.H. Saker, Oscillation criteria for a certain class of second order Emden–Fowler dynamic equations, *Electron. Trans. Numer. Anal.*, in press.
- [8] O. Došlý, S. Hilger, A necessary and sufficient condition for oscillation of the Sturm–Liouville dynamic equation on time scales, in: R.P. Agarwal, M. Bohner, D. O’Regan (Eds.), Special Issue on “Dynamic Equations on Time Scales”, *J. Comput. Appl. Math.* 141 (1–2) (2002) 147–158.
- [9] L. Erbe, Oscillation criteria for second order linear equations on a time scale, *Canad. Appl. Math. Quart.* 9 (2001) 1–31.
- [10] L. Erbe, A. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, *Math. Comput. Modelling* 32 (2000) 571–585.
- [11] L. Erbe, A. Peterson, Riccati equations on a measure chain, in: G.S. Ladde, N.G. Medhin, M. Sambandham (Eds.), *Proceedings of Dynamic Systems and Applications*, vol. 3, Dynamic Publishers, Atlanta, 2001, pp. 193–199.
- [12] L. Erbe, A. Peterson, Oscillation criteria for second order matrix dynamic equations on a time scale, in: R.P. Agarwal, M. Bohner, D. O’Regan (Eds.), Special Issue on “Dynamic Equations on Time Scales”, *J. Comput. Appl. Math.* 141 (1–2) (2002) 169–185.
- [13] L. Erbe, A. Peterson, Boundedness and oscillation for nonlinear dynamic equations on a time scale, *Proc. Amer. Math. Soc.* 132 (2004) 735–744.
- [14] L. Erbe, A. Peterson, S.H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, *J. London Math. Soc.* 67 (2003) 701–714.
- [15] G.Sh. Guseinov, B. Kaymakçalan, On a disconjugacy criterion for second order dynamic equations on time scales, in: R.P. Agarwal, M. Bohner, D. O’Regan (Eds.), Special Issue on “Dynamic Equations on Time Scales”, *J. Comput. Appl. Math.* 141 (1–2) (2002) 187–196.
- [16] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, second ed., Cambridge Univ. Press, 1952.
- [17] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, *Results Math.* 18 (1990) 18–56.
- [18] S.H.G. Olumolode, N. Pennington, A. Peterson, Oscillation of an Euler–Cauchy dynamic equation, in: *Proceeding of the Fourth International Conference on Dynamical Systems and Differential Equations*, Wilmington, 2002, pp. 24–27.
- [19] S.H. Saker, Oscillation of nonlinear dynamic equations on time scales, *Appl. Math. Comput.* 148 (2004) 81–91.
- [20] S.H. Saker, New oscillation criteria for second-order nonlinear neutral delay difference equations, *Appl. Math. Comput.* 142 (2003) 99–111.
- [21] B.G. Zhang, X. Deng, Oscillation of delay differential equations on time scales, *Math. Comput. Modelling* 36 (2002) 1307–1318.