



Comment on “Group solution of a time dependent chemical convective process” by M.M. Kassem and A.S. Rashed, Applied Mathematics and Computation, 215 (2009) 1671–1684

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ABSTRACT

The mathematical model formulated by M.M. Kassem and A.S. Rashed in their article: “Group solution of a time dependent chemical convective process, Applied Mathematics and Computation, 215 (2009) 1671–1684”, through group analysis, is reformulated and interpreted correctly so that it can represent a feasible situation. A perturbation analysis that replaces their incorrect analysis is performed and proved to compare well with a finite difference solution of the problem.

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1. Flow description and similarity transformation

The authors of [1] consider the unsteady problem of the incompressible viscous flow of a reacting fluid along a vertical surface. Their governing equations are the following boundary layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - g\beta c - \nu \frac{\partial^2 u}{\partial y^2} = 0, \quad (1.2)$$

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + kc^n - D \frac{\partial^2 c}{\partial y^2} = 0, \quad (1.3)$$

with appropriate boundary and initial conditions; in particular,

$$u = 0, \quad v = 0, \quad c = c_0 \quad \text{at} \quad y = 0, \quad (1.4)$$

$$u \sim 0, \quad c \sim 0 \quad \text{as} \quad y \sim \infty. \quad (1.5)$$

Using group analysis the authors of [1] look for cases that allow self similarity, and arrive at the following case.

- The longitudinal velocity component u and the concentration c do not depend on the longitudinal coordinate x . The continuity equation (1.1) and condition (1.4b) then necessitate that the normal velocity v vanishes. The convective terms disappear from the longitudinal momentum equation (1.2) and the concentration equation (1.3). Contrary to the title of [1],

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the process involves reaction, diffusion, buoyancy, but certainly no convection. Moreover, that u and c are independent of x means that the flow is not evolving in the x -direction, which necessitates that the surface be infinite not semi-infinite as stated in [1] and depicted in Fig. 1 of [1]. Furthermore, that $u_{xx} = 0$ and $c_{xx} = 0$, and that the momentum equation in the normal direction y is satisfied when $v = 0$; extend the case to the full Navier–Stokes equations, not just the boundary layer approximation.

- The concentration at the surface is given by

$$c_0 = t^{\frac{1}{1-n}}. \quad (1.6)$$

Thus, for c_0 to be finite at $t = 0$, the chemical reaction order n must not exceed 1. However, the authors of [1] present results for $n > 1$.

- The similarity coordinate is

$$\eta = t^{-\frac{1}{2}}y, \quad (1.7)$$

and u and c transform according to

$$u = t^{\frac{2-n}{1-n}}F'(\eta), \quad (1.8)$$

$$c = c_0C(\eta). \quad (1.9)$$

The process can now be described as follows. At $t = 0$, the fluid is stationary and non-reacting. For $t > 0$, a prescribed concentration c_0 is introduced at the surface, spreading in the y -direction by diffusion, and getting consumed (for $k > 0$) by chemical reaction with the fluid. The outcome is concentration variations which result in buoyancy effects that drive an upward flow (in the x -direction), for $\beta > 0$.

The transformations (1.7)–(1.9) lead to the problem described by Eqs. (3.46)–(3.49) of [1], which contain dimensional coefficients; forcing the authors of [1] to carry out calculations at specific values of the flow properties; e.g. $v = 10 \text{ in}^2/\text{s}$. This problem (as well as other problems with the dimensions of c , k and β) could have been avoided, had the authors of [1] used the following relations

$$c_0 = \alpha_c t^{\frac{1}{1-n}}, \quad (1.10)$$

$$\eta = (vt)^{-\frac{1}{2}}y, \quad (1.11)$$

$$u = \alpha_u t^{\frac{2-n}{1-n}}U(\eta), \quad (1.12)$$

$$c = c_0C(\eta), \quad (1.13)$$

where α_c and α_u are constant coefficients such that c_0 and U are non-dimensional. These relations lead to the following problem with non-dimensional coefficients.

$$\varepsilon \left(U'' + \frac{1}{2} \eta U' - U + \gamma C \right) - U = 0, \quad (1.14)$$

$$\varepsilon \left(\lambda C'' + \frac{1}{2} \eta C' - \kappa C^{1-\varepsilon} \right) - C = 0, \quad (1.15)$$

$$U(0) = 0, \quad C(0) = 1, \quad (1.16)$$

$$U(\infty) = 0, \quad C(\infty) = 0, \quad (1.17)$$

where dashes denote differentiation with respect to η , $\varepsilon = 1 - n$, $\gamma = g\beta\alpha_c/\alpha_u$, $\kappa = k/\alpha_c^{1-n}$, and $\lambda = D/v$ is the reciprocal of the Schmidt number.

2. Perturbation analysis

The authors of [1] present an analysis that they describe as being a perturbation analysis near $n = 1$, but that in no way can be classified as such. Note that the small parameter ε which is negative in their unfeasible case with $n > 1$, multiplies the highest derivative in Eq. (1.15). Matching inner and outer regions are, therefore, expected to exist [2].

The leading order term in their expansion for C

$$C \sim C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \varepsilon^3 C_3 \dots, \quad (2.1)$$

has the solution $C_0 = 0$; which they reject, for not satisfying Condition (1.16b); although it represents the outer solution.

The authors then go on an anomalous procedure that leads them to two approximate expressions (Eqs. (5.16) and (5.18) of [1]) for C_0 and, consequently, for C ; since $C_i = 0$ for $i > 0$. Interpreted correctly, the first approximation corresponds to the neglect of the last term on the left hand side of Eq. (1.15) and replacing $C^{1-\varepsilon}$ by C in the preceding term, while the second corresponds to the neglect of these two terms. The authors apply a similar procedure to Eq. (1.14), with C substituted for by the second approximation.

For the correct perturbation analysis, we explore the inner region. The coordinate η is stretched according to

$$\eta = \varepsilon^{\frac{1}{2}} \zeta, \quad (2.2)$$

transforming Eqs. (1.14) and (1.15) to

$$U'' - U = -\varepsilon \left(\frac{1}{2} \zeta U - U + \gamma C \right), \quad (2.3)$$

$$\lambda C'' - C = -\varepsilon \left(\frac{1}{2} \zeta C' - \kappa C^{1-\varepsilon} \right), \quad (2.4)$$

with the dashes, now, denoting differentiation with respect to ζ .

We start with the problem for the concentration field, as it is independent of the velocity field. Introducing Expansion (2.1) into Eq. (2.4), Condition (1.16b) and the matching condition $C(\infty) = 0$; equating like powers of ε on both sides; and solving the resulting problems; one obtains

$$C_0 = e^{-\zeta}, \quad (2.5)$$

$$C_1 = -\left(\frac{4\kappa + 1}{8} \zeta + \frac{1}{8} \zeta^2 \right) e^{-\zeta}, \quad (2.6)$$

$$C_2 = \left[\frac{16\kappa^2 - 24\kappa - 1}{128} (\zeta + \zeta^2) + \frac{12\kappa + 1}{192} \zeta^3 + \frac{1}{128} \zeta^4 \right] e^{-\zeta}, \quad (2.7)$$

where

$$\zeta = \lambda^{-\frac{1}{2}} \eta \quad (2.8)$$

is the stretched coordinate suitable to capture the fast changes in the concentration boundary layer, which forms when $\lambda \sim 0$.

Applying the same procedure to the problem for U , with

$$U \sim U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 \dots, \quad (2.9)$$

gives, for $\lambda \neq 1$,

$$U_0 = 0, \quad (2.10)$$

$$U_1 = \frac{\gamma \lambda}{1 - \lambda} (e^{-\zeta} - e^{-\varepsilon \zeta}), \quad (2.11)$$

$$U_2 = \frac{\gamma \lambda}{1 - \lambda} \left[-\left(\frac{\kappa + 1 - \lambda}{1 - \lambda} + \frac{5}{8} \zeta + \frac{1}{8} \zeta^2 \right) e^{-\zeta} + \left(\frac{\kappa + 1 - \lambda}{1 - \lambda} + \frac{4\kappa + 5}{8} \zeta + \frac{1}{8} \zeta^2 \right) e^{-\varepsilon \zeta} \right], \quad (2.12)$$

and, for $\lambda = 1$,

$$U_0 = 0, \quad (2.13)$$

$$U_1 = \frac{1}{2} \gamma \zeta e^{-\zeta}, \quad (2.14)$$

$$U_2 = -\gamma \left(\frac{2\kappa + 3}{16} \zeta + \frac{2\kappa + 3}{16} \zeta^2 + \frac{1}{16} \zeta^3 \right) e^{-\zeta}, \quad (2.15)$$

These inner solutions contain the outer solutions as $\zeta \sim \infty$, obviating the need for the outer region.

3. Results and comparisons

Being based on an unfeasible mathematical model (with $n > 1$) and incorrect perturbation analysis, the numerical results presented in [1] should be discarded. Two more observations need to be made. The authors of [1] present results for zero Schmidt number ($\lambda^{-1} = 0$), for which the resulting linear variation for C with η can never satisfy the farfield condition (1.17b). They also tolerate regions of negative (downward) velocity U that should be precluded for positive buoyancy factor γ .

Solutions for the feasible problem, with $n < 1$, are now calculated. Eqs. (1.14)–(1.17) are solved numerically using Keller's two-point, second-order-accurate, finite-difference scheme [3]. A uniform step-size $\Delta\eta = 0.01$ is used and the farfield conditions (1.17) are invoked at a large but finite distance $\eta = \eta_\infty$. A value of $\eta_\infty = 10$ is used throughout; as further increase in η_∞ does not have an effect on the presented results. The non-linear term $C^{1-\varepsilon}$ in Eq. (1.15) is linearized, and an iterative pro-

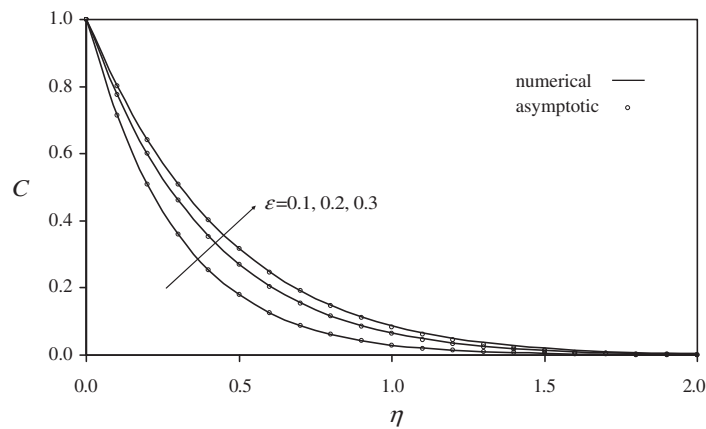


Fig. 1. Concentration profiles for different values of ε , with $\gamma = \kappa = \lambda = 1$.

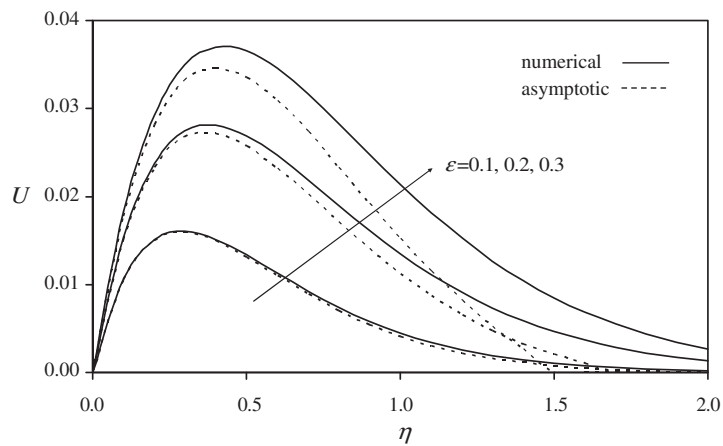


Fig. 2. Velocity profiles for different values of ε , with $\gamma = \kappa = \lambda = 1$.

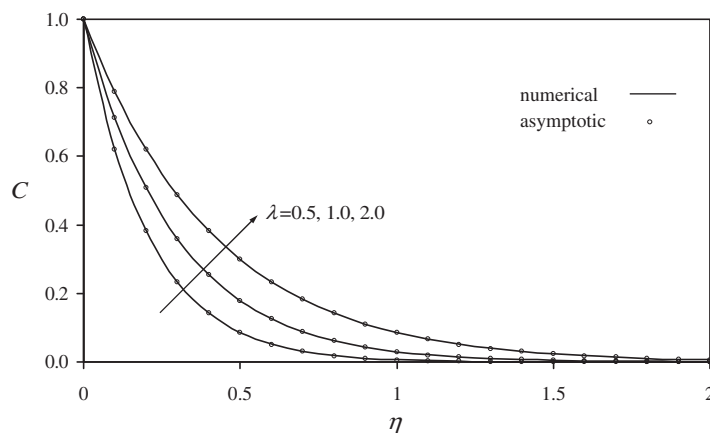


Fig. 3. Concentration profiles for different values of λ , with $\varepsilon = 0.1$ and $\gamma = \kappa = 1$.

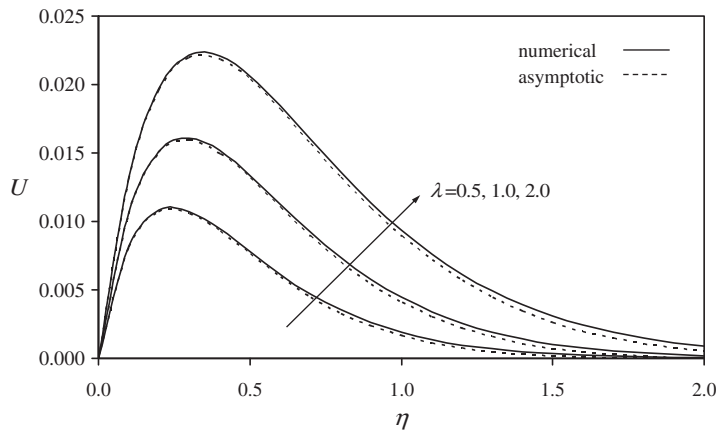


Fig. 4. Velocity profiles for different values of λ , with $\varepsilon = 0.1$ and $\gamma = \kappa = 1$.

cedure with under-relaxation is implemented; terminating when the maximum error in $U(0)$ and $C(0)$ becomes less than 10^{-10} .

Fig. 1 shows the variation of C with η obtained by the numerical method (solid lines), for different values of $\varepsilon = 0.1, 0.2, 0.3$ with $\gamma = \kappa = \lambda = 1$. The concentration diminishes quickly and monotonically from its maximum value of unity at $\eta = 0$ to almost zero at $\eta \approx 2$. Shown also (by markers) are results obtained by the three-term asymptotic expansion (2.1) with (2.5)–(2.7). The agreement is excellent. The corresponding results for U are shown in Fig. 2. The results (dashed lines) of the two-term asymptotic expansion (2.9) with (2.13)–(2.15) deviate from the numerical results (solid lines) as η grows, becoming even negative; thus requiring more terms to be included. The effect of λ on C and U is depicted in Figs. 3 and 4, respectively. It is observed that as λ decreases, the concentration boundary layer diminishes in size; so that, at any given η , the concentration C decreases. Since C drives the flow by buoyancy, the velocity U decreases as well.

4. Conclusion

The authors of [1] presented an involved group analysis that led to a self-similar problem, which involves chemical reaction, diffusion, and buoyancy. Unfortunately, the problem was interpreted and analyzed incorrectly. With few but significant changes to theirs, this comment has presented a feasible problem whose numerical solution and correct perturbation analysis as the chemical reaction index approaches unity has compared well.

References

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