Parameter Estimation for the New Weibull–Pareto Distribution under Type II Censored Samples

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Abstract— In this paper, parameters estimation under type II censored samples and the corresponding variance covariance matrix for the new Weibull–Pareto distribution are obtained. Nasiru and Luguterah (2015) Results may be considered as a special case from present results. An illustrative example is carried out by using a simulated data.

Index Terms— The new Weibull-Pareto distribution, type II censored samples, asymptotic variance covariance matrix.

I. INTRODUCTION

Many lifetime data used for statistical analysis follows a particular statistical distribution. Knowledge of the appropriate distribution that any phenomenon follows, greatly improves the sensitivity, power and efficiency of the statistical tests associated with it. Several distributions exist for modeling these lifetime data however, some of these lifetime data do not follow these existing distributions or are inappropriately described by them. This therefore creates room for developing new distributions which could better describe some of these phenomena and therefore provide greater flexibility in the modeling of lifetime data.

Gupta et al. (1998) developed the exponentiated exponential distribution, Mudholkar et al. (1995) proposed the exponentiated- Weibull distribution, Akinsete et al. (2008) developed the beta-Pareto distribution, Alzaatreh et al. (2012) developed the gamma-Pareto distribution and Alzaatreh al.(2013) developed the Weibull-Pareto distribution. Also, Merovci and Puka (2014) developed the transmuted Pareto distribution while Kareema and Boshi (2013), developed the Exponential Pareto distribution.

Nasiru and Luguterah (2015) introduced the probability density and the cumulative distribution functions of the new Weibull – Pareto distribution (NWPD) given as,

$$f(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, 0 < x < \infty, \beta > 0, \theta > 0, \delta > 0,$$

(1)

and

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$$F(x) = 1 - e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}}, 0 < x > \infty, \beta > 0, \theta > 0, \delta > 0,$$

(2)

Where β is a shape parameter, and δ and θ are some scale parameters. If $\delta = 1$ the distribution (1) reduced to Weibull distribution and if $\delta = 1, \beta = 1$, the distribution (1) reduced to Exponential distribution.

Nasiru and Luguterah (2015) presented some properties of New Weibull-Pareto Distribution such as Survival, Hazard functions as follows:

$$S(x) = e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}},$$

and

$$h(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1}.$$

Nasiru and Luguterah (2015) introduced moment and moment generating functions, Incomplete moments, Quantile function:

Moments

If X is a random variable distributed as NWPD, then the rth non-central moment is given by:

$$E(x^r) = \theta^r \delta^{\frac{-r}{\beta}} \Gamma\left(\frac{\beta+r}{r}\right),$$

Where $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \qquad a > 0,$

$$M_x(t) = \sum_{i=0}^\infty \frac{t^i}{i!} \theta^i \delta^{\frac{-i}{\beta}} \Gamma\left(\frac{\beta+i}{\beta}\right).$$

Incomplete Moment

$$M_r(z) = \int_0^z x^r \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} dx.$$

(3)

is the lower incomplete gamma and is given by:

$$\Gamma\frac{\beta+r}{\beta}, \delta\left(\frac{z}{\theta}\right)^{\beta}, \text{ Then}$$

$$M_r(z) = \theta^r \delta^{\frac{-r}{\beta}} \Gamma\frac{\beta+r}{\beta}, \delta\left(\frac{z}{\theta}\right)^{\beta}$$

Quantile function

$$x = \theta \left\{ \frac{1}{\delta} \ln\left(\frac{1}{1-u} \right) \right\}^{\frac{1}{\beta}}$$

Nasiru and Luguterah (2015) introduced Mode and Mean deviation,

The mode of the NWPD is obtained by finding the first derivate of $\ln f(x)$ with respect to x and equating it to zero. Therefore, the mode at $x = x_0$ is given by

$$x_0 = \theta \left[\frac{\beta - 1}{\beta \delta} \right]^{\frac{1}{\beta}}$$

The mean deviation about the mean and the median are useful measures of variation for a population. Let $\mu = E(X)$ and m be the mean and median of the NWPD respectively. The mean deviation about the mean is

$$D(\mu) = E(|x - \mu|) = \int_0^\infty |x - \mu| f(x)\,dx = 2\mu\, F(\mu) - 2\int_0^\mu xf(x)\,dx$$

$$= 2\mu\, F(\mu) - 2\,\theta\, \delta^{\frac{-1}{\beta}}\, \Gamma\left(\frac{\beta+1}{\beta},\ \delta\left(\frac{\mu}{\theta} \right)^{\beta} \right)$$

,
The mean deviation from the median is:

$$D(m) = E(|x - m|) = \int_0^\infty |x - m| f(x)\,dx = \mu - 2\int_0^m x\, g(x)\,dx$$

$$= \mu - 2\,\theta\, \delta^{\frac{-1}{\beta}}\, \Gamma\left(\frac{\beta+1}{\beta},\ \delta\left(\frac{m}{\theta} \right)^{\beta} \right)$$

Nasiru and Luguterah assumed $x_{(1)}$ denote the smallest of $\{x_1, x_2, ..., x_n\}$, $x_{(2)}$ denote the second smallest of $\{x_1, x_2, ..., x_n\}$, and similarly $x_{(r)}$ denote the rth smallest of $\{x_1, x_2, ..., x_n\}$. Then the random variables $x_{(1)}, x_2, ..., x_n$, has probability density function of the rth order statistics as

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_x(x) \left[F_x(x) \right]^{r-1} \left[1 - F_x(x) \right]^{n-r}, \ r = 1,2,3,...,n$$

The pdf of the rth order statistic of the NWPD is

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} \frac{\beta\delta}{\theta} \left(\frac{x}{\theta} \right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta} \right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta} \right)^{\beta}} \right]^{r-1} \left[e^{-\delta\left(\frac{x}{\theta} \right)^{\beta}} \right]^{n-r}$$

The pdf of the largest order statistic $x_{(n)}$ is therefore

$$f_{X_{(n)}}(x) = \frac{n\beta\delta}{\theta} \left(\frac{x}{\theta} \right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta} \right)^{\beta}} \left[1 - e^{-\delta\left(\frac{x}{\theta} \right)^{\beta}} \right]^{n-1},$$

and the pdf of the smallest order statistic $x_{(1)}$ is given by

$$f_{X_{(1)}}(x) = \frac{n\beta\delta}{\theta} \left(\frac{x}{\theta} \right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta} \right)\beta} \left[e^{-\delta\left(\frac{x}{\theta} \right)^{\beta}} \right]^{n-1}.$$

II. MAXIMUM LIKELIHOOD ESTIMATORS FOR TYPE II CENSORED SAMPLE:

In a typical life test, n specimens are placed under observation and as each failure occur the time is noted. When a predetermined total number of failures r is reached, the test is terminated. In this case the data collected consist of observations $x_{(1)} < x_{(2)} < ... < x_{(r)}$ plus the information that (n-r) items survived beyond the time of termination $x_{(r)}$, this censoring scheme is known as type II censoring and also the collected data is said to be censored type II data. Cohen (1965) gave the likelihood function for type II censoring:

$$L = C \prod_{i=1}^{r} f(x_{(i)}; \theta) \left[1 - F(x_{(r)}; \theta) \right]^{n-r},$$

Where C is a constant, r is the number of uncensored sample, $x_{(r)}$ is the observed value of the rth order statistic, $f(x_{(i)}; \theta)$ and $F(x_{(r)}; \theta)$ are the density function and the cumulative function of the underlying distribution, respectively.

For the new Weibull – Pareto distribution (1), the likelihood function will be

$$L = C \left(\frac{\beta\delta}{\theta} \right)^r \prod_{i=1}^{r} \left[\left(\frac{x_{(i)}}{\theta} \right)^{\beta-1} e^{-\delta\left(\frac{x_{(i)}}{\theta} \right)^{\beta}} \right] \left[1 - \left[1 - e^{\delta\left(\frac{x_{(r)}}{\theta} \right)^{\beta}} \right]^{n-r} \right]$$

(3)

Taking the logarithm, (3) becomes

$$\ln L \propto r\ln\beta + r\ln\delta - r\ln\theta + (\beta-1)\sum_{i=1}^{r} \ln\left(\frac{x_{(i)}}{\theta} \right) - \delta\sum_{i=1}^{r} \left(\frac{x_{(i)}}{\theta} \right)^{\beta} - (n-r)\,\delta\left(\frac{x_{(r)}}{\theta} \right)^{\beta}$$

(4)

Differentiate (4) with respect to the unknown parameters and equating to zero:

$$\frac{\partial \ln L}{\partial \delta} = \frac{r}{\delta} - \sum_{i=1}^{r} \left(\frac{x_{(i)}}{\theta} \right)^{\beta} - (n-r)\left(\frac{x_{(r)}}{\theta} \right)^{\beta} = 0,$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{r}{\beta} + \sum_{i=1}^{r} \ln\left(\frac{x_{(i)}}{\theta} \right) - \delta\sum_{i=1}^{r} \left(\frac{x_{(i)}}{\theta} \right)^{\beta} \ln\left(\frac{x_{(i)}}{\theta} \right) - \left[\delta(n-r)\left(\frac{x_{(r)}}{\theta} \right)^{\beta} \ln\left(\frac{x_{(r)}}{\theta} \right) \right] = 0,$$

and

$$\frac{\partial \ln L}{\partial \theta} = \frac{-r}{\theta} - \frac{(\beta-1)}{\theta} + \left(\frac{\delta\beta}{\theta^{\beta+1}} \right) \sum_{i=1}^{r} \left(x_{(i)} \right)^{\beta} + \left(\frac{\delta\beta(n-r)}{\theta^{\beta+1}} \left(x_{(r)} \right)^{\beta} \right) = 0.$$

Then, the maximum likelihood estimates of the parameters δ, β and θ can be obtained by solving the above system of equations. No explicit form for these parameters, we use a numerical technique using Mathcad 2001 Package to obtain $\hat{\delta}$, $\hat{\beta}$ and $\hat{\theta}$.

The element of the asymptotic variance covariance matrix can be approximated by Cohen's (1965) result:

$$-E\left[\frac{\partial^2 \ln L(\theta)}{\partial \theta_i \theta_j} \right] \approx -\frac{\partial^2 \ln L(\theta)}{\partial \theta_i \theta_j}\bigg|_{\theta = \hat{\theta}}$$

The second derivative can be written as follows:

$$\left.\frac{-\partial^2 \ln L}{\partial \delta^2}\right|_{\substack{\theta=\hat{\theta},\\ \beta=\hat{\beta},\\ \delta=\hat{\delta},}} = \frac{r}{\hat{\delta}^2} \quad,$$

$$\left.\frac{-\partial^2 \ln L}{\partial \delta \, \partial \theta}\right|_{\substack{\theta=\hat{\theta},\\ \beta=\hat{\beta},\\ \delta=\hat{\delta},}} = \frac{-\hat{\beta}}{\hat{\theta}^{\beta+1}} \sum_{i=1}^{r} \left(x_{(i)}\right)^{\hat{\beta}} - \frac{\hat{\beta}(n-r)}{\hat{\theta}^{\beta+1}} \left(x_{(r)}\right)^{\hat{\beta}},$$

$$\left.\frac{\partial^2 \ln L}{\partial \delta \, \partial \beta}\right|_{\substack{\theta=\hat{\theta},\\ \beta=\hat{\beta},\\ \delta=\hat{\delta},}} = \sum_{i=1}^{r}\left(\frac{x_{(i)}}{\hat{\theta}}\right)^{\hat{\beta}} \ln\left(\frac{x_{(i)}}{\hat{\theta}}\right) + (n-r)\left(\frac{x_{(r)}}{\hat{\theta}}\right)^{\hat{\beta}}\ln\left(\frac{x_{(r)}}{\hat{\theta}}\right),$$

$$\left.\frac{-\partial^2 \ln L}{\partial \beta^2}\right|_{\substack{\theta=\hat{\theta},\\ \beta=\hat{\beta},\\ \delta=\hat{\delta},}} = \frac{r}{\hat{\beta}^2} + \hat{\delta}\sum_{i=1}^{r}\left(\frac{x_{(i)}}{\hat{\theta}}\right)^{\hat{\beta}}\left(\ln\left(\frac{x_{(i)}}{\hat{\theta}}\right)\right)^2 + \hat{\delta}(n-r)\left(\frac{x_{(t)}}{\hat{\theta}}\right)^{\hat{\beta}}\left(\ln\left(\frac{x_{(r)}}{\hat{\theta}}\right)\right)^2,$$

$$\left.\frac{-\partial^2 \ln L}{\partial \beta \, \partial \theta}\right|_{\substack{\theta=\hat{\theta},\\ \beta=\hat{\beta},\\ \delta=\hat{\delta},}} = \frac{1}{\hat{\theta}} - \frac{\hat{\delta}}{\hat{\theta}}\sum_{i=1}^{r}\left(\frac{x_{(i)}}{\hat{\theta}}\right)^{\hat{\beta}}\left[1+\hat{\beta}\ln\left(\frac{x_{(i)}}{\hat{\theta}}\right)\right] - \frac{(n-r)\,\hat{\delta}}{\hat{\theta}}\left(\frac{x_{(t)}}{\hat{\theta}}\right)^{\hat{\beta}}\left[1+\hat{\beta}\ln\left(\frac{x_{(r)}}{\hat{\theta}}\right)\right],$$

and

$$\left.\frac{-\partial^2 \ln L}{\partial \theta^2}\right|_{\substack{\theta=\hat{\theta},\\ \beta=\hat{\beta},\\ \delta=\hat{\delta},}} = \frac{-r}{\hat{\theta}^2} - \frac{(\hat{\beta}-1)}{\hat{\theta}^2} + \frac{\hat{\delta}\hat{\beta}(\beta+1)}{\hat{\theta}^{\beta+2}}\sum_{i=1}^{r}\left(x_{(i)}\right)^{\hat{\beta}} + \left(\frac{(\hat{\beta}+1)\,\hat{\delta}\hat{\beta}\,(n-r)}{\hat{\theta}^{\beta+2}}\left(x_{(t)}\right)^{\hat{\beta}}\right).$$

Again, a numerical technique using Mathcad Package and computer facilities are used to obtain the variance – covariance matrix.

III. NUMERICAL ILLUSTRATION:

In this section, we present a numerical example to illustrate different maximum likelihood estimators, and their asymptotic variance covariance matrix. To generate random numbers from the new Weibull – Pareto distribution (1) we have:

$$x = \theta\left\{\frac{1}{\delta}\ \ln\left(\frac{1}{1-u}\right)\right\}^{\frac{1}{\beta}},$$

(5)

Where u comes from the uniform distribution (0, 1). Using Mathcad package, we generate a sample of 50 observations from (5) with $\beta = 0.5$ $\theta = 1.5$ and $\delta = 1$ under type II censored sample, take r=30, 40 and 50 respectively. All results are displayed in the following table.

Table1: The maximum likelihood estimates of the parameters of the new Weibull Pareto distribution, the asymptotic variances and covariance's based on type II censored samples.

Sample size"n"	r	$\hat{\theta}$	$\hat{\beta}$	$\hat{\delta}$	$Var(\hat{\theta})$	$Var(\hat{\beta})$	$Var(\hat{\delta})$
	30	2.04	0.16	2.7	0.46	0.013	1.04E5
50	40	0.4	0.08	0.7	5.7E-4	1.4E-4	1.3E-4
	50	0.6	0.09	2.05	0.27	0.03	1.2E6

According to these simulated data the variances decreases and then increasing with increasing sample size.

When r=n the results reduce to complete case [result of Nasiru and Luguterah (2015) as a special case].

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