Estimation of *P(Y < X)* for Weibull Distribution In the Presence of *k* Outliers

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Abstract: In this paper, we consider the problem of estimating $R=P(Y<X)$, where Y has Weibull distribution with parameters β and α and X has Weibull distribution with parameters β and p in presence of k outliers, such that X and Y are independent. The moment, maximum likelihood and mixture estimators of R are derived with help of numerical technique. At the end, we conclude that mixture estimators are better than the maximum likelihood and moment estimators. Some of the previous results in the literatures can be achieved as special case of our results.

Key words: Weibull distribution, the maximum likelihood estimator, moment estimator, mixture estimator and outliers.

1. Introduction

There has been continuous interest in the problem of estimating the probability that one random variable exceeds another, that is the quantity $R=P(Y<X)$ in reliability context, which is known as stress-strength model reliability, it describes life of a component which has a random strength X, and is subjected to random stress Y. This problem arises in the classical stress–strength reliability where one is interested in estimating the proportion of the times the random strength X of a component exceeds the random stress Y to which the component is subjected. If $X \leq Y$, then either the component fails or the system that uses the component may malfunction, and there is no failure when $Y<X$.

This problem also arises in situations where X and Y represent lifetimes of two devices and one wants to estimate the probability that one fails before the other.

The germ of this idea was introduced by Birnbaum (1956) and developed by Birnbaum and McCarty (1958). The latter paper does for the first time include $P(Y < X)$ in its title **[**A distribution-free upper confidence bound for $P\{Y < X\}$, based on independent samples of X and Y**]**, but the formal term "stress - strength" appeared in the title **[**The estimation of reliability from stress-strength relationships**]** of Church and Harris (1970). The component fails at the instant that stress applied to it exceeds the strength and the component will function satisfactorily whenever Y> X. The estimation of R is very common in the statistical literature, for example in mechanical reliability of a system if X is the strength of a component which is subject to stress Y, where X and Y independent distribution, then R is a measure of system performance. The system fails, if at any time the applied stress is greater than its strength Kotz, et el.(2003).

McCool (1991) presented inference on $P(Y < X)$ in the Weibull case, Kundu and Gupta (2006) presented the estimation of $R = P(Y < X)$ for Weibull distribution. Davarzani (2009) presented estimation of $P(Y < X)$ where X and Y are random variables from a bivariate Weibull distribution and X is censored at Y.

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Dixit, et al.(1996), assume that a set of random variables *(X_1, X_2, ..., X_n)* represent the distance of an infected sampled plant from a plant from a plot of plants inoculated with a virus. Some of the observations are derived from the airborne dispersal of the spores and are distributed according to the exponential distribution. The other observations out of *n* random variables (say *k*) are present because aphids which are known to be carriers of barley yellow mosaic dwarf virus (BYMDV) have passed the virus into the plants when the aphids feed on the sap.

Dixit and Nasiri (2001) considered estimation of parameters of the exponential distribution in the presence of outliers generated from uniform distribution. Deiri (2011c) presented the estimation of parameters of the gamma distribution with presence of two outliers, and presented (2011d) estimation of the parameters of minimax distribution with presence of two outliers.

Recently Nasiri and Pazira (2010) considered the problem of estimating of $R = P(Y < X)$, where *Y* and *X* have exponential distribution with presence of k outliers, Deiri (2011a) presented the same estimation of $P(Y<X)$ of the same distribution but in the presence of two outliers, then in the same year (2011e), he presented the estimation of $P(Y<X)$ but where *Y* and *X* has the rayleigh distribution with presence of two outliers, and he presented the estimation of $P(Y<X)$ for generalized exponential distribution in the presence of two outliers when scale parameter is known, also, Jafari (2011) presented the estimation of $P(Y<X)$ in the rayleigh distribution but with presence of one outlier, and Ghanizadeh (2011) presented the estimation of $P(Y<X)$ in the rayleigh distribution with presence of k outliers.

An outlier is a specific value that stands far away from the mean or from the rest of the values in a given set. When graphed, the outlier is visually far away from all the rest of the points, Barnett and Lewis (1994).

In this paper, we obtain the moment, maximum likelihood and mixture estimators of *R* in Weibull distribution with presence of *k* outliers generated from the same distribution and we compare them together.

The Weibull distributions with shape parameter β and scale parameter p will be denoted by WE (β; p) and the corresponding density function $f(x; \beta, p)$, for $\beta > 0$ and $p > 0$, is as follows;

$$f(x;\ \beta, p) = \frac{\beta}{p}\left(\frac{x}{p}\right)^{\beta-1} e^{-\left(\frac{x}{p}\right)^{\beta}} \qquad x > 0$$

with c.d.f $F(x;\ \beta, p)$

$$F(x;\ \beta, p) = 1 - e^{-\left(\frac{x}{p}\right)^{\beta}} \quad \text{for } x \geq 0, \text{and } F(x;\ k;\ \lambda) = 0 \text{ for } x < 0$$

Thus, we assume that the random variables *(X_1, X_2, ..., X_n)* are such that *k* of them are distributed with p.d.f $f_1(x;\ \beta, p)$,

$$f_1(x;\ p, \beta) = \frac{\beta}{p}\left(\frac{x}{p}\right)^{\beta-1} e^{-\left(\frac{x}{p}\right)^{\beta}} \qquad x > 0, \beta > 0 \ and\ p > 0$$

and the remaining *(n-k)* random variables are distributed with p.d.f $f_2(x;\gamma;\text{p})$,

$$f_2(x;\ \gamma,\beta) = \frac{\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} e^{-\left(\frac{x}{\gamma}\right)^{\beta}} \qquad x > 0, \beta\ >\ 0\ and\ \gamma\ >\ 0$$

then the marginal distribution of X is

$$f(x;\ \gamma,p,\beta) = \frac{k}{n}\frac{\beta}{p}\left(\frac{x}{p}\right)^{\beta-1} e^{-\left(\frac{x}{p}\right)^{\beta}} + \frac{(n-k)}{n}\frac{\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} e^{-\left(\frac{x}{\gamma}\right)^{\beta}} \qquad x > 0, \beta > 0, \gamma > 0\ and\ p > 0$$

Let $(Y_1, Y_2, \ldots, Y_m)$ be a random sample for Y with pdf,

$$g(y;\ \alpha;\beta) = \frac{\beta}{\alpha}\left(\frac{y}{\alpha}\right)^{\beta-1} e^{-\left(\frac{y}{\alpha}\right)^{\beta}}$$

where X, Y are independent.

With presence of k outliers, the parameter R we want to estimate is

$$R = P(y < x)$$
$$P(y < x) = \int_0^{\infty}\int_0^{x} g(y)f(x)\,dy\,dx$$
$$= \int_0^{\infty} f(x)\left[\int_0^{x} g(y)\,dy\ \right]dx$$
$$\int_0^{x} g(y)\,dy\ \ = 1 - e^{-\left(\frac{y}{\alpha}\right)^{\beta}}$$
$$P(y < x) = \int_0^{\infty} f(x)\left[1 - e^{-\left(\frac{y}{\alpha}\right)^{\beta}}\right]dx$$
$$P(y < x) = 1 - \int_0^{\infty} f(x)e^{-\left(\frac{y}{\alpha}\right)^{\beta}}\ dx = 1 - \mathrm{I}$$
$$\mathrm{I} = \int_0^{\infty} f(x)e^{-\left(\frac{y}{\alpha}\right)^{\beta}}\ dx$$
$$\mathrm{I} = \int_0^{\infty}\frac{k}{n}\frac{\beta}{p}\left(\frac{x}{p}\right)^{\beta-1} e^{-\left[\left(\frac{x}{p}\right)^{\beta}+\left(\frac{y}{\alpha}\right)^{\beta}\right]}dx + \int_0^{\infty}\frac{n-k}{n}\frac{\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1} e^{-\left[\left(\frac{x}{\gamma}\right)^{\beta}+\left(\frac{y}{\alpha}\right)^{\beta}\right]}dx$$
$$\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$$
$$\mathrm{I}_1 = \frac{k\beta}{np^{\beta}}\int_0^{\infty} x^{\beta-1}e^{-[p^{-\beta}+\alpha^{-\beta}]x^{\beta}} \quad dx$$
$$= \frac{k}{np^{\beta}[p^{-\beta}+\alpha^{-\beta}]}\int_0^{\infty}\beta\left[p^{-\beta}+\alpha^{-\beta}\right]x^{\beta-1}e^{-[p^{-\beta}+\alpha^{-\beta}]x^{\beta}} \quad dx$$
$$= \frac{k}{n\left[1+\left(\frac{p}{\alpha}\right)^{\beta}\right]} = \frac{k\alpha^{\beta}}{n[p^{\beta}+\alpha^{\beta}]}$$

And by the same way, we can obtain I_2

$$\mathrm{I}_2 = \frac{(n-k)\alpha^{\beta}}{n[\alpha^{\beta} + \gamma^{\beta}]}$$

Let

$\frac{k}{n} = b,$ $\frac{(n-k)}{n} = \overline{b}$

where $b + \overline{b} = 1$

$$\mathrm{I} = \left[\frac{b\alpha^{\beta}}{p^{\beta} + \alpha^{\beta}} + \frac{\overline{b}\alpha^{\beta}}{\alpha^{\beta} + \gamma^{\beta}}\right]$$

$$P(y < x) = 1 - \left[\frac{b\alpha^{\beta}}{p^{\beta} + \alpha^{\beta}} + \frac{\overline{b}\alpha^{\beta}}{\alpha^{\beta} + \gamma^{\beta}}\right]$$

Since $P(y < x) + P(y > x) = 1$, we consider $P(y > x)$,

$$R = P(y > x) = 1 - P(y < x)$$

$$R = \frac{b\alpha^{\beta}}{p^{\beta} + \alpha^{\beta}} + \frac{\overline{b}\alpha^{\beta}}{\alpha^{\beta} + \gamma^{\beta}}$$

Special cases

a. Let $\beta = 2$, $p^2 = \frac{\theta}{\beta}$, $\alpha^2 = \lambda$, $\gamma^2 = \theta$

Then

$$R = \frac{b\lambda\beta}{\lambda\beta + \theta} + \frac{\overline{b}\lambda}{\lambda + \theta}$$ see Ghanizadeh(2011)

Let $\beta = 1$, $p = \frac{\theta}{\beta}$, $\alpha = \lambda$, $\gamma = \theta$

Then

$$R = \frac{b\lambda\beta}{\lambda\beta + \theta} + \frac{\overline{b}\lambda}{\lambda + \theta}$$ see Deiri (2011e)

In section 2, 3 and 4, we deal with the methods of moment, maximum likelihood and mixture of methods of moment and maximum likelihood in estimating *R*. We compare these methods numerically, and finally the researcher represents conclusion in section 5.

2. Method of Moment

The moment estimator of *R* can be shown to be

$$\hat{R} = \frac{b\hat{\alpha}^{\beta}}{\hat{p}^{\beta} + \hat{\alpha}^{\beta}} + \frac{\overline{b}\hat{\alpha}^{\beta}}{\hat{\alpha}^{\beta} + \hat{\gamma}^{\beta}}$$

$\hat{\alpha}, \hat{\gamma}$ and $\hat{p}$ can be obtained as;

$$E(y^r) = \int_0^{\infty} y^r g(y) dy$$

$$= \int_0^{\infty} \frac{\beta}{\alpha^{\beta}} y^{\beta+r-1} e^{-\left(\frac{y}{\alpha}\right)^{\beta}} dy$$

$$= \alpha^r \Gamma \frac{r}{\beta} + 1$$

at r=1 then $E(y) = \alpha\Gamma \frac{1}{\beta} + 1 = \frac{\sum_{i=1}^{m} y_i}{m}$

$$\hat{\alpha} = \frac{\sum_{i=1}^{m} y_i}{m\,\Gamma \frac{1}{\beta} + 1}$$

$$E(x^r) = \int_0^{\infty} x^r f(x) dx$$

$$= b \int_0^{\infty} \frac{\beta}{p^{\beta}} (x)^{\beta+r-1} e^{-\left(\frac{x}{p}\right)^{\beta}} + \overline{b} \int_0^{\infty} \frac{\beta}{\gamma^{\beta}} (x)^{\beta+r-1} e^{-\left(\frac{x}{\gamma}\right)^{\beta}} dx$$

$$= \left(bp^r + \overline{b}\gamma^r\right)\Gamma \frac{r}{\beta} + 1$$

at r=1

$$\left(bp + \overline{b}\gamma\right)\Gamma \frac{1}{\beta} + 1 = \frac{\sum_{i=1}^{n} x_i}{n} = m_1^{'} \tag{1}$$

at r=2

$$\left(bp^2 + \overline{b}\gamma^2\right)\Gamma \frac{2}{\beta} + 1 = \frac{\sum_{i=1}^{n} x_i^2}{n} = m_2^{'} \tag{2}$$

From (1)

$$bp\Gamma \frac{1}{\beta} + 1 = m_1^{'} - \overline{b}\gamma\Gamma \frac{1}{\beta} + 1$$

$$p = \frac{m_1^{'}}{b\Gamma \frac{1}{\beta} + 1} - \frac{\overline{b}\gamma}{b} \tag{3}$$

From (3) in (2)

$$b\left(\frac{m_1'}{b\,\Gamma\frac{1}{\beta}+1}-\frac{\bar{b}\gamma}{b}\right)^2+\bar{b}\gamma^2=\frac{m_2'}{\Gamma\frac{2}{\beta}+1}$$

$$b\left[\frac{{m_1'}^2}{b^2\left(\Gamma\frac{1}{\beta}+1\right)^2}-2\frac{\bar{b}m_1'}{b^2\Gamma\frac{1}{\beta}+1}\gamma+\frac{\bar{b}^2}{b^2}\gamma^2\right]+\bar{b}\gamma^2=\frac{m_2'}{\Gamma\frac{2}{\beta}+1}$$

$$\frac{{m_1'}^2}{b\left(\Gamma\frac{1}{\beta}+1\right)^2}-2\frac{\bar{b}m_1'}{b\,\Gamma\frac{1}{\beta}+1}\gamma+\bar{b}\left(\frac{\bar{b}}{b}+1\right)\gamma^2=\frac{m_2'}{\Gamma\frac{2}{\beta}+1}$$

$$\bar{b}\left(\frac{\bar{b}}{b}+1\right)\gamma^2-2\frac{\bar{b}m_1'}{b\,\Gamma\frac{1}{\beta}+1}\gamma+\left[\frac{{m_1'}^2}{b\left(\Gamma\frac{1}{\beta}+1\right)^2}-\frac{m_2'}{\Gamma\frac{2}{\beta}+1}\right]=0$$

Let

$\xi_1=\bar{b}\left(\frac{\bar{b}}{b}+1\right)$, $\xi_2=-2\frac{\bar{b}m_1'}{b\,\Gamma\frac{1}{\beta}+1}$, and $\xi_3=\frac{{m_1'}^2}{b\left(\Gamma\frac{1}{\beta}+1\right)^2}-\frac{m_2'}{\Gamma\frac{2}{\beta}+1}$

$$\xi_1\gamma^2+\xi_2\gamma+\xi_3=0$$

if $\Delta={\xi_2}^2-4\xi_1\xi_3$ is non-negative then the roots are real. Therefore

$$\hat{\gamma}=\frac{-\xi_2+\sqrt{{\xi_2}^2-4\xi_1\xi_3}}{2\xi_1}$$

and, from (2)

$$\hat{p}=\frac{m_1'}{b\,\Gamma\frac{1}{\beta}+1}-\frac{\bar{b}}{b}\hat{\gamma}$$

3. Method of Maximum Likelihood

The maximum Likelihood estimator of R is shown to be

$$\hat{R}=\frac{b\hat{\alpha}^{\beta}}{\hat{p}^{\beta}+\hat{\alpha}^{\beta}}+\frac{\bar{b}\hat{\alpha}^{\beta}}{\hat{\alpha}^{\beta}+\hat{\gamma}^{\beta}}$$

Where $\hat{\alpha}=\left(\frac{1}{m}\sum_{i=1}^{m}y_i^{\beta}\right)^{\frac{1}{\beta}}$

Let y_1, y_2… y_m be a random sample and let $g(y_i, \alpha, \beta)$ denotes the Weibull distribution function

$$g(y_{i,} \alpha, \beta) = \frac{\beta}{\alpha}\left(\frac{y_i}{\alpha}\right)^{\beta-1} e^{-\left(\frac{y_i}{\alpha}\right)^{\beta}}$$

The likelihood function for weibull distribution given by

$$L = \prod_{i=1}^{m} g(y_{i,} \alpha, \beta) = \prod_{i=1}^{m}\left[\frac{\beta}{\alpha}\left(\frac{y_i}{\alpha}\right)^{\beta-1} e^{-\left(\frac{y_i}{\alpha}\right)^{\beta}}\right]$$

$$L = \left(\frac{\beta}{\alpha}\right)^{m} \left(\frac{1}{\alpha}\right)^{m(\beta-1)} e^{-\sum_{i=1}^{m}\left(\frac{y_i}{\alpha}\right)^{\beta}} \prod_{i=1}^{m}\left[(y_i)^{\beta-1}\right]$$

The log-likelihood given by

$$l = ln(L) = m\,ln(\beta) - m\,ln(\alpha) - m(\beta-1)\,ln(\alpha) - \alpha^{-\beta\sum_{i=1}^{m}(y_i)^{\beta}} + (\beta-1)\sum_{i=1}^{m} ln y_i$$

The partial derivatives of l with respect to α is

$$\frac{\partial l}{\partial \alpha} = \frac{-m}{\alpha} - \frac{m(\beta-1)}{\alpha} + \beta\alpha^{-\beta\sum_{i=1}^{m}(y_i)^{\beta}} = 0$$

$$\frac{\partial l}{\partial \alpha} = -\frac{m\beta}{\alpha} + \beta\alpha^{-\beta-1}\sum_{i=1}^{m}(y_i)^{\beta} = 0$$

$$\frac{1}{\alpha^{\beta}}\sum_{i=1}^{m} y_i^{\beta} = m$$

$$\hat{\alpha} = \left(\frac{1}{m}\sum_{i=1}^{m} y_i^{\beta}\right)^{\frac{1}{\beta}}$$

And to be the maximum Likelihood estimator of p and γ, we consider the joint distribution of X with presence of k outliers:

$$L(\underline{x}, p, \gamma) = \prod_{i=1}^{n}[f(x_i;\, \gamma, p, \beta)]$$

$$= \prod_{i=1}^{n}\left[\frac{k}{n}\frac{\beta}{p}\left(\frac{x_i}{p}\right)^{\beta-1} e^{-\left(\frac{x_i}{p}\right)^{\beta}} + \frac{(n-k)}{n}\frac{\beta}{\gamma}\left(\frac{x_i}{\gamma}\right)^{\beta-1} e^{-\left(\frac{x_i}{\gamma}\right)^{\beta}}\right]$$

$$= \prod_{i=1}^{n} \left[\frac{k}{n} \frac{\beta}{p^{\beta}} (x_i)^{\beta-1} e^{-\left(\frac{x_i}{p}\right)^{\beta}} \left[1 + \frac{(n-k)}{k} \left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\left(\frac{x_i}{\gamma}\right)^{\beta} - \left(\frac{x_i}{p}\right)^{\beta}\right)} \right] \right]$$

$$= \prod_{i=1}^{n} \left[\frac{k}{n} \frac{\beta}{p^{\beta}} (x_i)^{\beta-1} e^{-\left(\frac{x_i}{p}\right)^{\beta}} \left(\psi(x_i; p, \gamma)\right) \right]$$

where:

$$\psi(x_i; p, \gamma) = 1 + \frac{b}{\bar{b}} \left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}}$$

$$L(\underline{x}, p, \gamma) = \left(\frac{b\beta}{p^{\beta}}\right)^{n} e^{-\sum_{i=1}^{n} \left(\frac{x_i}{p}\right)^{\beta}} \left[\prod_{i=1}^{n} x_i{}^{\beta-1}\right] \left[\prod_{i=1}^{n} \psi(x_i; p, \gamma)\right]$$

$$l(p, \gamma) = ln\, L(x, p, \gamma)$$

$$l(p, \gamma) = n \ln(b\beta) - n\beta\, lnp - \frac{1}{p^{\beta}} \sum_{i=1}^{n} x_i{}^{\beta} + (\beta - 1) \sum_{i=1}^{n} lnx_i + \sum_{i=1}^{n} ln\psi(x_i; p, \gamma)$$

The solve for our MLEs of p and γ we take the derivative of the log likelihood ($l(p, \gamma)$) with respect to each parameter set the partial derivatives equal to zero and solve for $\hat{\gamma}$ and $\hat{p}$:

$$\frac{\partial l(p, \gamma)}{\partial p} = \frac{-n\beta}{p} + \beta p^{-\beta-1} \sum_{i=1}^{n} x_i{}^{\beta} + \frac{b\beta}{p\bar{b}} \left(\frac{p}{\gamma}\right)^{\beta} \sum_{i=1}^{n} \frac{(1 - \left(\frac{x_i}{p}\right)^{\beta}) e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x^{\beta}}}{\psi(x_i; p, \gamma)} = 0$$

$$= -n + \left(\frac{1}{p}\right)^{\beta} \sum_{i=1}^{n} x_i{}^{\beta} + \frac{b}{\bar{b}} \left(\frac{p}{\gamma}\right)^{\beta} \sum_{i=1}^{n} \frac{(1 - \left(\frac{x_i}{p}\right)^{\beta}) e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}}}{\psi(x_i; p, \gamma)} = 0$$

$$= -n + \left(\frac{1}{p}\right)^{\beta} \sum_{i=1}^{n} x_i{}^{\beta} + \frac{b}{\bar{b}} \left(\frac{p}{\gamma}\right)^{\beta} \left[\sum_{i=1}^{n} \frac{\left(\frac{x_i}{p}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}}}{\psi(x_i; p, \gamma)} - \sum_{i=1}^{n} \frac{e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}}}{\psi(x_i; p, \gamma)} \right] = 0 \qquad \textbf{(A)}$$

where

$$\frac{\partial ln\psi}{\partial p} = \frac{1}{\psi} \frac{\partial \psi}{\partial p}$$

$$\frac{\partial \psi}{\partial p} = \frac{b}{\bar{b}} \left[\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}} (-\beta) p^{-\beta-1} x_i{}^{\beta} + \frac{\beta}{\gamma} e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}} \left(\frac{p}{\gamma}\right)^{\beta-1} \right]$$

$$= \frac{b\beta}{\bar{b}p}\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} \left[1-\left(\frac{x_i}{p}\right)^{\beta}\right]$$

$$= \frac{\frac{b\beta}{\bar{b}p}\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} \left[1-\left(\frac{x_i}{p}\right)^{\beta}\right]}{1+\frac{b}{\bar{b}}\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}$$

$$\frac{\partial\psi}{\partial\gamma} = \frac{b}{\bar{b}}\left[\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} \beta\gamma^{-\beta-1}{x_i}^{\beta} - \beta e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} - \beta e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} \left(\frac{p}{\gamma}\right)^{\beta-1} p\gamma^{-2}\right]$$

$$= \frac{b\beta}{\bar{b}\gamma}\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} \left[\left(\frac{x_i}{\gamma}\right)^{\beta}-1\right]$$

$$\frac{\partial ln\psi}{\partial\gamma} = \frac{1}{\psi}\frac{\partial\psi}{\partial\gamma}$$

$$= \frac{\frac{b\beta}{\bar{b}\gamma}\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}} \left[\left(\frac{x_i}{\gamma}\right)^{\beta}-1\right]}{1+\frac{b}{\bar{b}}\left(\frac{p}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}$$

$$\frac{\partial l(p,\gamma)}{\partial\gamma} = \frac{b\beta}{\bar{b}\gamma}\left(\frac{p}{\gamma}\right)^{\beta}\sum_{i=1}^{n}\frac{\left(\left(\frac{x_i}{\gamma}\right)^{\beta}-1\right)e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} = 0$$

$$\frac{\partial l(p,\gamma)}{\partial\gamma} = \sum_{i=1}^{n}\frac{\left(\left(\frac{x_i}{\gamma}\right)^{\beta}-1\right)e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} = 0$$

$$\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} - \sum_{i=1}^{n}\frac{e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} = 0$$

$$\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} = \sum_{i=1}^{n}\frac{e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} \qquad (\mathbf{B})$$

From (B) in (A)

$$-n+\left(\frac{1}{p}\right)^{\beta}\sum_{i=1}^{n}{x_i}^{\beta} + \frac{b}{\bar{b}}\left(\frac{p}{\gamma}\right)^{\beta}\left[\sum_{i=1}^{n}\frac{\left(\frac{x_i}{p}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)} - \sum_{i=1}^{n}\frac{\left(\frac{x_i}{\gamma}\right)^{\beta} e^{-\left(\frac{1}{\gamma^{\beta}}-\frac{1}{p^{\beta}}\right){x_i}^{\beta}}}{\psi(x_i;p,\gamma)}\right] = 0$$

$$\frac{\partial l(p,\gamma)}{\partial p} = -n + \left(\frac{1}{p}\right)^{\beta} \sum_{i=1}^{n} x_i{}^{\beta} + \frac{b}{\bar{b}}\left(\frac{p}{\gamma}\right)^{\beta} \sum_{i=1}^{n} \frac{\left(\left(\frac{x_i}{p}\right)^{\beta} - \left(\frac{x_i}{\gamma}\right)^{\beta}\right) e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}}}{\psi(x_i; p, \gamma)} = 0 \qquad \textbf{(C)}$$

There is no closed-form solution to this system of equations (A, B), so we will solve for $\hat{p}, \hat{\gamma}$ iteratively. In case of no outlier presence, *R*, $\hat{p}$ and $\hat{\gamma}$ were proposed by Kundu, D., Gupta, R.D. (2006).

4. Mixture of method of Moment and Maximum likelihood

Read (1981) proposed the methods, which avoid the difficulty of complicated equations. According to Read (1981), replacement of some, but not all, of the equations in the system of likelihood may make it more manageable. From moment estimation, we have (21)

$$\hat{\gamma} = \frac{-\xi_2 + \sqrt{\xi_2{}^2 - 4\xi_1\xi_3}}{2\xi_1}$$

and, from equation (C)

$$= -n + \left(\frac{1}{p}\right)^{\beta} \sum_{i=1}^{n} x_i{}^{\beta} + \frac{b}{\bar{b}}\left(\frac{p}{\gamma}\right)^{\beta} \sum_{i=1}^{n} \frac{\left(\left(\frac{x_i}{p}\right)^{\beta} - \left(\frac{x_i}{\gamma}\right)^{\beta}\right) e^{-\left(\frac{1}{\gamma^{\beta}} - \frac{1}{p^{\beta}}\right) x_i{}^{\beta}}}{\psi(x_i; p, \gamma)} = 0$$

We obtain $\hat{p}$.

5. Conclusion

In this paper, the researcher has addressed the problem of estimating $P(Y<X)$ for the Weibull distribution with presence of *k* outliers. The moment, maximum likelihood and mixture estimators of *R* are derived using MathCAD Package. All the results are base on 10000 replications and are given in Tables 1 to 6.

From Tables, the researcher infers that the moment estimator of *R* is asymptotically unbiased and as expected when $m=n$ and m, n increase then the average biases and the MSEs decrease, and the MSEs of any three estimators are tending to zero and when $m=n$.

Tables show the estimates of $R=P(Y<X)$ with presence of one and two outliers under different values of β, γ, p and α, on the other hand they show some of the previous results in the literatures such as Ghanizadeh (2011) and Deiri (2011e) can be achieved as special case of our results.

Tables show that almost the mixture estimator has the smallest estimated MSEs as compared with the moment and maximum likelihood estimators, so the researcher strongly feels that mixture estimator is better and easy to calculate than the maximum likelihood and moment estimations. From the previous observations, the researcher suggests to use mixture method for estimating $R = P(Y < X)$ in the Weibull distribution in the presence of *k* outliers because it is easy to calculate than the rest.

Table (1)

Estimates of R, Biases and Mean Squared Errors MSE's of the point estimates from Weibull Distribution, when $k=1$, $\beta=1$, $\alpha=0.5$, $\gamma=2$ and $p=1$

	$\hat{R}$			Bias			MSE		
n,m	MLE	Mom	Mix	MLE	Mom	Mix	MLE	Mom	Mix
15,15	0.207	0.260	0.317	0.059	0.053	-0.110	0.209	0.025	0.018
20,20	0.203	0.253	0.288	0.043	0.032	-0.085	0.196	0.012	0.010
25,25	0.201	0.244	0.257	0.039	-0.011	-0.071	0.155	0.009	0.007
15,20	0.205	0.254	0.315	0.042	0.037	-0.092	0.214	0.031	0.013
20,15	0.213	0.261	0.332	0.054	0.046	-0.088	0.238	0.053	0.027
15,25	0.215	0.243	0.298	0.047	-0.048	0.098	0.241	0.037	0.031
25,15	0.221	0.253	0.302	0.053	-0.061	-0.077	0.247	0.067	0.026
20,25	0.241	0.282	0.317	0.044	-0.077	-0.103	0.211	0.045	0.022
25,20	0.238	0.279	0.322	0.048	0.081	-0.113	0.216	0.057	0.053

|Table (2)

Estimates of R, Biases and Mean Squared Errors MSE's of the point estimates from Weibull Distribution, when $k=2$, $\beta=1$, $\alpha=0.5$, $\gamma=2$ and $p=1$

	$\hat{R}$			BIAS			MSE		
n,m	MLE	Mom	Mix	MLE	Mom	Mix	MLE	Mom	Mix
15,15	0.220	0.274	0.322	0.091	0.063	0.160	0.254	0.045	0.027
20,20	0.217	0.268	0.303	0.075	-0.044	-0.091	0.234	0.033	0.019
25,25	0.216	0.272	0.272	0.078	0.023	-0.083	0.201	0.034	0.016
15,20	0.221	0.282	0.328	0.065	0.051	0.094	0.264	0.053	0.021
20,15	0.226	0.291	0.352	0.084	-0.062	-0.092	0.286	0.066	0.039
15,25	0.231	0.269	0.314	0.085	0.065	0.106	0.297	0.051	0.038
25,15	0.236	0.283	0.317	0.098	-0.083	0.095	0.298	0.076	0.034
20,25	0.255	0.312	0.336	0.087	-0.107	-0.114	0.252	0.054	0.035
25,20	0.257	0.306	0.341	0.104	-0.113	0.121	0.262	0.068	0.061

Table (3)

Estimates of R, Biases and Mean Squared Errors MSE's of the point estimates from Weibull Distribution, when $k=1$, $\beta=2$, $\alpha=0.5$, $\gamma=2$ and $p=1$

	$\hat{R}$			BIAS			MSE		
n,m	MLE	Mom	Mix	MLE	Mom	Mix	MLE	Mom	Mix
15,15	0.321	0.374	0.402	0.108	0.059	0.142	0.152	0.072	0.087
20,20	0.319	0.358	0.425	-0.098	-0.048	0.123	0.144	0.065	0.092
25,25	0.321	0.362	0.392	0.087	-0.027	0.115	0.134	0.084	0.081
15,20	0.331	0.371	0.418	-0.084	0.048	0.126	0.178	0.091	0.087
20,15	0.342	0.382	0.457	-0.099	0.059	0.134	0.155	0.069	0.093
15,25	0.341	0.369	0.445	0.105	-0.068	0.152	0.192	0.081	0.104
25,15	0.346	0.384	0.427	-0.112	0.079	0.143	0.171	0.064	0.084
20,25	0.365	0.374	0.432	-0.107	0.096	0.137	0.182	0.074	0.113
25,20	0.367	0.398	0.465	0.123	-0.107	0.141	0.148	0.097	0.109

Table (4)

Estimates of R, Biases and Mean Squared Errors MSE's of the point estimates from Weibull Distribution, when $k=2$, $\beta=2$, $\alpha=0.5$, $\gamma=2$ and $p=1$

	$\hat{R}$			BIAS			MSE		
n,m	MLE	Mom	Mix	MLE	Mom	Mix	MLE	Mom	Mix
15,15	0.337	0.353	0.377	0.077	0.108	0.088	0.057	0.213	0.066
20,20	0.325	0.382	0.412	-0.078	-0.098	0.067	0.049	0.192	0.089
25,25	0.343	0.394	0.418	-0.059	-0.075	0.081	0.052	0.184	0.086
15,20	0.352	0.397	0.414	-0.064	-0.069	0.055	0.078	0.211	0.072
20,15	0.361	0.377	0.413	0.076	0.087	0.072	0.086	0.231	0.061
15,25	0.365	0.386	0.423	0.086	-0.096	0.078	0.092	0.207	0.087
25,15	0.359	0.378	0.433	-0.097	0.103	0.086	0.077	0.198	0.055
20,25	0.381	0.386	0.425	0.081	-0.109	0.091	0.087	0.215	0.067
25,20	0.373	0.397	0.423	0.092	0.113	0.086	0.069	0.197	0.093

Table (5)

Estimates of R, Biases and Mean Squared Errors MSE's of the point estimates from Weibull Distribution, when $k=1$, $\beta=1.5$, $\alpha=0.5$, $\gamma=2$ and $p=1$

	$\hat{R}$			BIAS			MSE		
n,m	MLE	Mom	Mix	MLE	Mom	Mix	MLE	Mom	Mix
15,15	0.231	0.287	0.312	0.077	0.057	0.121	0.092	0.065	0.081
20,20	0.249	0.276	0.341	0.065	-0.043	-0.107	0.083	0.068	0.087
25,25	0.247	0.303	0.362	0.055	-0.037	0.109	0.094	0.079	0.065
15,20	0.256	0.292	0.382	-0.057	0.035	0.113	0.088	0.088	0.077
20,15	0.251	0.298	0.354	0.068	0.049	0.126	0.087	0.067	0.083
15,25	0.234	0.308	0.322	0.086	0.057	-0.125	0.092	0.076	0.083
25,15	0.256	0.287	0.367	-0.092	-0.069	0.131	0.071	0.059	0.071
20,25	0.248	0.294	0.353	0.079	0.067	-0.117	0.094	0.067	0.090
25,20	0.262	0.314	0.361	0.108	-0.087	0.132	0.092	0.085	0.057

Table (6)

Estimates of R, Biases and Mean Squared Errors MSE's of the point estimates from Weibull Distribution, when $k=2$, $\beta=1.5$, $\alpha=0.5$, $\gamma=2$ and $p=1$

	$\hat{R}$			BIAS			MSE		
n,m	MLE	Mom	Mix	MLE	Mom	Mix	MLE	Mom	Mix
15,15	0.306	0.331	0.375	0.105	0.114	-0.091	0.083	0.156	0.085
20,20	0.311	0.342	0.352	-0.098	-0.123	0.086	0.069	0.134	0.093
25,25	0.321	0.352	0.348	0.076	0.108	-0.065	0.066	0.144	0.103
15,20	0.335	0.367	0.394	0.089	0.099	0.076	0.087	0.161	0.075
20,15	0.347	0.356	0.371	-0.114	-0.093	0.077	0.093	0.187	0.098
15,25	0.351	0.385	0.388	0.086	-0.104	0.079	0.067	0.154	0.077
25,15	0.337	0.382	0.403	0.109	0.103	0.089	0.059	0.134	0.083
20,25	0.363	0.397	0.396	-0.071	0.107	-0.091	0.058	0.162	0.096
25,20	0.342	0.403	0.408	0.096	-0.097	0.076	0.087	0.167	0.109

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