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Comparison of Estimators for Exponentiated Inverted Weibull Distribution Based on Grouped Data

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ABSTRACT
In many situations, instead of complete sample, data is available only in grouped form. This paper presents estimation of population parameters for the exponentiated inverted Weibull distribution based on grouped data with equi and unequi-spaced grouping. Several alternative estimation schemes, such as, the method of maximum likelihood, least lines, least squares, minimum chi-square, and modified minimum chi-square are considered. Since the different methods of estimation didn't provide closed form solution, thus numerical procedure is applied. The root mean squared error resulting estimators used as comparison criterion to measure both the accuracy and the precision for each parameter.
Keywords - exponentiated inverted Weibull, grouped data, least lines, least squares, maximum likelihood, minimum chi square, modified minimum chi square.

I. INTRODUCTION

In different fields of science such as engineering, biology, and medicine it is not possible to obtain the measurements of a statistical experiment correctly but is possible to classify them into intervals, rectangles or disjoint subsets [Alodat and Al-Saleh [1]; Shadrouk and Nasiri [2]]. This means that a raw sample grouped into a frequency distribution with equi or unequi-spaced intervals, for example in life testing experiments, the failure time of a component is observed to the nearest hour, day or month. In this case the values of individual observations are not known, but the number of observations that fall in each group is only known.

Exact measurements often require costly skilled, personnel and complex instruments; whereas grouped data is usually quicker, easier and cheaper.

The estimation problem of the unknown parameters from different distributions based on grouped samples has considered by many authors. Earlier works on grouped samples can be found in Kulldorff [3], who got the maximum likelihood estimate (MLE) of the exponential distribution parameter for grouped data. Some other works can be found in, see for example, Archer [4], Cheng and Chen [5], Rosaiah et al. [6], Rao et al [7], Chen and Mi [8], Komori and Hirose [9] , Kantam et al. [10].

More recently, Shadrokh and Pazira [11] obtained the classical and Bayesian estimation from grouped and un-grouped data when the underlying distribution is exponentiated gamma. Shadrokh and Nasiri [12] considered the ungrouped and grouped data problems for the minimax distribution.

Marwa et al. [13] obtained the MLE of the unknown parameter for the exponentiated Fréchet distribution based on grouped data. Also, asymptotic optimum group limits in the case of unequi-spaced groupings is worked out.

Adding one or more parameters to a distribution makes it richer and more flexible for modeling data. There are different ways for adding parameters to a distribution. Exponentiated (generalized) inverted Weibull distribution is a generalization to inverted Weibull through adding a new shape parameter by exponentiation to distribution function.

The standard two-parameter exponentiated inverted Weibull distribution (EIW) distribution has been proposed by Flaih et al. [14] with the following cumulative distribution function (CDF)

$$F(x;\theta,\beta) = (e^{-x^{-\beta}})^{\theta}\,; x\,, \beta\,, \theta > 0 \qquad (1)$$

which is simply the θ-th power of the distribution function of the standard inverted Weibull distribution with two shape parameters β and θ. Therefore, the probability density function(PDF) is:

$$f(x;\theta,\beta) = \theta\,\beta\,x^{-(\beta+1)}\,(e^{-x^{-\beta}})^{\theta}\,; x > 0 \qquad (2)$$

For θ=1, it represents the standard inverted Weibull distribution, and for β = 1 it represents the exponentiated standard inverted exponential distribution.

The exponentiated inverted Weibull distribution has been studied by a few authors, for example; Hassan [15] concerned with the optimal designing of failure step-stress partially accelerated life tests based on EIW distribution with two stress levels under type-I censoring. Aljuaid [16] obtained the Bayesian and classical estimators for the two parameters EIW distribution when sample is available from complete and type II censoring scheme.

This paper is organized as follows. In section 2 basic assumptions are described. Section 3 deals with MLE of the unknown parameters for the EIW distribution for equi and unequi spaced grouped data. Section 4 presents least lines and least square estimators in cases of equi and unequi spaced grouped data. Minimum and modified minimum chi-square estimators for EIW are discussed in section 5. Using simulation techniques, the biases and root mean square errors (RMSEs) in the estimation methods are analyzed and the guideline of the best estimation method is laid down in Section 6. Finally, conclusion is presented in Section 7

The main aim of this paper is to study how the different estimators of the EIW distribution behave for different parameter values and different number of groups. Comparison study between estimators made with respect to their biases and RMSEs using extensive simulation techniques.

II. ASSUMPTIONS OF THE MODEL

1. Let n independent and identical observations are made on a random variable X following the EIW with PDF and CDF defined in equations (1) and (2) respectively.
2. Let x_0 be the lower bound of the distribution, such that $F(x_o)=0$, and let x_k, analogously, be the upper bound of the distribution. It will be assumed that the range of variation (x_0, x_k) is divided (partitioned) into k intervals (these intervals could be equally or not equally).
3. The dividing points $x_1, x_2, ..., x_{k-1}$, such that $x_0 < x_1 < x_2 < ... < x_{k-1} < x_k$ are predetermined.
4. The interval (x_{i-1}, x_i) will be referred to as the ith interval (or the ith group); the points $x_i \, (i = 0,1,...,k)$ are called the group limits; the first and last group limits are $x_0 > 0$ and $x_k = \infty$ respectively.
5. The number of observations falling in the ith group are recorded and will be denoted by n_i, such that $\sum_{i=1}^{k} n_i = n$.
6. $P_i = P_i(\underline{\theta})$ is the probability of an observation falling in the ith group; i.e.

$$P_i = \int_{x_{i-1}}^{x_i} f(x)dx = F(x_i) - F(x_{i-1}) = \left(\exp - x_i^{-\beta}\right)^{\theta} - \left(\exp - x_{i-1}^{-\beta}\right)^{\theta};$$

where $F(x_o)=0$, and $F(x_k)=1$.

III. MAXIMUM LIKELIHOOD METHOD

Based on the form of the data described in the previous section, the maximum likelihood estimators (MLEs) will be obtained.

According to Kulldorff [3] the likelihood function based on grouped data described above takes the following form:

$$L(X;\underline{\theta}) = C \prod_{i=1}^{k} P_i^{n_i}, \quad (3)$$

where, $\underline{\theta}$ is the vector of the unknown parameters, $C = \dfrac{n!}{n_1! n_2! ... n_k!}$ is a constant with respect to the parameters.

Depending on the definition of P_i and the CDF of the EIW distribution; the likelihood function (3) may be written as follows:

$$L(X;\theta,\beta) = C \prod_{i=1}^{k} [(e^{-x_i^{-\beta}})^{\theta} - (e^{-x_{i-1}^{-\beta}})^{\theta}]^{n_i}. \quad (4)$$

The log-likelihood function becomes:

$$\ell nL(X;\theta,\beta) = \ln C + \sum_{i=1}^{k} n_i \ln\left[\begin{matrix}(e^{-x_i^{-\beta}})^{\theta} - \\ (e^{-x_{i-1}^{-\beta}})^{\theta}\end{matrix}\right]. \quad (5)$$

For simplicity $\ell nL(X;\theta,\beta)$ will be written as ℓnL.

To get the MLEs of the parameters θ and β, the log-likelihood (5) must be maximized by differentiate it with respect to θ and β, respectively, and set it equal to zero. The MLEs for the two unknown parameters θ and β will be obtained under two different cases, equi- and unequi-spaced grouped samples.

(3.1) MLE in the case of unequi-spaced grouped samples

The first partial derivatives of ℓnL with respect to θ and β are given respectively by:

$$\frac{\partial \ell nL}{\partial \theta}=\sum_{i=1}^{k}\frac{n_i[(x_{i-1})^{-\beta}(\mathrm{e}^{-(x_{i-1})^{-\beta}})^{\theta}-(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}(x_i)^{-\beta}]}{[(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}-(\mathrm{e}^{-(x_{i-1})^{-\beta}})^{\theta}]}, \tag{6}$$

$$\frac{\partial \ell nL}{\partial \beta}=\sum_{i=1}^{k}\frac{n_i[\theta(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}(x_i)^{-\beta}\ln(x_i)-\theta(\mathrm{e}^{-(x_{i-1})^{-\beta}})^{\theta}(x_{i-1})^{-\beta}\ln(x_{i-1})]}{[(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}-(\mathrm{e}^{-(x_{i-1})^{-\beta}})^{\theta}]} \tag{7}$$

The maximum likelihood estimates of θ and β denoted by $\hat{\theta}_{UG}$ and $\hat{\beta}_{UG}$ can be obtained by maximizing (6) and (7) with respect to θ and β respectively, by setting $\frac{\partial \ell nL}{\partial \beta}=0$ and $\frac{\partial \ell nL}{\partial \theta}=0$ and solve for the values of θ and β. As it seems the likelihood equations have no closed forms solutions in θ and β. Therefore a numerical technique method will be used to get the solution.

(3.2) MLE based on equi-spaced grouped samples

The special but not unusual case where the group limits $0, x_1, x_2, ..., x_{k-1}$ (except the last group limit $(x_{k-1}, x_k=\infty)$) are equi-distance should be considered. This situation will be referred to as equi-spaced grouping, where $x_i = i\delta$; $\delta > 0$.

The log likelihood function denoted by ℓnL^* in this case will reduced to

$$\ell nL^* = \ln C + \sum_{i=1}^{k} n_i \ln[(e^{-(i\delta)^{-\beta}})^{\theta} - (e^{-((i-1)\delta)^{\beta}})^{\theta}].$$

The first partial derivatives for ℓnL^* with respect to θ and β are given respectively by:

$$\frac{\partial \ell nL^*}{\partial \theta}=\sum_{i=1}^{k}\frac{n_i((\mathrm{i}-1)\delta)^{-\beta}(\mathrm{e}^{-((\mathrm{i}-1)\delta)^{-\beta}})^{\theta}-(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}.(i\delta)^{-\beta}}{[(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}-(\mathrm{e}^{-((\mathrm{i}-1)\delta})^{-\beta})^{\theta}]}, \tag{8}$$

$$\frac{\partial \ell nL^*}{\partial \beta}=\theta\sum_{i=1}^{k}\frac{n_i[(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}.(i\delta)^{-\beta}.\ln(i\delta)-(\mathrm{e}^{-((i-1)\delta)^{-\beta}})^{\theta}.((i-1)\delta)^{-\beta}.\ln((i-1)\delta)]}{[(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}-(\mathrm{e}^{-((i-1)\delta)^{-\beta}})^{\theta}]}. \tag{9}$$

The MLEs of the parameters θ and β denoted by $\hat{\theta}_{EG}$ and $\hat{\beta}_{EG}$ in the case of equi-spaced grouping are simultaneous iterative solutions of the equations (8) and (9) when they are set equal to zero.

IV. LEAST LINES AND LEAST SQUARES METHODS

The least square (LS) method is a very popular technique. It is one of the oldest techniques of modern statistics for estimation of the probability distribution. Least square method is extensively used in reliability engineering and mathematics problems; it is perhaps the most widely used technique in geophysical data analysis. Nowadays, the least square method is widely used to find or estimate the numerical values of the parameters to fit a function to a set of data and to characterize the statistical properties of estimates.

Suppose that the unknown vector of the parameters is ($\underline{\theta}$); according to Macdonald and Ransom [17] the least lines and the least squares estimator based on grouped can be defined by the value of $\underline{\theta}$ which minimizes the following amount:

$$SE(\underline{\theta};\rho)=\sum_{i=1}^{k}\left|P_i-\frac{n_i}{n}\right|^{\rho}. \tag{10}$$

If $\rho=1$ then the associated estimator is referred to as least lines (LL), and for $\rho=2$ the associated estimator is referred to as the least square estimator. Minimize (10) to select estimators of $\underline{\theta}$ so as to provide the best fit between the empirical and theoretical density functions.

Depending on the value of P_i defined above, the least lines estimators for θ and β denoted by $LL\theta_{UG}$ and $LL\beta_{UG}$ in the case of unequi-spaced grouping, can be obtained by minimizing the following

$$SE_{UG}(\theta,\beta,1)=\sum_{i=1}^{k}\left|(e^{(x_i)^{-\beta}})^{\theta}-(\mathrm{e}^{-(x_{i-1})^{-\beta}})^{\theta}-\frac{n_i}{n}\right|, \tag{11}$$

with respect to θ and β.

For the case of equi-spaced grouping the least lines estimators for θ and β denoted by $LL\theta_{EG}$ and $LL\beta_{EG}$ can be obtained by minimizing the following,

$$SE_{EG}(\theta,\beta,1)=\sum_{i=1}^{k}\left|(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}-(\mathrm{e}^{-((\mathrm{i}-1)\delta)^{-\beta}})^{\theta}-\frac{n_i}{n}\right|, \tag{12}$$

with respect to θ and β.

Similarly, for $\rho=2$, the least squares estimators for θ and β can be obtained in the case of unequi-spaced grouping by minimizing

$$SE_{UG}(\theta,\beta,2)=\sum_{i=1}^{k}\left|(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}-(\mathrm{e}^{-(x_{i-1})^{-\beta}})^{\theta}-\frac{n_i}{n}\right|^2, \quad (13)$$

with respect to θ and β respectively to obtain the least squares estimators $LS\theta_{UG}$ and $LS\beta_{UG}$. On the other hand, for the equi-spaced grouping

$$SE_{EG}(\theta,\beta,2)=\sum_{i=1}^{k}\left|(e^{-(i\delta)^{-\beta}})^{\theta}-(e^{-((i-1)\delta)^{-\beta}})^{\theta}-\frac{n_i}{n}\right|^2. \quad (14)$$

(14) has to be minimized with respect to θ and β to obtain $LS\theta_{EG}$ and $LS\beta_{EG}$ (least squares estimator).

A simulation study will be carried out to obtain the least lines and the least squares estimates for the parameters θ and β.

V. Minimum Chi-Square and Modified Minimum Chi-Square Methods

Minimum chi-square (chi) method of estimation is applicable to situations where the data arise, or are rendered to be, in the form of frequencies of a number of mutually exclusive and exhaustive events. In statistics, minimum chi-square estimation is a method of estimation of unobserved quantities based on observed data, [see Agarwal [18]].

According to Macdonald & Ransom [17], and Kedem and Wu [19], the minimum chi-square estimators can be obtained by choosing the parameter vector $\underline{\theta}$ such that,

$$\chi^2(\underline{\theta})=\sum_{i=1}^{k}\frac{\left(n_i-nP_i(\underline{\theta})\right)^2}{nP_i(\underline{\theta})}, \quad (15)$$

is minimized.

Based on the EIW distribution, the chi-square estimator of θ and β, denoted by $\chi^2_{UG}(\theta)$ and $\chi^2_{UG}(\beta)$ can be obtained by minimizing

$$\chi^2_{UG}(\theta,\beta)=\sum_{i=1}^{k}\frac{\{n_i-n[(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}-(\mathrm{e}^{--(x_{i-1})^{-\beta}})^{\theta}]\}^2}{n[(\mathrm{e}^{-(x_i)^{-\beta}})^{\theta}-(\mathrm{e}^{--(x_{i-1})^{-\beta}})^{\theta}]}, \quad (16)$$

with respect to θ and β in the case of unequi-spaced grouped samples, and by minimizing:

$$\chi^2_{EG}(\theta,\beta)=\sum_{i=1}^{k}\frac{\{n_i-n[(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}-(\mathrm{e}^{-((i-1)\delta)^{-\beta}})^{\theta}]\}^2}{n[(\mathrm{e}^{-(i\delta)^{-\beta}})^{\theta}-(\mathrm{e}^{-((i-1)\delta)^{-\beta}})^{\theta}]}, \quad (17)$$

with respect to θ and β, to find the chi-square estimator $\chi^2_{EG}(\theta)$ and $\chi^2_{EG}(\beta)$ in the case of equi-spaced grouping.

The expected frequency $nP_i(\underline{\theta})$ in the dominator of χ^2 statistics causes certain difficulties. Hence, a modification has been suggested which make the process of differentiation easier. The modified minimum chi-square (M-chi) can be obtained by choosing the parameters $(\underline{\theta})$ which minimize

$$\chi^2_1(\underline{\theta})=\sum_{i=1}^{k}\frac{\left(n_i-nP_i(\underline{\theta})\right)^2}{n_i}, \quad (18)$$

where $P_i(\underline{\theta})=(\mathrm{e}^{-x_i^{-\beta}})^{\theta}-(\mathrm{e}^{-x_{i-1}^{-\beta}})^{\theta}$

The reset of the procedure of modified minimum chi-square remains same as that of minimum chi-square.

VI. Simulation Procedure

The main objective of this section is to illustrate the theoretical results of estimation problem for the EIW distribution based on equi- and unequi-spaced grouped data. Five different methods of estimation, namely, LL, LS, chi, M-chi and ML methods are considered.

In order to compare the performance of different estimators, some statistical measures are evaluated such as, biases and RMSEs. The simulation procedures are described through the following two algorithms:

(6.1) First Algorithm

Step (1): A random sample of size 100 is generated from EIW distribution.

Step (2): Different values of the shape parameters θ and β of the EIW distribution be selected as, θ =0.5 (0.5) 1.5, and β = 0.5 (0.5) 1.5. The number of groups selected as $k=$ 3 (1) 10.

Step (3): Construct a frequency distribution (grouping) from the generated random numbers by enter a pre-specified group limits (not equal), and determine the number of observations falling in each group sample (n_i).

Step (4): The frequency distribution obtained in step (3) is used to determine the probability of an observation falling in the ith group P_i

Step (5): Depending on the values of n_i, P_i and equations(6), and (7) the MLE of θ and β in the case of unequi-grouping is calculated. The values of the LL and LS estimators are calculated depending on (11) and (13). Chi and M-chi estimators are obtained using (16) and (18).

Step (6): This procedure is repeated N times where $N = 1000$, representing N different samples.

Step (7): The biases and RMSEs of θ and β depending on the selected number of groups and sets of parameters are computed for the five methods of estimation and will be compared to decide which method is the best for estimating the unknown parameters.

(6.2) Second Algorithm

Step (1) and **Step (2):** will be as in the previous algorithm.

Step (3): Construct a frequency distribution (grouping) from the generated random numbers by evaluating equi-spaced group limits, and determine the number of observations falling in each group sample.

Step (4): The frequency distribution obtained in step (3) is used to determine the probability of an observation falling in the ith group (P_i)

Step (5): Depending on the values of n_i, P_i and equations (8), and (9) the MLE of θ and β in the case of equi-grouping is calculated. The values of the LL and LS estimators are calculated depending on equations (12) and (14). Chi and M-chi estimators are obtained using (17) and (18).

Step (6): This procedure is repeated N times where $N = 1000$, representing N different samples.

Step (7): The same as the previous algorithm.

All simulation studies presented here are obtained via the MathCAD (14). Simulation results are reported in Tables 1 and 2. Table 1 contains biases and RMSEs for the different estimated parameters based on unequi-spaced grouped samples. Table 2 contains biases and RMSEs for the different estimated parameters based on equi-spaced grouped samples. Figures (1-4) represent RMSEs for parameters θ and β for different methods of estimation and different number of groups in the case of unequi-spaced grouping samples. Figures (5-8) represent RMSEs for parameters θ and β for different methods of estimation and different number of groups in the case of equi-spaced grouping samples.

From simulation studies many observations can be made on the performance of different estimated parameters from EIW based on grouped data:

1. The biases associated with MLE are larger than achievable using other grouped data estimation techniques based on unequi-spaced grouping. (see Table 1)
2. Based on unequi-grouped samples; the chi and M-chi estimation methods yield the smallest sample biases. (see Table 1).
3. For the equi-spaces grouped sample the methods of LL and LS have the smallest biases. (see Table 2).
4. In the case of equi-spaced grouping; the biases of ML and chi are larger than the biases associated with the other estimation methods. (see Table 2).
5. In the case of unequi-spaced grouping the RMSEs for the chi estimates of both parameters θ and β are smaller than the RMSEs for the other methods in almost all the cases. (see Figures from 1to 4)
6. In the case of equi-spaced grouping the RMSEs for the MLEs of both parameters θ and β are smaller than the RMSEs for the other methods in almost all the cases.(see Figures from 5 to 8)
7. For all methods of estimation, it is clear that RMSEs decrease as the number of groups increases.(see Figure from 1to 8)
8. In the case of unequi-spaced grouping, for a fixed value of θ, the RMSEs of the parameter β increases as the value of parameter β increases for all the methods in most all the cases. (see Figures 2, 3 and Table 1)
9. In the case of equi-spaced grouping, for $\theta = 0.5$, the RMSEs of the parameter β decreases, and then increases with the increase of the parameter β for all the methods in most all the cases. For $\theta = 1$ the RMSEs of the parameter β increases, and then decreases with the increase of the

parameter β for all the methods in most all the cases. For $\theta = 1.5$ the RMSEs of the parameter β decreases, and then increases with the increase of the parameter β for all the methods in most all the cases. (see Figure 5 and Table 2)

10. In the case of equi-spaced grouping, for a fixed value of β, the RMSEs of the parameter θ increases with the increase of the parameter θ for all the methods in most all the cases (except when $\beta = 0.5$ the RMSEs of the parameter θ decreases and then increases with the increase of the parameter θ.(see Figure 5)

VII. Conclusion

In this paper, the estimation problem for EIW distribution based on grouped data has been discussed. Five different estimators (least lines, least square, minimum chi square, modified minimum chi square and maximum likelihood) for equi- as well as unequi-grouped samples have been obtained.

Based on grouped data, the estimates with unequi-spaced sample have smaller RMSE's compared with the equi-spaced grouped sample. It is immediate to note that RMSEs decrease as number of groups increase in both cases (equi- and unequi-spaced grouped samples).

This study revealed that the estimators based on minimum chi square method in the case of unequi-spaced grouping perform the best among rest estimators, in most situations, with respect to their biases and RMSEs . On the other hand, in case of equi-spaced grouping the MLEs perform the best among other estimators.

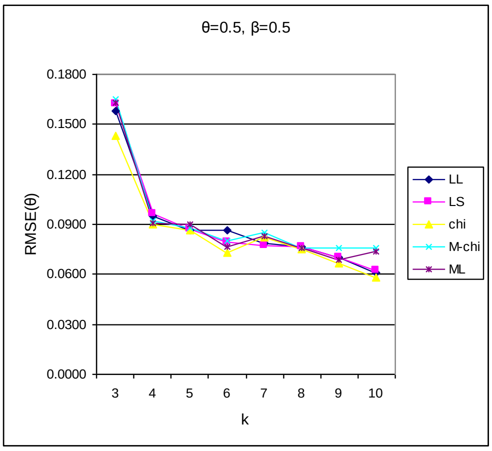

Figure (1): RMSEs of θ in the case of unequi-grouping for $\theta = 0.5$, $\beta = 0.5$.

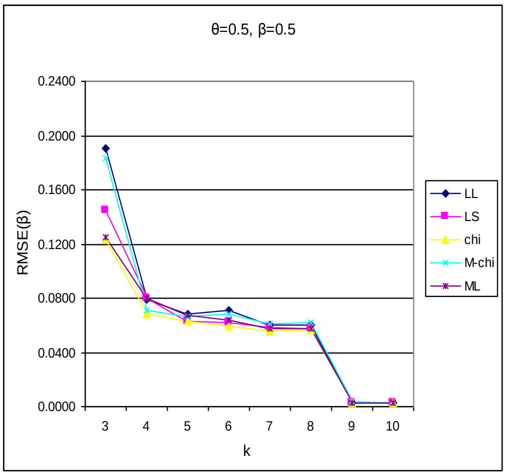

Figure (2): RMSEs of β in the case of unequi-grouping for $\theta = 0.5$, $\beta = 0.5$.

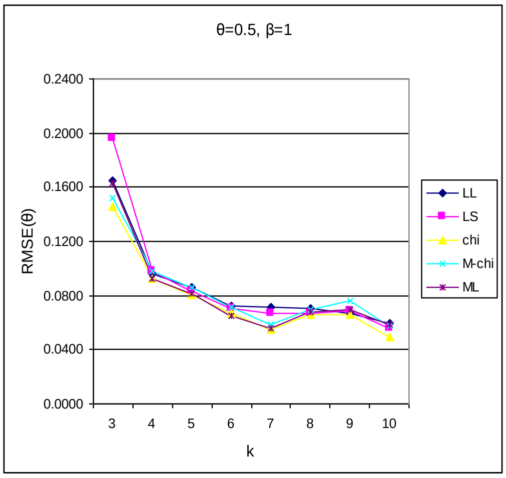

Figure (3): RMSEs for the parameter θ in the case of unequi-grouping for $\theta = 0.5$, $\beta = 1$.

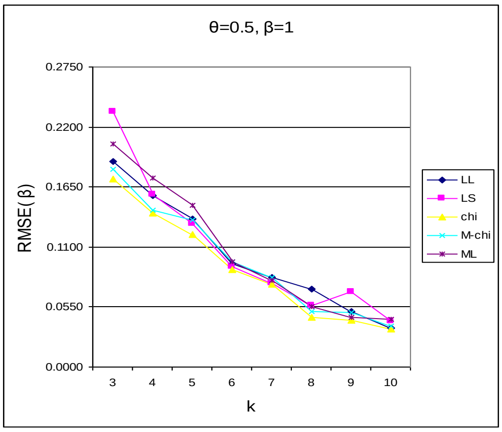

Figure (4): RMSEs for the parameter β in the case of unequi-grouping for $\theta = 0.5$, $\beta = 1$.

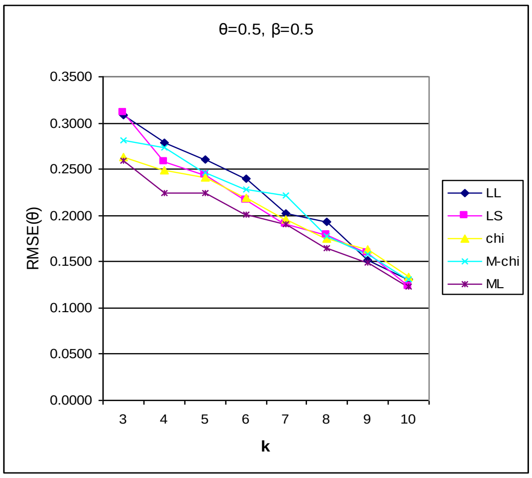

Figure (5): RMSEs of θ in the case of equi-grouping for $\theta = 0.5$, $\beta = 0.5$.

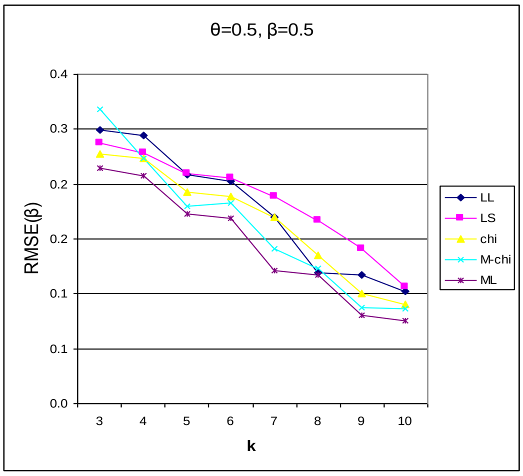

Figure (6): RMSEs of β in the case of equi-grouping for $\theta = 0.5$, $\beta = 0.5$.

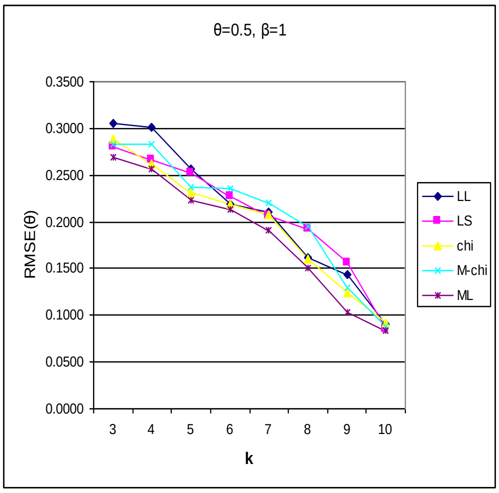

Figure (7): RMSEs for the parameter θ in the case of equi-grouping for $\theta = 0.5$, $\beta = 1$.

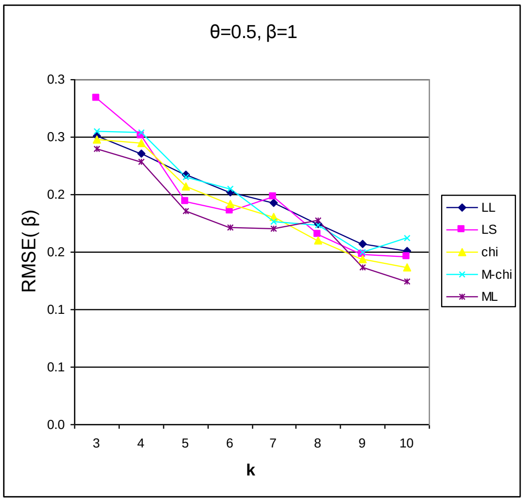

Figure (8): RMSEs of β in the case of equi-grouping for $\theta = 0.5$, $\beta = 1$.

Table (1): The RMSEs and Biases of the Estimators of the Parameters (θ, β) for Different Number of Groups and Different Methods of Estimation Based on Unequi-spaced Grouping

k	Method	θ=0.5, β=0.5				θ=0.5, β=1				θ=0.5, β=1.5			
		Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)
3	LL	0.0323	0.1579	0.0539	0.1909	-0.0172	0.1648	0.0233	0.1885	0.0322	0.1193	-0.0162	0.2738
	LS	0.013	0.1624	0.031	0.1447	0.0253	0.1960	0.0767	0.2336	0.0175	0.1094	0.0276	0.2396
	chi	-0.0274	0.1434	0.0167	0.1233	-0.031	0.1455	0.0161	0.1726	-0.0026	0.1088	-0.0075	0.2113
	M-chi	-0.0038	0.1653	0.0184	0.1831	0.0078	0.1525	0.0174	0.1809	0.0263	0.1104	-0.0107	0.2150
	ML	0.037	0.1632	0.0543	0.1254	0.0392	0.1628	0.1077	0.2040	0.0622	0.1177	0.1894	0.2629
4	LL	0.0106	0.0952	-0.021	0.0788	0.0026	0.0961	-0.0203	0.1568	0.0107	0.0853	-0.0631	0.2364
	LS	0.0073	0.0962	-0.0134	0.0799	0.0008	0.0977	0.0138	0.1578	0.0073	0.0791	-0.017	0.2129
	chi	0.0087	0.0898	-0.0057	0.0682	-0.0004	0.0923	0.0071	0.1412	0.0067	0.0744	0.0051	0.2047
	M-chi	0.0067	0.0924	-0.0025	0.0706	-0.0008	0.0977	-0.0061	0.1440	0.0087	0.0778	-0.0402	0.2118
	ML	-0.0178	0.0909	0.0591	0.0804	-0.025	0.0924	0.1317	0.1732	-0.0178	0.0779	0.1773	0.2413
5	LL	-0.0059	0.0867	-0.0154	0.0682	-0.0044	0.0865	-0.0095	0.1357	0.0053	0.0766	-0.0229	0.2065
	LS	0.0058	0.0874	-0.0199	0.0633	-0.004	0.0834	-0.0172	0.1310	0.0038	0.0755	-0.005	0.1924
	chi	0.0025	0.0862	-0.0071	0.0626	0.0045	0.0806	0.007	0.1215	-0.0027	0.0726	-0.0014	0.1831
	M-chi	0.0029	0.0878	-0.0069	0.0661	-0.0008	0.0863	0.0053	0.1347	-0.0038	0.0746	-0.0397	0.2017
	ML	0.0117	0.0902	0.0445	0.0670	-0.0142	0.0811	0.105	0.1478	0.0119	0.0709	0.1456	0.2083
6	LL	-0.0021	0.0862	-0.0184	0.0708	-0.0082	0.0727	0.0071	0.0949	0.0046	0.0724	0.0106	0.2074
	LS	0.0008	0.0792	-0.0074	0.0616	0.0048	0.0706	0.0073	0.0916	-0.0055	0.0734	0.0109	0.1765
	chi	-0.0013	0.0731	-0.011	0.0590	0.0003	0.0666	0.0057	0.0897	0.0013	0.0700	0.0086	0.1704
	M-chi	0.0007	0.0800	-0.0196	0.0686	-0.0029	0.0710	0.0024	0.0965	-0.0033	0.0740	-0.0035	0.1885
	ML	0.0162	0.0763	0.0396	0.0642	-0.0128	0.0647	0.0086	0.0966	0.0055	0.0699	0.0129	0.1731
7	LL	0.0057	0.0788	0.0028	0.0600	0.003	0.0711	0.0056	0.0825	0.0092	0.0767	0.0084	0.1800
	LS	0.0055	0.0769	0.0027	0.0583	-0.0037	0.0671	0.0055	0.0766	0.0045	0.0708	0.0082	0.1748
	chi	0.0065	0.0825	0.0049	0.0561	0.0046	0.0553	0.0097	0.0761	-0.0051	0.0695	0.0146	0.1682
	M-chi	0.0008	0.0851	0.0023	0.0607	-0.0022	0.0588	0.0046	0.0821	-0.0037	0.0734	-0.0069	0.1821
	ML	0.0116	0.0829	-0.0089	0.0572	0.0098	0.0556	-0.0177	0.0794	-0.0098	0.0704	-0.0266	0.1717
8	LL	0.0083	0.0763	0.0094	0.0607	-0.0051	0.0703	0.0263	0.0714	0.0066	0.0730	0.0323	0.1802
	LS	0.0043	0.0766	0.0038	0.0573	-0.0042	0.0672	0.0261	0.0566	0.0035	0.0700	0.0358	0.1710
	chi	0.0043	0.0752	0.0037	0.0570	0.0032	0.0655	0.0250	0.0459	0.0026	0.0682	0.0151	0.1649
	M-chi	0.0039	0.0760	0.0034	0.0623	0.0017	0.0696	0.0110	0.0508	0.0027	0.0729	-0.0222	0.1788
	ML	0.0105	0.0760	-0.0191	0.0572	-0.0052	0.0678	0.0356	0.0556	0.0096	0.0690	-0.0552	0.1655
9	LL	0.0063	0.0703	0.0047	0.0560	0.0111	0.0673	0.0042	0.0511	0.0055	0.0679	0.0385	0.1683
	LS	0.0053	0.0700	0.0026	0.0530	0.0059	0.0684	0.0007	0.0686	-0.0013	0.0685	0.0153	0.1652
	chi	0.0011	0.0669	0.0003	0.0512	-0.0044	0.0663	0.0095	0.0432	0.0012	0.0674	0.0171	0.1400
	M-chi	0.0028	0.0755	0.0002	0.0605	-0.0021	0.0756	0.0004	0.0504	0.0007	0.0712	-0.0058	0.1692
	ML	0.0214	0.0688	-0.0239	0.0527	0.0211	0.0692	-0.0243	0.0457	0.0205	0.0681	-0.0425	0.1434
10	LL	0.011	0.0606	0.0019	0.0521	0.0052	0.0591	0.0252	0.0356	0.005	0.0507	0.0136	0.1558
	LS	-0.0084	0.0627	0.0030	0.0494	0.0011	0.0558	0.033	0.0419	-0.0033	0.0535	0.0287	0.1466
	chi	0.0030	0.0583	-0.0002	0.0481	-0.0036	0.0490	0.0294	0.0351	-0.0032	0.0463	0.0072	0.1392
	M-chi	0.0058	0.0755	-0.0006	0.0501	-0.0001	0.0577	0.0243	0.0377	-0.0011	0.0592	-0.0601	0.1683
	ML	0.0177	0.0740	0.0038	0.0495	0.0194	0.0587	-0.0484	0.0437	0.0204	0.0600	0.0367	0.1450

Continued Table (1)

k	method	θ=1, β=0.5				θ=1, β=1				θ=1, β=1.5			
		Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)
3	LL	0.0406	0.2727	-0.0187	0.0832	-0.004	0.2046	-0.0218	0.1646	0.064	0.2417	0.0596	0.2268
	LS	0.0247	0.2474	-0.0112	0.0728	-0.0074	0.2136	-0.0108	0.1450	0.0598	0.2559	0.0119	0.2283
	chi	0.0127	0.2384	0.0107	0.0715	-0.0017	0.1991	0.0021	0.1446	0.056	0.2382	0.0278	0.2262
	M-chi	0.0111	0.2472	-0.0047	0.0744	-0.016	0.1857	-0.0015	0.1456	0.0195	0.2611	-0.0113	0.2282
	ML	0.0567	0.2301	0.0702	0.0915	0.0993	0.2023	0.1309	0.1736	0.1944	0.2388	0.2033	0.2386
4	LL	0.0032	0.2449	0.0219	0.0721	0.0555	0.1523	0.0132	0.1417	-0.0039	0.2240	0.0281	0.2063
	LS	0.0117	0.2408	0.0013	0.0721	0.0364	0.1805	-0.0113	0.1427	0.0016	0.2256	0.0264	0.1752
	chi	0.0154	0.2385	0.0053	0.0712	0.0214	0.1405	0.0097	0.1424	0.0015	0.2156	0.0265	0.1732
	M-chi	-0.0022	0.2399	0.0054	0.0721	0.0218	0.1486	-0.0012	0.1450	0.0014	0.2271	-0.0045	0.2012
	ML	0.0517	0.2416	0.0598	0.0820	0.1231	0.1469	0.0313	0.1589	0.0041	0.2243	0.0367	0.2015
5	LL	0.0079	0.2150	0.0126	0.0691	0.0249	0.1437	0.0214	0.1390	0.0373	0.1852	0.0039	0.1935
	LS	0.0083	0.2139	0.011	0.0653	0.0195	0.1448	0.0208	0.1306	0.0376	0.1803	0.0035	0.1544
	chi	0.0087	0.2132	0.0098	0.0612	0.0186	0.1399	0.0198	0.1230	0.0286	0.1744	0.0112	0.1471
	M-chi	-0.0064	0.2137	0.011	0.0645	0.005	0.1479	0.0072	0.1301	0.0192	0.1775	-0.0011	0.1582
	ML	0.0157	0.2136	-0.017	0.0630	0.034	0.1450	0.0256	0.1240	0.0532	0.1864	-0.0203	0.1783
6	LL	0.0130	0.0854	0.008	0.0669	0.0162	0.1338	0.0244	0.1323	-0.0155	0.1201	0.0397	0.1723
	LS	0.0119	0.0811	0.0068	0.0610	0.0147	0.1314	0.0252	0.1226	0.0071	0.1185	0.0387	0.1530
	chi	0.0110	0.0806	0.0066	0.0587	0.0145	0.1308	0.0221	0.1119	0.008	0.1178	0.0291	0.1324
	M-chi	-0.0079	0.0839	0.0012	0.0628	-0.0055	0.1329	0.0193	0.1301	0.0067	0.1222	0.0342	0.1332
	ML	0.0226	0.0800	0.0118	0.0593	0.0322	0.1338	0.0257	0.1140	0.0186	0.1187	0.0399	0.1373
7	LL	-0.0607	0.0853	0.0063	0.0648	0.0089	0.1263	0.022	0.0986	0.0213	0.0960	0.0385	0.0986
	LS	-0.0621	0.0786	0.0036	0.0588	-0.0149	0.1240	0.0209	0.0977	0.006	0.0977	0.0353	0.0966
	chi	-0.0460	0.0625	0.0043	0.0568	0.0076	0.1225	0.0188	0.0919	0.0052	0.0951	0.0322	0.0937
	M-chi	-0.0616	0.0759	0.0023	0.0607	0.007	0.1267	-0.0061	0.0926	0.0047	0.0962	-0.0046	0.0966
	ML	-0.0937	0.0722	-0.0089	0.0572	0.0194	0.1234	0.0244	0.0954	-0.0236	0.0972	0.0407	0.0945
8	LL	0.0148	0.0712	0.0043	0.0631	-0.0172	0.0921	0.0274	0.0767	-0.0151	0.0785	0.0400	0.0925
	LS	0.0148	0.0670	0.0039	0.0583	0.0066	0.0919	0.0187	0.0734	0.0037	0.0851	0.0263	0.0871
	chi	0.0149	0.0661	0.0029	0.0561	0.0072	0.0905	0.0174	0.0653	0.0033	0.0794	0.0234	0.0856
	M-chi	-0.0050	0.0722	0.0027	0.0640	0.0065	0.0924	-0.0003	0.0682	0.0026	0.0839	-0.0243	0.0922
	ML	0.0325	0.0719	0.0049	0.0577	0.0181	0.0919	-0.0284	0.0825	0.0203	0.0831	-0.0457	0.0862
9	LL	0.0114	0.0655	-0.0029	0.0608	0.0048	0.0725	0.0243	0.0574	0.0120	0.0651	-0.0441	0.0866
	LS	0.0102	0.0759	0.0028	0.0551	0.0063	0.0670	0.0197	0.0644	0.0115	0.0527	0.0309	0.0827
	chi	0.0099	0.0626	-0.0027	0.0541	0.0082	0.0655	0.0019	0.0557	0.0107	0.0476	0.0292	0.0783
	M-chi	0.0058	0.0760	-0.0027	0.0629	0.0047	0.0733	0.0190	0.0599	0.0070	0.0615	-0.0305	0.0792
	ML	0.0289	0.0656	-0.0293	0.0548	0.0255	0.0683	-0.0286	0.0822	0.0279	0.0485	0.0415	0.0800
10	LL	0.0241	0.0386	0.0062	0.0588	0.0097	0.0396	0.0204	0.0506	-0.0317	0.0430	0.0327	0.0604
	LS	0.0110	0.0443	0.0052	0.0528	0.0099	0.0387	0.0136	0.0509	-0.0237	0.0482	0.0236	0.0590
	chi	0.0097	0.0374	0.0036	0.0498	0.0117	0.0388	0.0129	0.0460	0.0188	0.0404	0.0236	0.0569
	M-chi	0.0079	0.0452	0.0032	0.0537	0.0248	0.0460	0.0028	0.0568	0.0190	0.0558	-0.0231	0.0641
	ML	0.0272	0.0371	-0.0205	0.0509	0.0286	0.0467	-0.0320	0.0642	0.0356	0.0354	-0.0494	0.0641

Continued Table (1)

k	method	θ=1.5, β=0.5				θ=1.5, β=1				θ=1.5, β=1.5			
		Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)
3	LL	0.0154	0.2591	0.0263	0.1252	0.0382	0.2430	-0.0402	0.1887	0.0208	0.2267	0.0114	0.2180
	LS	0.0159	0.2478	0.0628	0.1261	0.0161	0.2411	0.0177	0.1441	0.0172	0.2434	0.0112	0.2197
	chi	0.0109	0.2396	0.0263	0.1208	0.0158	0.2317	0.0426	0.1438	0.0164	0.2033	-0.0074	0.2119
	M-chi	0.0061	0.2406	0.0040	0.1256	0.0128	0.2421	0.0064	0.1581	0.0109	0.2215	-0.0069	0.2179
	ML	0.1765	0.2725	0.0678	0.1362	0.0640	0.2390	0.1377	0.1833	0.0221	0.2455	0.0522	0.2150
4	LL	0.0174	0.2213	0.0094	0.0922	0.0332	0.2175	0.0176	0.1449	-0.0378	0.2064	-0.0543	0.1964
	LS	0.0167	0.2198	0.0088	0.0851	0.0333	0.2182	0.006	0.1593	-0.0166	0.2182	-0.0180	0.1944
	chi	-0.0004	0.2184	0.0088	0.0797	0.0001	0.2159	-0.0057	0.1388	0.0093	0.1767	-0.0404	0.1733
	M-chi	0.0165	0.2215	0.0015	0.0899	0.0325	0.2204	0.0030	0.1498	-0.0027	0.2015	-0.0147	0.1758
	ML	0.0249	0.2196	0.0122	0.0925	0.0477	0.2169	0.0245	0.1491	0.1338	0.2254	0.1498	0.2024
5	LL	0.0241	0.1953	0.0026	0.0787	0.0965	0.2148	0.0137	0.1371	0.037	0.1797	0.0401	0.1543
	LS	0.0224	0.1975	0.0013	0.0638	-0.0569	0.2044	0.0098	0.1261	0.0369	0.2033	0.0393	0.1737
	chi	0.0222	0.1931	0.0037	0.0593	-0.0437	0.1987	0.0090	0.1224	0.0354	0.1473	0.0352	0.1480
	M-chi	-0.0100	0.1937	0.0004	0.0655	-0.0312	0.2222	-0.0049	0.1278	0.0060	0.1700	-0.0196	0.1649
	ML	0.0372	0.1934	-0.0068	0.0611	-0.1043	0.2025	0.0154	0.1230	0.0495	0.1733	0.0432	0.1840
6	LL	0.0363	0.1220	0.0074	0.0718	0.0379	0.1834	0.0249	0.0906	0.0277	0.1644	0.0287	0.1229
	LS	0.0335	0.1572	0.0073	0.0637	0.0361	0.1439	0.0238	0.0957	0.0130	0.1885	0.0243	0.1451
	chi	0.0325	0.1210	0.0060	0.0583	-0.0057	0.1357	0.0211	0.0902	0.0089	0.1103	0.0147	0.1176
	M-chi	-0.0051	0.1374	0.0058	0.0646	0.0335	0.1696	0.0189	0.0958	0.0117	0.1266	-0.0186	0.1441
	ML	0.0507	0.1213	0.0112	0.0597	0.0521	0.1747	0.0303	0.0919	-0.0296	0.1261	0.0356	0.1781
7	LL	0.0441	0.0954	0.0086	0.0701	0.0339	0.0913	0.0235	0.0860	0.0330	0.0971	0.0225	0.1116
	LS	0.0405	0.0923	0.0075	0.0619	-0.0294	0.0944	0.0208	0.0846	0.0095	0.1557	0.0242	0.1356
	chi	0.0401	0.0804	0.0064	0.0570	0.0289	0.0892	0.0185	0.0783	0.0075	0.0879	0.0157	0.1142
	M-chi	-0.0202	0.0962	-0.0060	0.0631	0.0286	0.0943	-0.0041	0.0845	0.0066	0.0914	-0.021	0.1157
	ML	0.0675	0.0942	0.0116	0.0588	0.0556	0.1358	0.0282	0.0795	-0.0519	0.0908	0.0272	0.1521
8	LL	-0.0375	0.0875	0.0133	0.0633	0.0328	0.0741	0.0256	0.0848	0.0430	0.0794	0.0341	0.0967
	LS	0.0205	0.0867	0.0116	0.0590	0.0291	0.0769	-0.0255	0.0841	0.0404	0.0822	0.0309	0.1152
	chi	0.0183	0.0803	0.0088	0.0485	0.0283	0.0692	0.0196	0.0716	0.0402	0.0804	0.0199	0.0950
	M-chi	0.0180	0.0847	0.0078	0.0623	-0.0266	0.0718	0.0184	0.0825	-0.0172	0.0813	-0.0188	0.0957
	ML	0.0504	0.0930	-0.0152	0.0586	0.0602	0.0944	0.0281	0.0778	0.0738	0.0822	-0.0521	0.1144
9	LL	0.0360	0.0742	0.0108	0.0584	0.0283	0.0697	0.0116	0.0732	0.0159	0.0593	0.0179	0.0902
	LS	0.0358	0.0724	0.0084	0.0543	0.0271	0.0670	-0.0045	0.0610	0.0168	0.0588	0.0008	0.0908
	chi	0.0007	0.0647	0.0056	0.0464	0.0270	0.0662	0.0037	0.0596	0.0103	0.0567	0.0104	0.0724
	M-chi	0.0344	0.0794	0.0055	0.0593	-0.0068	0.0656	0.0033	0.0607	-0.0159	0.0599	-0.0681	0.0913
	ML	0.0482	0.0726	-0.0190	0.0573	0.0298	0.0658	0.0257	0.0614	0.0190	0.0588	-0.0016	0.0908
10	LL	-0.0229	0.0661	0.0032	0.0501	-0.0315	0.0593	0.0236	0.0577	-0.005	0.0424	0.0040	0.0693
	LS	0.0193	0.0636	0.0016	0.0462	0.0303	0.0626	0.0227	0.0496	0.0045	0.0453	0.0038	0.0649
	chi	0.0059	0.0578	-0.0013	0.0323	0.0028	0.0536	0.0091	0.0344	-0.0028	0.0410	0.0030	0.0548
	M-chi	-0.0165	0.0652	-0.0011	0.0429	0.0258	0.0652	-0.0225	0.0368	0.0035	0.0477	-0.0030	0.0619
	ML	-0.0287	0.0625	-0.0247	0.0335	-0.0349	0.0617	0.0329	0.0360	0.0051	0.0451	-0.0049	0.0562

Table (2): The RMSEs and Biases of the Estimators of the Parameters (θ, β) for Different Number of Groups and Different Methods of Estimation Based on Equi-spaced Grouping

k	method	θ=0.5, β=0.5				θ=0.5, β=1				θ=0.5, β=1.5			
		Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)
3	LL	0.0148	0.3082	0.0103	0.2989	0.0025	0.3054	0.0500	0.2512	0.0110	0.3093	0.0671	0.3552
	LS	0.0212	0.3117	0.0142	0.2848	-0.0045	0.2801	0.0550	0.2835	0.0268	0.3036	0.0332	0.4177
	chi	0.0607	0.2633	0.0341	0.2734	0.0161	0.2889	0.0897	0.2482	0.0596	0.2896	0.0729	0.3664
	M-chi	-0.0225	0.2807	0.0244	0.3221	0.0634	0.2825	0.1930	0.2555	-0.0275	0.3152	0.1178	0.3212
	ML	0.0992	0.2593	-0.1930	0.2575	0.0981	0.2693	0.2339	0.2403	-0.0601	0.2834	0.2811	0.3117
4	LL	0.0001	0.2786	-0.0004	0.2931	-0.0098	0.3008	0.0353	0.2362	-0.0047	0.2898	0.0510	0.3175
	LS	0.0090	0.2583	0.0354	0.2740	0.0165	0.2661	0.0102	0.2510	0.0082	0.2923	0.0304	0.3412
	chi	0.0245	0.2490	0.0175	0.2680	-0.0206	0.2615	0.0383	0.2451	0.0352	0.2677	0.0655	0.3268
	M-chi	-0.0122	0.2731	0.0361	0.2682	0.0258	0.2836	0.0657	0.2542	0.0093	0.2866	0.0763	0.3181
	ML	0.0321	0.2246	0.1153	0.2488	0.0435	0.2573	0.1924	0.2284	0.0994	0.2611	0.1425	0.2798
5	LL	0.0093	0.2606	0.0037	0.2509	-0.0245	0.2566	0.0369	0.2177	0.0198	0.2196	0.1262	0.2950
	LS	-0.0087	0.2436	0.003	0.2515	0.0005	0.2518	0.0261	0.1937	0.017	0.1768	0.1061	0.2916
	chi	0.0193	0.2417	0.0219	0.2311	0.0373	0.2320	0.1628	0.2070	0.0291	0.1787	0.0277	0.3023
	M-chi	0.0097	0.2461	-0.0332	0.2158	-0.0134	0.2376	0.0023	0.2150	0.0005	0.1985	0.1161	0.2876
	ML	0.0092	0.2245	0.0045	0.2072	-0.0234	0.2235	0.0335	0.1859	0.0166	0.1726	0.0462	0.2778
6	LL	0.0093	0.2403	0.0136	0.2435	0.0006	0.2192	0.0104	0.2024	0.0042	0.1586	0.0094	0.2370
	LS	0.0092	0.2167	0.0162	0.2472	0.0109	0.2274	0.0116	0.1857	0.0031	0.1532	0.0155	0.2342
	chi	0.0248	0.2187	0.0238	0.2264	0.0327	0.2190	0.0254	0.1915	0.0042	0.1367	0.0678	0.2531
	M-chi	-0.0126	0.2277	-0.025	0.2188	-0.0270	0.2356	0.0286	0.2049	0.0139	0.1721	0.0208	0.2691
	ML	0.0097	0.2007	0.0905	0.2029	0.0113	0.2133	0.0314	0.1713	-0.0172	0.1347	0.0100	0.2189
7	LL	0.0046	0.2019	0.0017	0.2035	-0.0022	0.2107	0.0024	0.1932	0.0040	0.1228	0.0023	0.2287
	LS	0.0272	0.1908	0.0014	0.2263	0.0001	0.2057	0.0259	0.1984	0.0031	0.1234	0.0252	0.2299
	chi	0.0391	0.1952	0.0215	0.2033	0.0264	0.2077	0.0269	0.1808	0.0200	0.1236	0.0168	0.2259
	M-chi	-0.0145	0.2212	0.0026	0.1691	0.0172	0.2207	0.0296	0.1762	0.0054	0.1275	0.0223	0.2584
	ML	-0.0149	0.1904	-0.0392	0.1453	-0.0271	0.1917	0.0326	0.1702	-0.0213	0.1209	0.0504	0.1915
8	LL	0.0093	0.1929	0.0055	0.1432	0.0001	0.1615	0.0211	0.1748	-0.0029	0.0976	-0.0089	0.2184
	LS	0.0094	0.1791	0.0059	0.2003	0.0008	0.1927	0.0521	0.1656	-0.0023	0.0972	0.0170	0.2290
	chi	-0.0137	0.1754	0.0063	0.1617	0.0098	0.1608	0.0238	0.1601	0.0191	0.1001	0.0170	0.2074
	M-chi	0.0094	0.1775	-0.0279	0.1478	-0.0130	0.1947	0.0590	0.1733	0.0032	0.0964	0.0303	0.2481
	ML	0.0344	0.1645	0.0305	0.1411	0.0306	0.1499	-0.4828	0.1779	-0.0289	0.0917	0.0534	0.1864
9	LL	0.0002	0.1517	0.0111	0.1404	0.0008	0.1435	0.0146	0.1574	0.0003	0.0922	0.011	0.2086
	LS	0.0003	0.1591	0.0122	0.1697	-0.0036	0.1562	0.0136	0.1484	0.0004	0.0923	0.0234	0.2223
	chi	0.0305	0.1631	0.0115	0.1209	0.0182	0.1246	0.0036	0.1437	-0.001	0.0910	0.0248	0.1993
	M-chi	0.0006	0.1581	-0.0216	0.1050	0.0237	0.1304	0.0064	0.1502	0.0291	0.0967	0.0493	0.2138
	ML	-0.0221	0.1497	0.0305	0.0965	-0.0022	0.1032	0.0233	0.1367	0.0252	0.0915	-0.0971	0.1838
10	LL	0.0002	0.1307	0.0106	0.1232	0.0048	0.0910	0.0033	0.1509	-0.0015	0.0906	0.0089	0.1805
	LS	0.0001	0.1235	0.0114	0.1275	0.0053	0.0851	-0.0042	0.1455	-0.0027	0.0892	0.0101	0.1728
	chi	0.0350	0.1338	0.0374	0.1090	0.0127	0.0914	0.0231	0.1370	0.0124	0.0923	0.0124	0.1638
	M-chi	-0.0217	0.1305	0.0106	0.1032	0.0068	0.0877	0.0086	0.1622	0.0028	0.0943	0.0137	0.1907
	ML	0.0003	0.1235	-0.0207	0.0906	-0.0152	0.0838	0.0265	0.1240	-0.0242	0.0839	0.0157	0.1542

Continued Table (2)

k	method	θ=1, β=0.5				θ=1, β=1				θ=1, β=1.5			
		Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)
3	LL	-0.0010	0.2506	0.0006	0.2905	0.0003	0.2784	-0.1032	0.2928	0.0026	0.2904	0.0210	0.2830
	LS	0.0083	0.2703	-0.0017	0.2834	0.0079	0.2896	-0.0889	0.3067	0.0031	0.2967	0.0357	0.2718
	chi	0.0598	0.2495	0.1268	0.2496	0.0699	0.3061	0.1112	0.2562	0.0564	0.2842	-0.0444	0.2464
	M-chi	-0.0091	0.2485	0.0202	0.2700	-0.0095	0.2863	-0.1056	0.2799	-0.0133	0.2831	0.0395	0.2847
	ML	0.0610	0.2473	0.0292	0.2468	0.0085	0.2679	0.1168	0.2521	0.1696	0.2645	0.2128	0.2425
4	LL	-0.0073	0.2506	0.0187	0.2491	0.0031	0.2681	0.0033	0.2628	0.0274	0.2573	0.0055	0.2467
	LS	0.0178	0.2703	-0.0004	0.2438	-0.0033	0.2624	-0.0669	0.2756	-0.0046	0.2679	-0.0265	0.2487
	chi	0.0631	0.2495	-0.0346	0.2308	-0.0089	0.2604	0.1235	0.2604	0.0724	0.2480	0.0247	0.2421
	M-chi	0.0193	0.2485	0.0143	0.2498	0.0074	0.2611	-0.0765	0.2653	0.0330	0.2783	-0.0483	0.2539
	ML	0.0233	0.2473	0.0980	0.2239	0.0366	0.2604	0.1435	0.2471	0.0374	0.2462	0.0595	0.2165
5	LL	-0.0106	0.2447	0.0059	0.2381	-0.0057	0.2137	-0.0014	0.2171	-0.0032	0.2399	-0.0176	0.2085
	LS	0.0129	0.2453	0.0061	0.2367	0.0079	0.1993	0.0645	0.2161	0.0119	0.2380	0.0378	0.2016
	chi	0.0261	0.2487	0.0135	0.2139	0.0087	0.1975	0.0595	0.2133	0.0196	0.2165	-0.0397	0.1995
	M-chi	0.0131	0.2450	0.0066	0.2338	0.0083	0.2056	-0.0243	0.2234	0.0129	0.2389	0.0341	0.2093
	ML	0.0133	0.2443	-0.0389	0.2039	0.0155	0.1940	-0.0614	0.2124	0.0125	0.2160	0.0532	0.1947
6	LL	0.0098	0.2165	0.0106	0.2183	-0.0008	0.1382	0.0035	0.1973	0.0091	0.1926	0.0162	0.2091
	LS	0.0100	0.2160	0.0106	0.2191	-0.0125	0.1372	0.004	0.1966	0.0108	0.2163	0.0121	0.1939
	chi	0.0115	0.2136	0.0277	0.2086	-0.0151	0.1362	0.0205	0.1795	0.0294	0.1654	0.0300	0.1849
	M-chi	0.0109	0.2090	0.0111	0.2065	-0.0135	0.1480	0.0054	0.2102	0.0116	0.1631	-0.0268	0.2088
	ML	0.0104	0.2089	-0.0276	0.2021	0.0398	0.1346	-0.0401	0.1715	-0.0296	0.1622	0.0383	0.1830
7	LL	0.0091	0.1608	0.003	0.2110	0.0145	0.1240	0.0162	0.1288	0.0096	0.1641	0.0068	0.1472
	LS	0.0090	0.1892	0.0026	0.2041	0.0153	0.1221	0.0177	0.1254	0.0088	0.1916	-0.007	0.1616
	chi	0.0096	0.1777	0.0251	0.2056	0.0214	0.1204	0.0229	0.1223	0.0364	0.1297	0.0261	0.1492
	M-chi	0.0052	0.1613	0.0027	0.1890	0.0168	0.1230	0.0162	0.1443	0.0112	0.1324	0.0232	0.1683
	ML	0.0129	0.1592	-0.0446	0.1809	-0.0356	0.1199	-0.042	0.1201	-0.0403	0.1272	0.0263	0.1451
8	LL	0.0043	0.1588	-0.0003	0.1977	0.0034	0.1187	0.0012	0.1269	0.004	0.1285	0.0003	0.1429
	LS	0.0014	0.1556	-0.0003	0.1610	0.0014	0.1152	0.0021	0.1214	0.009	0.1595	0.0057	0.1305
	chi	0.0081	0.1537	0.0202	0.1622	0.0028	0.1175	0.0178	0.1215	0.0312	0.1207	0.0307	0.1368
	M-chi	0.0017	0.1176	-0.0012	0.1376	0.0229	0.1179	0.0159	0.1404	0.0099	0.1142	-0.0317	0.1527
	ML	-0.0069	0.1117	-0.0426	0.1259	-0.0288	0.1140	-0.065	0.1197	-0.0378	0.1029	0.0332	0.1237
9	LL	-0.0010	0.1193	0.0084	0.0982	-0.0017	0.1149	0.0011	0.1256	0.0113	0.1223	0.0035	0.1224
	LS	0.0017	0.1164	0.0087	0.0991	0.0029	0.1128	-0.0001	0.1179	0.0076	0.1284	0.001	0.0930
	chi	0.0191	0.1247	0.0252	0.1006	0.0194	0.1132	0.0175	0.1193	0.0218	0.1100	0.0214	0.1277
	M-chi	0.0097	0.1108	0.0089	0.0983	0.0054	0.1222	-0.0197	0.1384	-0.0185	0.1001	0.0177	0.1247
	ML	0.0103	0.1103	-0.0429	0.0956	0.0038	0.1113	0.0215	0.1177	0.0081	0.0989	-0.4636	0.0922
10	LL	0.0029	0.1082	0.0003	0.0920	0.0025	0.1140	0.0017	0.0957	0.0059	0.1184	0.0031	0.0877
	LS	0.0033	0.1161	-0.0001	0.0970	0.0037	0.1074	0.0226	0.0961	0.0089	0.1133	-0.0039	0.0852
	chi	0.0108	0.1184	0.0165	0.0990	0.0044	0.1047	0.0285	0.0991	0.0071	0.0957	0.0076	0.0850
	M-chi	-0.0091	0.1081	-0.0164	0.0983	0.0046	0.1153	0.0025	0.1035	-0.0146	0.0967	0.0051	0.0874
	ML	0.0313	0.1036	0.0109	0.0892	0.0128	0.1026	-0.0392	0.0955	0.0171	0.0922	-0.0208	0.0855

Continued Table (2)

k	method	θ=1.5, β=0.5				θ=1.5, β=1				θ=1.5, β=1.5			
		Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)	Biases(θ)	RMSE(θ)	Biases(β)	RMSE(β)
3	LL	0.0316	0.3224	-0.0040	0.2898	-0.0022	0.2778	0.0123	0.2878	0.0026	0.2890	0.0142	0.2534
	LS	0.0920	0.3032	0.0113	0.2816	0.0032	0.2678	0.0182	0.2919	0.0231	0.3650	-0.0302	0.3055
	chi	0.1562	0.2922	0.0113	0.2752	0.1771	0.2867	0.0184	0.2514	0.1958	0.3043	0.2007	0.2272
	M-chi	-0.0930	0.3037	-0.0144	0.2858	0.0078	0.2610	-0.0168	0.2504	0.0252	0.2685	0.0351	0.2527
	ML	0.1835	0.2877	0.1340	0.2744	0.008	0.2604	0.0731	0.2502	0.2394	0.2662	0.2478	0.2695
4	LL	0.0315	0.3025	-0.0062	0.2872	0.0022	0.2331	0.0051	0.2622	0.0083	0.2665	0.008	0.2443
	LS	0.0376	0.2970	0.0081	0.2301	0.0297	0.2345	0.0511	0.2447	0.038	0.2645	0.0364	0.2321
	chi	0.0688	0.2923	0.0195	0.2629	0.0428	0.2353	0.0636	0.2418	0.0504	0.2641	0.0419	0.2217
	M-chi	-0.0387	0.2923	-0.0136	0.2817	-0.017	0.2329	-0.0211	0.2426	0.0373	0.2603	-0.0179	0.2359
	ML	-0.0603	0.2720	0.1078	0.2512	0.0317	0.2313	0.0677	0.2312	0.0454	0.2601	0.049	0.2259
5	LL	0.0105	0.2875	-0.0053	0.2841	0.0296	0.2326	0.0081	0.2248	-0.0077	0.2646	-0.0062	0.2416
	LS	0.0125	0.2697	-0.007	0.2286	0.0304	0.2319	0.0061	0.2049	0.0375	0.2614	0.0364	0.2238
	chi	0.0272	0.2710	0.0099	0.2618	0.0514	0.2349	0.0201	0.2057	0.0564	0.2612	0.0327	0.2144
	M-chi	-0.0172	0.2704	0.0056	0.2506	0.0298	0.2317	0.0062	0.2292	0.0477	0.2628	0.037	0.2248
	ML	0.0133	0.2688	-0.0425	0.2456	0.0311	0.2295	-0.0279	0.2048	0.0353	0.2563	0.0533	0.2097
6	LL	0.0116	0.2434	-0.0014	0.2553	0.0127	0.1937	0.0119	0.2199	-0.0240	0.2105	0.0121	0.1904
	LS	0.0127	0.2424	0.0111	0.2122	0.0154	0.1889	0.0111	0.2190	0.0246	0.1999	0.0147	0.1834
	chi	0.0136	0.2457	0.0329	0.2280	0.0351	0.1861	0.0315	0.2200	0.0486	0.2013	0.0265	0.1761
	M-chi	0.0122	0.2461	0.0111	0.2289	0.0138	0.1910	0.0111	0.1948	0.0306	0.2039	-0.0216	0.2042
	ML	0.0217	0.2407	-0.0378	0.2257	-0.0408	0.1844	0.0386	0.1846	0.0265	0.1966	0.0165	0.1790
7	LL	0.0181	0.1758	0.0060	0.2439	0.0138	0.1886	0.0001	0.1559	0.0054	0.1923	-0.0082	0.1837
	LS	0.0170	0.1749	0.0052	0.2027	0.0098	0.1868	0.0002	0.1547	0.0038	0.1904	0.0116	0.1757
	chi	0.0187	0.1732	0.0285	0.2036	0.0287	0.1859	0.0209	0.1547	0.0218	0.1871	0.0128	0.1746
	M-chi	0.0172	0.1792	0.0062	0.2059	-0.0151	0.1883	-0.0520	0.1718	0.0062	0.1922	0.0181	0.1954
	ML	0.0232	0.1713	-0.0402	0.2017	0.0108	0.1822	-0.0011	0.1542	0.0296	0.1857	0.0277	0.1728
8	LL	0.0078	0.1587	0.0037	0.2027	0.0228	0.1790	0.0038	0.1176	0.0248	0.1873	0.0031	0.1755
	LS	0.0080	0.1479	0.0034	0.1452	0.0090	0.1758	0.0039	0.1156	0.0266	0.1869	0.0040	0.1610
	chi	0.0139	0.1437	-0.0520	0.1471	0.0309	0.1753	0.0258	0.1135	0.054	0.1915	-0.0237	0.1573
	M-chi	0.0094	0.1506	0.0037	0.1532	0.0251	0.1803	0.0050	0.1365	-0.0356	0.1914	0.0065	0.1928
	ML	0.0101	0.1445	0.0304	0.1439	-0.0320	0.1748	-0.0514	0.1124	0.0265	0.1839	0.0435	0.1541
9	LL	0.0167	0.1414	0.0059	0.1464	0.0054	0.1717	0.0089	0.0961	0.0086	0.1783	0.0190	0.1746
	LS	0.0214	0.1405	0.0056	0.1271	0.0227	0.1754	0.0096	0.0972	0.0088	0.1781	0.0014	0.1684
	chi	0.0211	0.1394	0.0281	0.1068	0.0271	0.1709	0.0535	0.1048	0.0435	0.1783	0.0173	0.1561
	M-chi	0.0225	0.1449	0.0056	0.1145	0.0081	0.1763	-0.0434	0.1074	0.0093	0.1914	-0.0135	0.1911
	ML	0.0281	0.1381	-0.0490	0.1031	0.0089	0.1683	0.0100	0.0907	-0.0647	0.1731	0.0336	0.1537
10	LL	-0.0080	0.1065	0.0069	0.0960	0.0098	0.1505	0.0093	0.0939	0.0103	0.1510	0.0049	0.1687
	LS	0.0229	0.0993	0.0065	0.0967	0.0032	0.1635	0.0042	0.0972	0.0313	0.1612	0.0038	0.1607
	chi	0.0247	0.0989	0.0415	0.1014	0.0222	0.1132	-0.0773	0.0925	0.0279	0.1153	0.0078	0.1544
	M-chi	0.0270	0.1146	0.0075	0.0972	0.0189	0.1177	0.0388	0.1211	0.0195	0.1446	0.0128	0.1899
	ML	0.0623	0.0911	-0.0519	0.0909	0.0414	0.1037	0.0106	0.0784	0.0317	0.1136	-0.0537	0.1518

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