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Generalized Alpha-Power Transformation Family of Distributions with an Application to Exponential Model

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We propose a new generated family of distributions that is known as the *generalized alpha-power transformation-G (GAPT-G) family* of distributions. The new class of family can be more flexible since the density shapes are left skewed, symmetrical and reversed-J. Some special models derived and discussed. Several of its important properties are derived. The maximum likelihood equations are derived for GAPT-G family parameters. The importance and flexibility of the derived models is assessed using one real dataset.

Keywords: Alpha-Power Transformation Family of Distributions, Moments, Maximum Likelihood.

1. INTRODUCTION

The most popular traditional distributions often do not characterize and do not predict most of the interesting data sets. Generated family of continuous distributions are a new improvement for creating and extending the usual classical distributions. The newly generated families have been broadly studied in several areas as well as yield more flexibility in applications. Eugene et al. [17] studied the beta-family of distributions. Zografos and Balakrishnan [26] suggested a generated family using gamma distribution. Kumaraswamy generalized family provided by Cordeiro and de Castro (2011). Weibull-G by Bourguignon et al. [11], exponentiated half-logistic by Cordeiro et al. [14], the type I half-logistic by Cordeiro et al. [12], The new Kumaraswamy Kumaraswamy family of generalized distributions with application has been presented by Mahmoud et al. [25], Garhy generated family of distributions introduced by Elgarhy et al. [16], the Kumaraswamy Weibull by Hassan and Elgarhy [20], Type II half logistic-G by Hassan et al. [21], exponentiated extended by Elgarhy et al. [15]. Muth-G by Almarashi and Elgarhy [4], odd Frèchet-G by Haq and Elgarhy [19], a new power Topp-Leone-G by Bantan et al. [10], truncated Cauchy power-G studied by Aldahlan et al. [2], Type II exponentiated half logistic-G in Al-Mofleh et al. [7], transmuted odd Fréchet-G by Badr et al. [8], exponentiated M-G by Bantan et al. [9], exponentiated truncated inverse Weibull-G in Almarashi et al. [5], Topp-Leone odd Fréchet-G by AlMarzouki et al. [6], Sine Topp-Leone-G by Al-Babtain et al. [1], odd generalized N-H by Ahmad et al. [3], generalized truncated Fréchet-G by ZeinEldin et al. [27].

Recently, Mahdavi and Kundu [24] proposed a new method for introducing new statistical distributions defined by the following expression

$$F(x)=\frac{\alpha^{G(x)}-1}{\alpha-1}\quad x\in R,\quad \alpha>0,\quad \alpha\neq 1 \tag{1}$$

Using (5), Mahdavi and Kundu [24] introduced the alpha power exponential (APE) distribution by considering *F(x)* be the CDF of exponential distribution. Later on, Nassar et al. [22] studied the alpha power transformed Weibull (APTW) distribution by using *F(z)* as the cdf of the Weibull random variable.

In this article, we propose a new method for generating lifetime distributions. We call this new method is generalized alpha power transformation (GAPT) method. The new family defined by the following cdf

$$F(x)=\frac{\alpha^{G(x)^{\beta}}-1}{\alpha-1},\quad \alpha,\beta>0,\quad \alpha\neq 1,\quad x\in R \tag{2}$$

Clearly, when $\beta=1$, then (2) reduces to (1). The pdf corresponding to (2) is given by

$$f(x)=\frac{\beta\log(\alpha)}{\alpha-1}g(x)\,G(x)^{\beta-1}\,\alpha^{G(x)^{\beta}} \tag{3}$$

The survival, hazard rate, reverse hazard and cumulative hazard rate functions of APT family is given respectively by:

$$\bar{F}(x)=1-F(x)=1-\frac{\alpha^{G(x)^{\beta}}-1}{\alpha-1}=\frac{\alpha-\alpha^{G(x)^{\beta}}}{\alpha-1}$$

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$$h(x) = \frac{f(x)}{R(x)} = \frac{\beta \log(\alpha)\, g(x)\, G(x)^{\beta-1}\, \alpha^{G(x)^{\beta}}}{\alpha - \alpha^{G(x)^{\beta}}}$$

$$\tau(x) = \frac{f(x)}{F(x)} = \frac{\beta \log(\alpha)\, g(x)\, G(x)^{\beta-1}\, \alpha^{G(x)^{\beta}}}{\alpha^{G(x)^{\beta}} - 1}$$

and

$$H(x) = -\ln R(x) = -\ln\left(\frac{\alpha - \alpha^{G(x)^{\beta}}}{\alpha - 1}\right)$$

The rest of this article is organized as follows: Section 2, provides the new submodels of the family. Section 3, contains some structural properties of the family. In Section 4, estimation of the parameters of the family is implemented through maximum likelihood method. Simulation study is carried out to estimate the model parameters of distribution in Section 5. An illustrative purpose on the basis of real data is investigated, in Section 6. Finally, Section 7 offer concluding remarks.

2. SUBMODELS OF THE NEW FAMILY

In this section, we define and describe four special models of the GAPT generated family namely, GAP-uniform, GAP-Frechet, GAP -Rayleigh and GAP-exponential.

2.1. GAP-Uniform Distribution

The pdf of GAP- uniform GAPU is derived from (3), by taking $g(x, \theta) = 1/\theta$, $0 < x < \theta$ and $G(x, \theta) = x/\theta$ as the following

$$f(x) = \frac{\beta \log(\alpha)}{\theta^{\beta}(\alpha - 1)} \alpha^{\theta^{-\beta}} \quad 0 < x < \theta, \quad \alpha, \theta, \beta > 0, \quad \alpha \neq 1$$

The corresponding cdf takes the following form

$$F(x) = \frac{\alpha^{\theta^{-\beta}} - 1}{\alpha - 1}$$

Moreover, the survival and the hazard rate functions are given, respectively, as follows

$$\bar{F}(x) = \frac{\alpha - \alpha^{\theta^{-\beta}}}{\alpha - 1}$$

and

$$h(x) = \frac{\beta \log(\alpha)}{\theta^{\beta}\left(\alpha^{\theta^{-\beta}} - 1\right)} \alpha^{\theta^{-\beta}}$$

The plots of pdf and hazard rate function for the GAPU are showed in Figures 1 and 2 respectively.

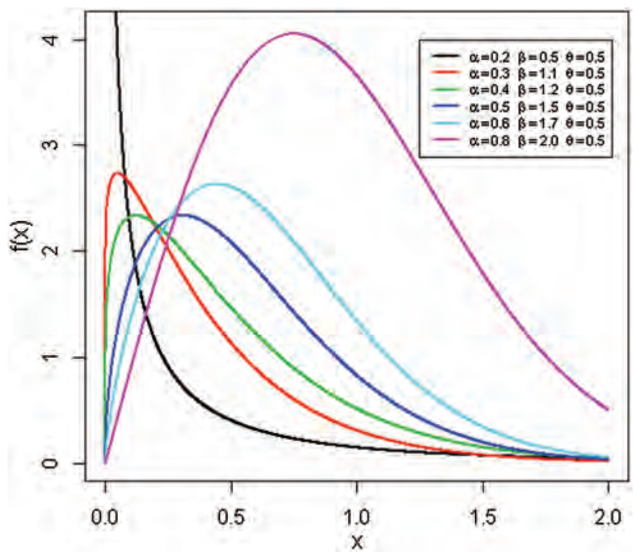

Fig. 1. Pdf of GAPU distribution.

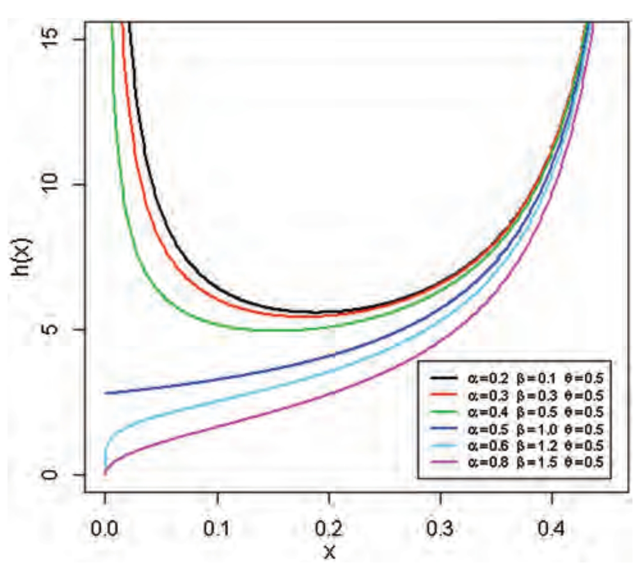

Fig. 2. Hazard rate function of GAPU distribution.

2.2. GAP-Frechet Distribution

Let us consider the Frechet distribution with distribution functions given by $G(x) = e^{-(\mu/x)^{\delta}}$ Then GAPF distribution has the following cdf, pdf, survival, and hazard rate functions.

$$F(x) = \frac{\alpha^{\left(1-e^{-(\mu/x)^{\delta}}\right)^{\beta}} - 1}{\alpha - 1}, \quad x, \alpha, \beta, \mu, \delta > 0, \quad \alpha \neq 1$$

$$f(x) = \frac{\delta\mu^{\delta}\beta \log(\alpha)}{\alpha - 1} x^{-\delta-1} e^{-(\mu/x)^{\delta}} \left(1 - e^{-(\mu/x)^{\delta}}\right)^{\beta-1} \times \alpha^{\left(1-e^{-(\mu/x)^{\delta}}\right)^{\beta}}$$

$$\bar{F}(x) = \frac{\alpha - \alpha^{\left(1-e^{-(\mu/x)^{\delta}}\right)^{\beta}}}{\alpha - 1}$$

and

$$h(x) = \frac{\delta\mu^{\delta}\beta \log(\alpha)}{\alpha - \alpha^{\left(1-e^{-(\mu/x)^{\delta}}\right)^{\beta}}} x^{-\delta-1} e^{-(\mu/x)^{\delta}} \left(1 - e^{-(\mu/x)^{\delta}}\right)^{\beta-1} \times \alpha^{\left(1-e^{-(\mu/x)^{\delta}}\right)^{\beta}}$$

The plots of pdf and hazard rate function for the GAPF are displayed in Figures 3 and 4 respectively.

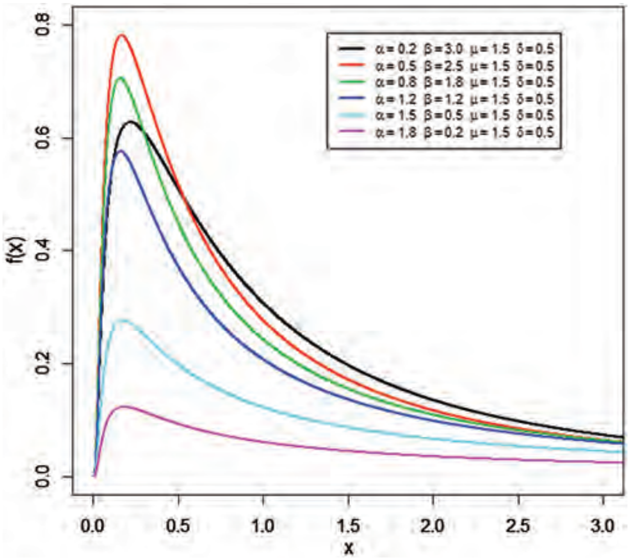

Fig. 3. Pdf of GAPF distribution.

2.3. GAP-Rayleigh Distribution

The cdf and pdf of GAP-Rayleigh (GAPR) distribution are derived from (3) taking $G(x,\lambda) = 1 - e^{-\lambda x^2}$ as the following

$$F(x) = \frac{\alpha^{\left(1-e^{-\lambda x^2}\right)^{\beta}} - 1}{\alpha - 1}, \quad x, \beta, \alpha, \lambda > 0, \quad \alpha \neq 1$$

and,

$$f(x) = \frac{2\beta\lambda \log(\alpha)\, x}{\alpha - 1} e^{-\lambda x^2} \left(1 - e^{-\lambda x^2}\right)^{\beta-1} \alpha^{\left(1-e^{-\lambda x^2}\right)^{\beta}}$$

Further, the survival and hazard rate functions are as follows

$$\bar{F}(x) = \frac{\alpha - \alpha^{\left(1-e^{-\lambda x^2}\right)^{\beta}}}{\alpha - 1}$$

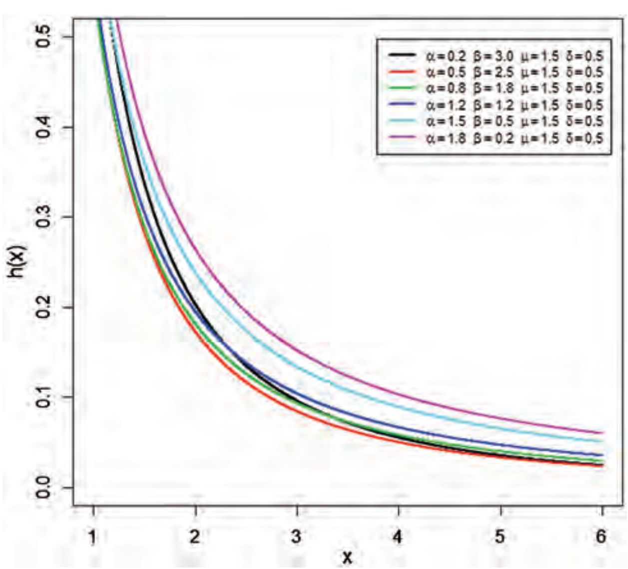

Fig. 4. Hazard rate function of GAPF distribution.

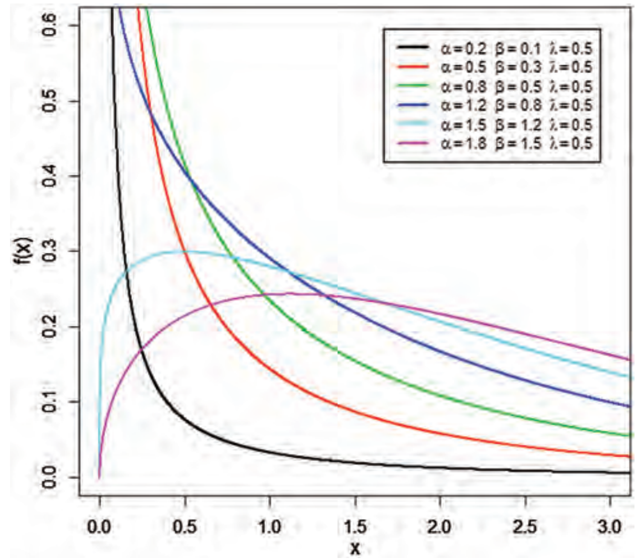

Fig. 5. Pdf of GAPR distribution.

and

$$h(x) = \frac{2\beta\lambda \log(\alpha)\, x}{\alpha - \alpha^{\left(1-e^{-\lambda x^2}\right)^{\beta}}} e^{-\lambda x^2} \left(1 - e^{-\lambda x^2}\right)^{\beta-1} \alpha^{\left(1-e^{-\lambda x^2}\right)^{\beta}}$$

The plots of pdf and hazard rate function for the GAPR are presented in Figures 5 and 6 respectively.

2.4. GAP-Exponential Distribution

The cdf and pdf of GAP-exponential (GAPE) distribution are derived from (2) and (3) taking $G(x,\lambda) = 1 - e^{-\lambda x}$ as the following

$$F(x) = \frac{\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}} - 1}{\alpha - 1}, \quad x, \alpha, \beta, \lambda > 0, \quad \alpha \neq 1$$

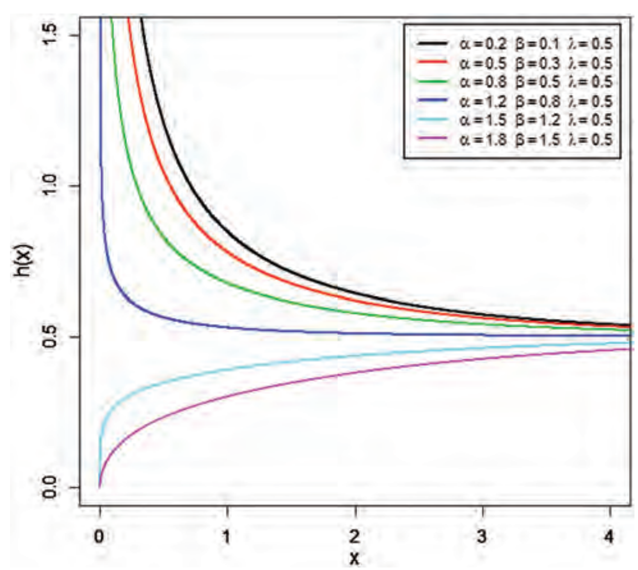

Fig. 6. Hazard rate function of GAPR distribution.

and,

$$f(x) = \frac{\beta\lambda\log(\alpha)}{\alpha-1} e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\beta-1}\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}$$

Further, the survival and hazard rate functions are as follows

$$\bar{F}(x) = \frac{\alpha - \alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}}{\alpha-1}$$

and

$$h(x) = \frac{\beta\lambda\log(\alpha)}{\alpha - \alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}} e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\beta-1}\alpha^{\left(1-e^{-\lambda x}\right)^{\beta}}$$

The plots of pdf and hazard rate function for the GAPE are presented in Figures 5 and 6 respectively.

3. BASIC MATHEMATICAL PROPERTIES

This section considers the basic mathematical properties of GAPT-G family of distributions.

3.1. Quantile Function

Let X denotes a random variable has the pdf (2), the quantile function; say $Q(u)$ of X is given by

$$Q(u) = G^{-1}\left(\frac{\ln(u\alpha - u + 1)}{\ln\alpha}\right)^{1/\beta} \tag{4}$$

where, u is a uniform distribution on the interval $(0, 1)$ and $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. In particular, the median is obtained by putting $u = 0.50$ in (4).

3.2. Useful Expansion

In this section some representations of the cdf and pdf for the GAPT family of distributions will be presented. The mathematical relation given below will be useful in this section.

By using the following expansion

$$\alpha^{\vartheta} = \sum_{i=0}^{\infty} \frac{(\log\alpha)^i}{i!}\vartheta^i \tag{5}$$

Then, by substituting (5) in (3), the distribution function of GAPT-G distributions becomes

$$f(x) = \sum_{i=0}^{\infty} A_i g(x) G(x)^{\beta(i+1)-1} \tag{6}$$

where, $A_i = (\beta(\log\alpha)^{i+1})/(i!(\alpha-1))$.

Further, an expansion for $[F(x)]^h$ is derived, for h is integer, again, the binomial expansion is worked out.

It is well-known that, if $\beta > 0$ and $|z| < 1$ the generalized binomial theorem is written as follows

$$(1-z)^{\beta-1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta-1}{i} z^i \tag{7}$$

Then,

$$[F(x)]^h = \sum_{k=0}^{\infty} w_k G(x)^{\beta k} \tag{8}$$

where,

$$w_k = \sum_{j=0}^{h} (-1)^h (j)^k \binom{h}{j} \frac{(\log\alpha)^k}{k!(1-\alpha)^h}$$

3.3. The Probability Weighted Moments

For a random variable X the probability-weighted moments (PWMs), denoted by $\tau_{r,s}$, can be calculated through the following relation

$$\tau_{r,s} = E\left(X^r F(x)^s\right) = \int_{-\infty}^{\infty} x^r f(x) F(x)^s \, dx \tag{9}$$

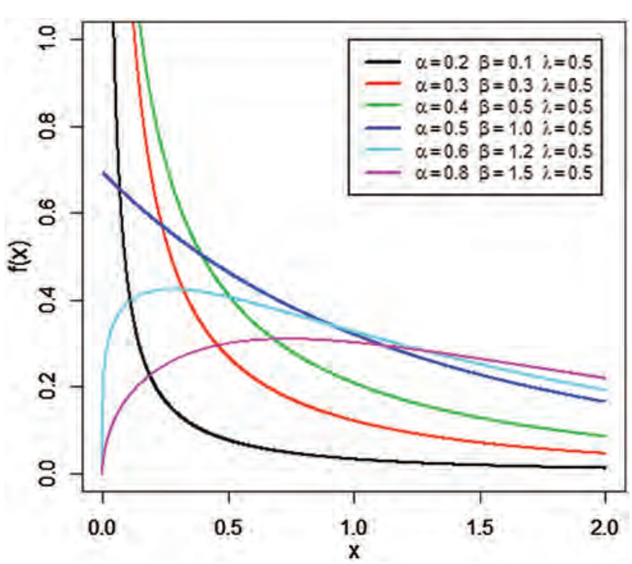

Fig. 7. Pdf of GAPE distribution.

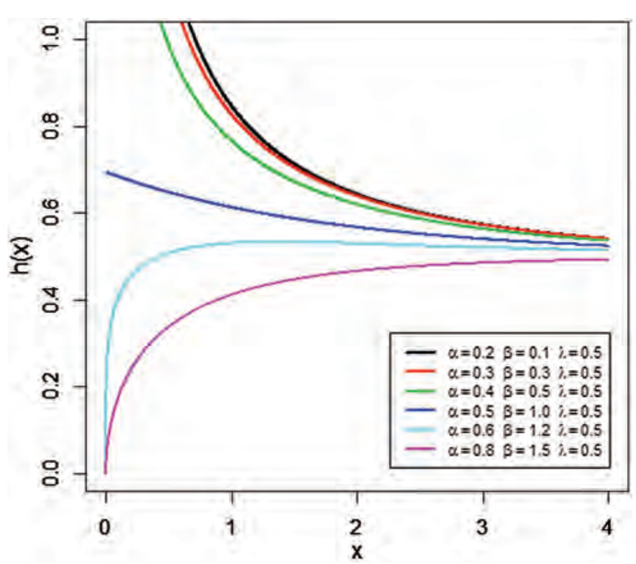

Fig. 8. Hazard rate function of GAPE distribution.

The PWMs of GAPT -G is obtained by substituting (6) and (8) into (9), and replacing h with s, as follows

$$\tau_{r,s} = \sum_{i,k=0}^{\infty} A_i w_k \int_{-\infty}^{\infty} x^r g(x) G(x)^{\beta(i+1)-1} dx$$

Then,

$$\tau_{r,s} = \sum_{i,k=0}^{\infty} A_i w_k \tau_{r,\beta(i+k+1)-1}$$

3.4. Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the rth moment for the GAPT-G family. If X has the pdf (10), then rth moment is obtained as follows

$$\grave{\mu}_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx$$

$$= \sum_{i=0}^{\infty} A_i \int_{-\infty}^{\infty} x^r g(x) G(x)^{\beta(i+1)-1} dx$$

Then,

$$\grave{\mu}_r = \sum_{i=0}^{\infty} A_i \tau_{r,\beta(i+1)-1}$$

3.5. Moment Generating Function

For a random variable X it is known that, the moment generating function is defined as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \grave{\mu}_r = \sum_{i,r=0}^{\infty} \frac{t^r}{r!} A_i \tau_{r,\beta(i+1)-1}.$$

3.6. The Mean Deviation

In statistics, mean deviation about the mean and mean deviation about the median measure the amount of scattering in a population. For random variable X with pdf $f(x)$, cdf $F(x)$, the mean deviation about the mean and mean deviation about the median, are defined by

$$\delta_1 = 2\mu F(\mu) - 2T(\mu) \quad \text{and} \quad \delta_2 = \mu - 2T(M)$$

where,

$$\mu = E(X), \quad M = \text{Median}(X)$$

and

$$T(q) \int_{-\infty}^{q} x f(x) dx$$

which is the first incomplete moment.

3.7. Order Statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ the corresponding ordered random sample from a population of size n. According to [18], the pdf of the kth order statistic, is defined as

$$fx_{(k)}(x) = \frac{f(x)}{B_{(k,n-k+1)}} \sum_{v=0}^{n-k} (-1)^v \binom{n-k}{v} F(x)^{v+k-1} \quad (10)$$

$B(.,.)$ stands for beta function. The pdf of the kth order statistic for GAPT -G family is derived by substituting (6) and (8) in (10), replacing h with $\nu + k - 1$

$$fx_{(k)}(x) = \frac{g(x)}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{i,k=0}^{\infty} W^* G(x)^{\beta(i+1)-1} \quad (11)$$

where, $W^* = (-1)^v \binom{n-k}{v} A_i w_k$, $g(\cdot)$ and $G(\cdot)$ are the pdf and cdf of the GAPT-G family, respectively.

Further, the rth moment of kth order statistics for GAPT-G is defined family by:

$$E\left(X^r{}_{(k)}\right) = \int_{-\infty}^{\infty} x^r{}_{(k)} f\left(x_{(k)}\right) dx_{(k)} \quad (12)$$

By substituting (11) in (12), leads to

$$E\left(X^r{}_{(k)}\right) = \frac{1}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{i,k=0}^{\infty} W^* \times \int_{-\infty}^{\infty} x^r g(x) G(x)^{\beta(i+1)-1} dx$$

Then,

$$E\left(X^r{}_{(k)}\right) = \frac{1}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{i,k=0}^{\infty} W^* \tau_{r,\beta(i+1)-1}$$

4. MAXIMUM LIKELIHOOD ESTIMATORS

In this section, we derive the maximum likelihood estimators (MLE's) of a three-parameter GAP family of distributions.

Let $X_1, X_2, X_3, \ldots X_k$ be a random sample from GAP-G of distributions, then the log likelihood function can be written as:

$$\ln L = n \ln \beta + n \ln(\log \alpha) - n \ln(\alpha - 1) + \sum_{i=1}^{n} \ln g(x_i, \varepsilon) + (\beta - 1) \sum_{i=1}^{n} \ln G(x_i, \varepsilon) + \ln \alpha \sum_{i=1}^{n} G^{\beta}(x_i, \varepsilon) \quad (13)$$

Obtaining the partial derivatives of (13), we get

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha \log \alpha} - \frac{n}{\alpha - 1} + \frac{1}{\alpha} \sum_{i=1}^{n} G^{\beta}(x_i, \varepsilon) \quad (14)$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} \ln G(x_i, \varepsilon) + \ln \alpha \sum_{i=1}^{n} G^{\beta}(x_i, \varepsilon) \ln G(x_i, \varepsilon) \quad (15)$$

and

$$\frac{\partial \ln L}{\partial \varepsilon_k} = \sum_{i=1}^{n} \frac{(\partial/\partial\varepsilon_k) g(x_i, \varepsilon)}{g(x_i, \varepsilon)} + (\beta - 1) \sum_{i=1}^{n} \frac{(\partial/\partial\varepsilon_k) G(x_i, \varepsilon)}{G(x_i, \varepsilon)} + \ln\alpha \sum_{i=1}^{n} \beta G^{\beta-1}(x_i, \varepsilon) \frac{\partial}{\partial\varepsilon_k} G(x_i, \varepsilon) \quad (16)$$

The expressions provided from (14)–(16) are not closed forms. Therefore, we use the iterating method such as newton Raphson method to estimate the model parameters numerically. In the present paper, the Mathematica (9) is used to have numerical estimates of the model parameters.

5. SIMULATION STUDY

It is very difficult to compare the theoretical performances of the ML method of estimation (MLE) for the GAPE distribution. Therefore, simulation is needed to compare the performances of the different methods of estimation mainly with respect to their biases, mean square errors and Variances (MLEs) for different sample sizes. A numerical study is performed using Mathematica 9 software. Different sample sizes are considered through the experiments at size $n = 50$, 100 and 150. In addition, the different values of parameters α, β and λ.

The experiment will be repeated 3000 times. In each experiment, the estimates of the parameters will be obtained by maximum likelihood methods of estimation. The means, MSEs and biases for the different estimators will be reported from these experiments.

6. REAL LIFE APPLICATION

In this section, we analyze one data set to show the performance of the proposed model. The uncensored data set corresponding to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875–1951, published by Maguire et al. [25]. The sorted data are given as follows:

1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120, 123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338, 345, 348, 354, 361, 364, 369, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613, 1630.

The minimum, maximum, first quartile, median, third quartile of the above data set are: 1, 1630, 54, 145, 312, respectively. It clearly indicates that the data are skewed.

Now we would like to fit the proposed GAPE (α, β, λ) to the above data set. For comparison purposes, we have fitted four other two-parameter models to the same data set. In all these four distributions,

They respective pdfs are as follows:

• Alpha power exponential distribution with PDF

$$f(x) = \frac{\lambda \log(\alpha)}{\alpha - 1} e^{-\lambda x} \alpha^{\left(1 - e^{-\lambda x}\right)}, \quad x > 0$$

• Weibull distribution with PDF

$$f_{GA}(x; \alpha, \lambda) = \alpha\lambda (\lambda x)^{\alpha-1} e^{-(\lambda x)^{\alpha}}, \quad x > 0$$

• Exponentiated exponential distribution (EE) with PDF

$$f_{EE}(x; \alpha, \lambda) = \alpha\lambda \left(1 - e^{-\lambda x}\right)^{\alpha-1} e^{-\lambda x}, \quad x > 0$$

see for example Gupta et al. (1998) or Gupta and Kundu (1999).

• Weighted exponential distribution (WE) proposed by Gupta and Kundu (2009) with the PDF

$$f(x) = \frac{\alpha + 1}{\alpha} \lambda e^{-\lambda x} \alpha^{\left(1 - e^{-\alpha\lambda x}\right)}, \quad x > 0$$

Table I. The parameter estimation from GAPE distribution using MLE.

N	Par	Init	MLE	Bais	MSE	Init	MLE	Bais	MSE
50	Α	0.3	0.6447	0.3447	2.0284	0.5	0.4009	−0.0991	1.0057
	Β	0.9	0.9031	0.0031	0.0284	1.2	1.2732	0.0732	0.0667
	Λ	0.5	0.4621	−0.0379	0.0516	0.5	0.3715	−0.1285	0.0460
100	Α	0.3	0.3994	0.0994	0.6833	0.5	0.2555	−0.2445	0.3736
	Β	0.9	0.9282	0.0282	0.0207	1.2	1.2838	0.0838	0.0337
	Λ	0.5	0.4457	−0.0543	0.0300	0.5	0.3482	−0.1519	0.0449
150	Α	0.3	0.2686	−0.0314	0.1459	0.5	0.2198	−0.2802	0.1668
	Β	0.9	0.9233	0.0233	0.0122	1.2	1.2628	0.0627	0.0200
	Λ	0.5	0.4198	−0.0802	0.0291	0.5	0.3487	−0.1513	0.0402
50	Α	0.5	0.2170	−0.2830	0.5825	0.5	0.1441	−0.3559	0.2903
	Β	1.5	1.5834	0.0834	0.0771	2.0	1.7325	−0.2675	0.2184
	Λ	0.5	0.3333	−0.1667	0.0437	0.5	0.2793	−0.2207	0.0627
100	Α	1.2	0.2108	−0.2892	0.1834	0.5	0.1051	−0.3949	0.2033
	Β	0.4	1.5633	0.0633	0.0372	2.0	1.7994	−0.2006	0.1603
	Λ	0.5	0.3584	−0.1416	0.0375	0.5	0.2815	−0.2185	0.0621
150	Α	1.2	0.1654	−0.3346	0.1618	0.5	0.0903	−0.4097	0.1875
	Β	0.4	1.5201	0.0201	0.0218	2.0	1.7942	−0.2058	0.1223
	Λ	0.5	0.3318	−0.1682	0.0315	0.5	0.2823	−0.2177	0.0570

Table II. The maximum likelihood estimates and Kolmogorov-Smirnov statistics and *p*-values for coal-mining data.

The model	MLEs of parameters	Log-liklihood	K–S statistic	*p*-value
GAPE	$\hat{\alpha} = 0.016$, $\hat{\lambda} = 1.44 \times 10^{-3}$ $\hat{\beta} = 1.054$	−700.568	0.0595	0.835
Weibull	$\hat{\alpha} = 0.8848$, $\hat{\lambda} = 0.0046$	−701.7724	0.0784	0.5135
EE	$\hat{\alpha} = 0.8605$, $\hat{\lambda} = 0.0039$	−702.5524	0.0830	0.4402
WE	$\hat{\alpha} = 35.2748$, $\hat{\lambda} = 0.0045$	−705.1641	0.0836	0.4313
APE	$\hat{\alpha} = 0.2807$, $\hat{\lambda} = 0.0030$	−701.2132	0.0742	0.5852

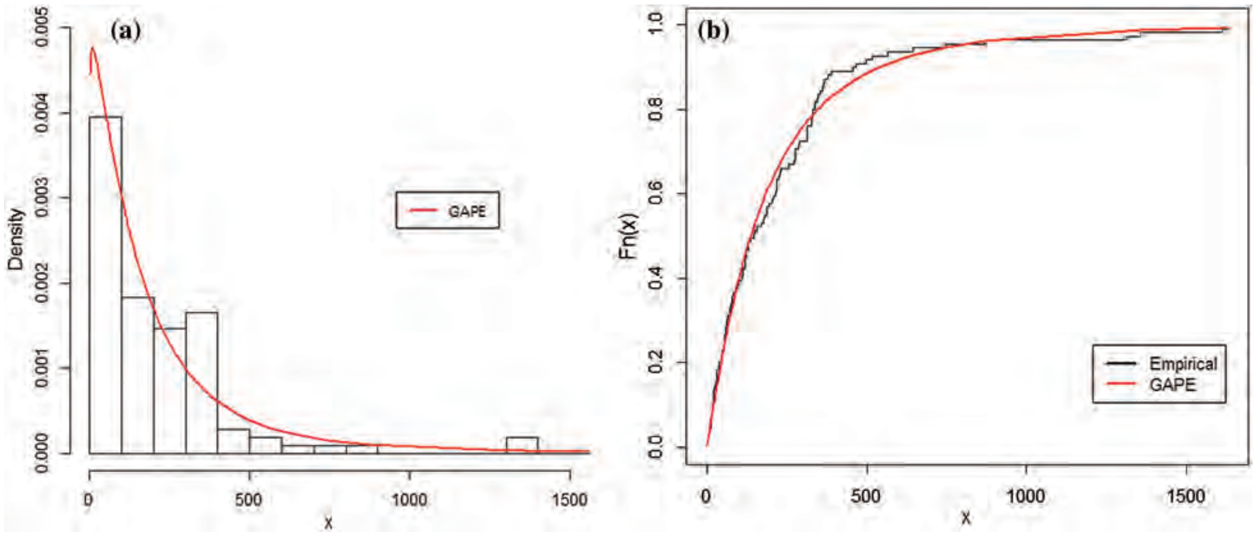

Fig. 9. (a) The histogram and the fitted APE distribution. (b) The fitted APE cdf and empirical cdf for coal-mining data.

We have computed the MLEs of α, β and λ and the associated log-likelihood values in all these cases. We have also obtained the Kolmogorov-Smirnov (K–S) distance between the empirical cumulative distribution function and the fitted distribution function in each case and the associated p value. The results are reported in Table II.

It is clear from the Table II that based on the log-likelihood v alue and also based on the K-S statistic, the proposed APE model provides a better fit than Weibull, EE, WE and APE models to this specific data set. The relative histogram and the fitted GAPE distribution are plotted in Figure 9(a). In order to assess if the model is appropriate, the plots of the fitted APE cdf and empirical cdf are displayed in Figure 9(b). Although, it is not guaranteed that the proposed model always provides a better fit than the other models, but at least in certain cases it definitely can provide a better fit. Therefore, the GAPE model can be used as a possible alternative to the well known, Weibull or EE and APE models.

7. CONCLUSION

In this article, we have introduced a new method to generate new statistical distributions called generalized Alpha Beta power transformation of distributions has been studied in detail. The proposed family have several desirable statistical properties. A real data set has been analyzed and the analytical measures of the GAPE distribution have been compared with other well-known statistical distributions. Although, it is not guaranteed that the proposed method will always provide best fit. But, at least in certain situations, the subject model may work better than the other well-known aging distributions. The work is in progress and it will be reported later.

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