

## The Exponentiated Generalized Alpha Power Exponential Distribution: Properties and Applications

ElSayed A. ElSherpieny<sup>1</sup>, Ehab M. Almetwally<sup>2,3,\*</sup>



\* Corresponding Author

1. Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt, [Elsayed.Elsherpiny@cu.edu.eg](mailto:Elsayed.Elsherpiny@cu.edu.eg)
2. Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt, [ehab.almetwally@pg.cu.edu.eg](mailto:ehab.almetwally@pg.cu.edu.eg)
3. Faculty of Business Administration, Delta University of Science and Technology, Gamasa 11152, Egypt, [ehab.metwaly@deltauniv.edu.eg](mailto:ehab.metwaly@deltauniv.edu.eg)

### Abstract

In this paper, we first introduce the exponentiated generalized alpha power family of distributions to extend several other distributions. We use the new family and develop a new distribution, called the exponentiated generalized alpha power exponential (EGAPEX) distribution. The proposed EGAPEX distribution provides greater flexibility in modeling data from a practical point of view. The new model includes the exponential; alpha power exponential; alpha power generalized exponential, generalized exponential, standardized generalized exponential and exponentiated generalized exponential distributions as a special cases. This distribution exhibits four hazard rate shapes such as L shaped, increasing, decreasing, and upside-down bathtub. Some statistical properties of the EGAPEX distribution are obtained. The model parameters are obtained by maximum likelihood, maximum product spacing, and Bayesian estimation methods. In addition, we have obtained approximate confidence intervals, two bootstrap confidence intervals, and Bayes credible intervals. A Monte Carlo Simulation is performed to compare the different methods of estimation. We illustrate the performance of the proposed distribution through two real data sets; one is related to economic data and another is failure time data and the data sets show the proposed distribution is superior in its ability to model the data sets as compared to the exponentiated generalized exponential, alpha power generalized exponential, alpha power exponential, generalized exponential and exponential distributions.

**Key Words:** Exponentiated Generalized distribution; Alpha Power distribution; Maximum Likelihood Estimation; Maximum Product Spacing; Bayesian estimation; Reliability Analysis.

**Mathematical Subject Classification:** 62F10, 62F15, 62E10, 62E15, 62F40

### 1. Introduction

The concept of developing new statistical distributions is not new in the statistical literature. The pioneering work of Pearson (1895) using the system of differential equation approach, set the trend for generating statistical distributions. Thereafter, numerous methods have been employed by various authors to generate a family of distributions. In this regard, readers may refer to the works of Burr (1942), Johnson (1949), Hastings et al. (1947), Tukey (1960), Azzalini (1985, 1986), Mudholkar and Srivastava (1993), Marshall and Olkin (1997), Eugene et al. (2002); Cooray and Ananda (2005); Ferreira and Steel (2006), Alzaatreh et al. (2013); and so on.

Of late, Cordeiro et al. (2013) proposed an important method wherein two new parameters were added to a continuous distribution resulting in a new distribution called the Exponentiated Generalized (EG) Family of distributions. The two extra parameters in the EG family of distributions can control both tail weights of the EG density function.

Let  $G(x)$  and  $g(x)$  be the cumulative distribution function (cdf) and probability density function (pdf) of a given random variable. Then, the cdf and pdf of the EG family class are, respectively, given by:

$$F(x) = [1 - (1 - G(X))^\theta]^\beta, \beta, \theta > 0, \tag{1}$$

and

$$f(x) = \theta\beta [1 - (1 - G(X))^\theta]^{\beta-1} (1 - G(X))^{\theta-1} g(x). \tag{2}$$

The EG family of densities (2) allows for greater flexibility of its tails and can be widely applied in many areas of engineering and biology. In this regard, readers may refer to the works of Cordeiro and Lemonte (2014), Andrade et al. (2015), Aryal and Elbatal (2015), and De Andrade et al. (2016), Elgarhy et al. (2018), and Reyad et al. (2019). Mahdavi and Kundu (2017) proposed a transformation of the baseline cdf by adding a new parameter to obtain a family of distributions. The proposed method is called alpha power (AP) transformation. If  $G(x)$  is the cdf of any distribution, then the cumulative distribution function(x) is given by:

$$W(x) = \begin{cases} \frac{\alpha^{G(x)} - 1}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1, \\ G(x) & \text{if } \alpha = 1 \end{cases}, \tag{3}$$

and the corresponding pdf takes the form

$$w(x) = \begin{cases} \frac{\ln(\alpha) g(x) \alpha^{G(x)}}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1 \\ g(x) & \text{if } \alpha = 1 \end{cases} \tag{4}$$

It is clear that for  $\alpha \neq 1$ ,  $W(x)$  is a weighted version of  $g(x)$ , where the weight function is  $u(x) = \alpha^{G(x)}$ .

This approach of generalization has been adopted frequently by several researchers in recent times. The applicability of this kind of generalized distribution can be found in the works of Dey et al. (2017a, 2017b, 2018, 2019), Nassar et al. (2017), Elbatal et al. (2018), Hassan et al. (2019), Basheer (2019), Eghwerido et al. (2020) amongst others in diverse areas like analysis of ozone data, earthquake data, failure time data, etc.

In this paper, our first objective is to obtain a new distribution, called the exponentiated generalized alpha power exponential (EGAPEx) distribution that contains some special submodels. One important feature of the new model is that the failure rate function can be L-shaped, increasing, decreasing, and upside-down bathtub. Owing to its flexibility of the failure rate function, this new model provides a good alternative to many existing lifetime distributions in modeling positive real data sets in practice. In particular, it can be an alternative to the well-known distributions like exponentiated generalized exponential (EGEx), Alpha Power Generalized Exponential (APGEx), Alpha Power Exponential (APEX), Generalized Exponential (GEx), and exponential (Ex) distributions in modeling economic and failure data. Besides, we provide some statistical properties of the proposed distribution. The second objective is to evaluate and study the behavior of maximum likelihood estimators (MLE) and maximum product spacing estimators (MPSE) as classical estimators. The third objective is to construct approximate confidence intervals (ACIs) and two bootstrap confidence intervals (BCIs) of the model parameters. The fourth objective is to obtain Bayes estimators of the unknown parameters of the model using the squared error loss function under the assumptions of independent gamma priors. Besides, we present a Metropolis-Hastings (MH) algorithm to compute the Bayes estimates and the associated credible intervals. Although the utility and usefulness of an estimator are subject to the area of study, in practice users look for their desired estimator under the different settings of parameters and sample sizes. As it is tedious to compare the performances of these estimators theoretically, we conduct extensive simulations for assessing the performances of the said estimators in terms of their bias and mean squared error (MSE). The novelty of this study is that thus far no study has been carried out on EGAPEx distribution using all these estimation methods.

The layout of the paper is as follows. In Section 2, we provide the genesis of the EGAPEx distribution and also plots for its HRF and PDF. In Section 3, we obtain some properties of this distribution. In Section 4, two classical estimation methods and the Bayesian estimation method of the unknown parameters are presented. In Section 5, asymptotic confidence intervals, two bootstrap confidence intervals (percentile and t bootstraps), and Bayes credible intervals are provided. In Section 6, we perform a simulation study to evaluate the performance of the aforementioned estimation methods. The potentiality of the EGAPEx distribution is also illustrated through two real data sets in Section 7. Finally, in Section 8, we provide some conclusions.

## 2. Model description of EGAP Family

The CDF of the EGAP family is obtained by replacing  $F(x)$  in Eq. (1) by the EG family and Eq. (3) of the AP class, thus, we have

$$F_{EGAP}(x) = \begin{cases} \left[ 1 - \left( 1 - \frac{\alpha^{G(x)} - 1}{\alpha - 1} \right)^{\theta} \right]^{\beta} & ; \alpha, \beta, \theta > 0, \alpha \neq 1, \\ \left[ 1 - (1 - G(X))^{\theta} \right]^{\beta} & ; \alpha = 1 \end{cases}, \tag{5}$$

The pdf of the EGAP family is obtained by replacing pdf of Eq. (2) by the EG family and Eq. (4) of the AP class, we have

$$f_{EGAP}(x) = \begin{cases} \theta \beta \left[ 1 - \left( 1 - \frac{\alpha^{G(x)} - 1}{\alpha - 1} \right)^{\theta} \right]^{\beta-1} \left( 1 - \frac{\alpha^{G(x)} - 1}{\alpha - 1} \right)^{\theta-1} \frac{\ln(\alpha) g(x) \alpha^{G(x)}}{\alpha - 1} & \\ \theta \beta \left[ 1 - (1 - G(X))^{\theta} \right]^{\beta-1} (1 - G(X))^{\theta-1} g(x) & ; \alpha = 1 \end{cases}. \tag{6}$$

Figure 1 displays plots of the pdf of the *uniform*(0, 1) based on EGAP, then EGAPU distribution for some values of parameters as follows

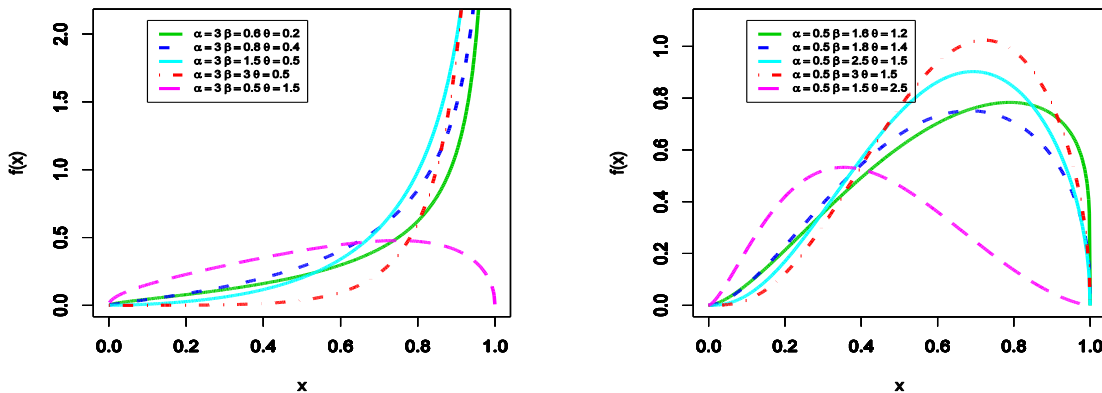


Figure 1: Plots of the EGAPU pdf for some values of parameters.

In Figure 1, we note the new family can have symmetrical, n-shaped, increasing and decreasing densities.

Let  $x_u = Q(u)$ , for  $0 < u < 1$  denote the quantile function of EGAP family. Then

$$x_u = G^{-1} \left[ \frac{1}{\ln(\alpha)} \ln \left( 1 + (\alpha - 1) \left[ 1 - \left( 1 - u^{1/\beta} \right)^{1/\theta} \right] \right) \right]. \tag{7}$$

In particular, the first three quantiles, Q1, Q2, and Q3 can be obtained by setting  $u=0.25$ ,  $u=0.5$ , and  $u=0.75$  in Eq. (7), respectively.

### 2.1. The EGAPEx distribution

In this subsection, we apply the EGAP family to the exponential distribution. The generated distribution is called the EGAPEx distribution and it is represented by the random variable  $X \sim EGAPEx(\alpha, \beta, \theta, \lambda)$ . The pdf and cdf of the exponential distribution are  $g(x; \lambda) = \lambda e^{-\lambda x}$ ,  $\lambda > 0, x > 0$  and  $G(x) = 1 - e^{-\lambda x}$ , respectively.

Using Eq. (6) and the exponential pdf, the density function of EGAPEx distribution becomes

$$f(x; \Omega) = \theta \beta \left[ 1 - \left( 1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1} \right)^{\theta} \right]^{\beta-1} \left( 1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1} \right)^{\theta-1} \frac{\ln(\alpha) \lambda e^{-\lambda x} \alpha^{1-e^{-\lambda x}}}{\alpha - 1}, \tag{8}$$

where  $\Omega = (\alpha, \beta, \theta, \lambda)$ ,  $\Omega > 0$  and  $\alpha \neq 1$ . By using Eq. (5) and the exponential cdf, the cdf of EGAPEx distribution takes the form

$$F(x; \Omega) = \left[ 1 - \left( 1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1} \right) \theta \right]^\beta \tag{9}$$

Using pdf, the binomial series and exponential series, the EGAPEx density can be expressed as a linear combination of exponential densities, namely

$$f(x; \Omega) = \mathfrak{P}(\alpha, \beta, \theta) g(x; \lambda(p + 1)), \tag{10}$$

where  $\mathfrak{P}(\alpha, \beta, \theta) = \frac{\theta \beta \ln(\alpha)}{(p+1)(\alpha-1)} \sum_{k=0}^{\infty} \binom{\beta-1}{k} \sum_{j=0}^{\infty} \binom{\theta(k+1)-1}{j} (1-\alpha)^j \sum_{h=0}^{\infty} \binom{j}{h} \sum_{q=0}^{\infty} \frac{((h+1)\ln(\alpha))^q}{q!} \sum_{p=0}^{\infty} \binom{q}{p} (-1)^{k+j+h+p}$

It can be denoted by exponential density with scale parameter  $\lambda(p + 1)$ . This formula is very useful to obtain various structural properties of the EGAPEx distribution directly from the exponential distribution.

Plots of the PDF of X of the EGAPEx distribution for selected parameter values of  $\alpha, \beta, \theta, \lambda$  are shown in Figure 2. Figure 2 shows the EGAPEx density can be right-skewed, almost symmetrical, and unimodal shaped, where  $\lambda$  is responsible for the tapered form, and the higher the value of  $\lambda$ , the greater the tapered form of the distribution. While higher the value of the parameter  $\alpha$ , the shape of the density is closer to normal distribution.

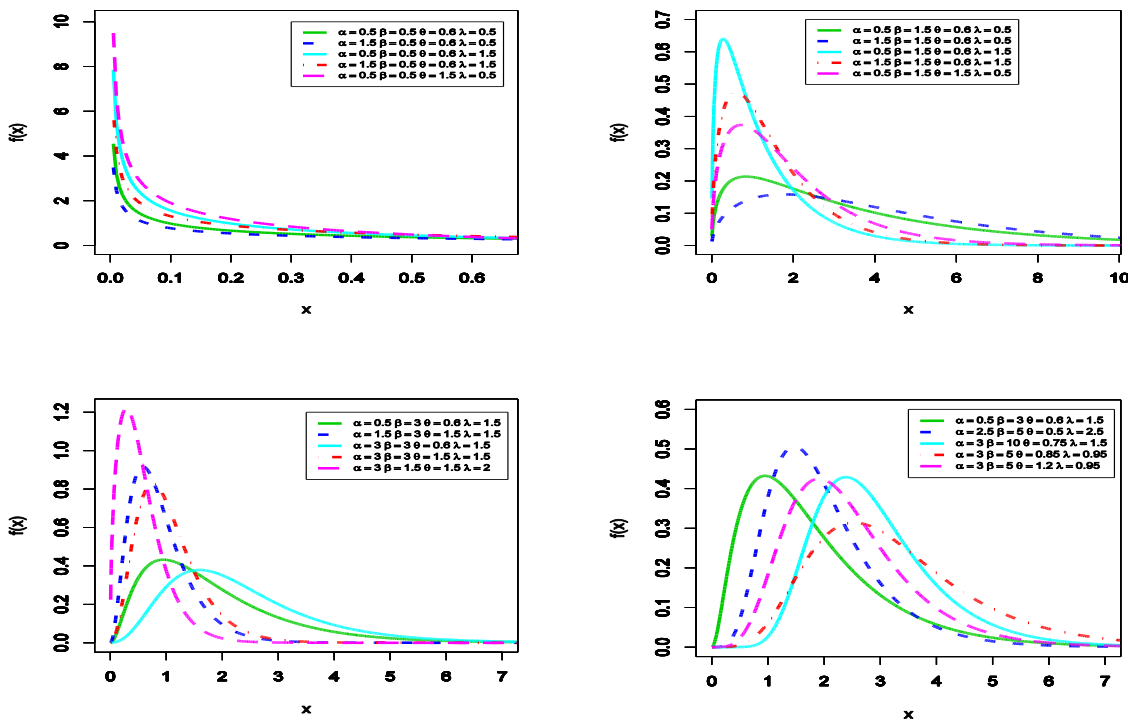


Figure 2: Plots of the EGAPEx pdf for some values of the parameters  $\alpha, \beta, \theta, \lambda$

### 2.2. EGAPEx Sub-Models

From Eq. (5) pdf and Eq. (6) cdf of EGAPEx distribution, we study the sub-models of EGAPEx distribution. Table 1 summarizes the sub-models of the EGAPEx distribution.

Table 1: Sub-models of the EGAPEx distribution.

| Distributions                                | $\alpha$ | $\beta$ | $\theta$ | $\lambda$ |
|--|----------|---------|----------|-----------|
| Exponentiated Generalized Exponential (EGEx) | 1        | $\beta$ | $\theta$ | $\lambda$ |
| Generalized Exponential (GE)                 | 1        | 1       | $\theta$ | $\lambda$ |
| Standardized Generalized Exponential (GE)    | 1        | 1       | $\theta$ | 1         |
| Alpha Power Generalized Exponential (APGEx)  | $\alpha$ | 1       | $\theta$ | $\lambda$ |
| Alpha Power Exponential (APEX)               | $\alpha$ | 1       | 1        | $\lambda$ |
| Exponential (Ex)                             | 1        | 1       | 1        | $\lambda$ |

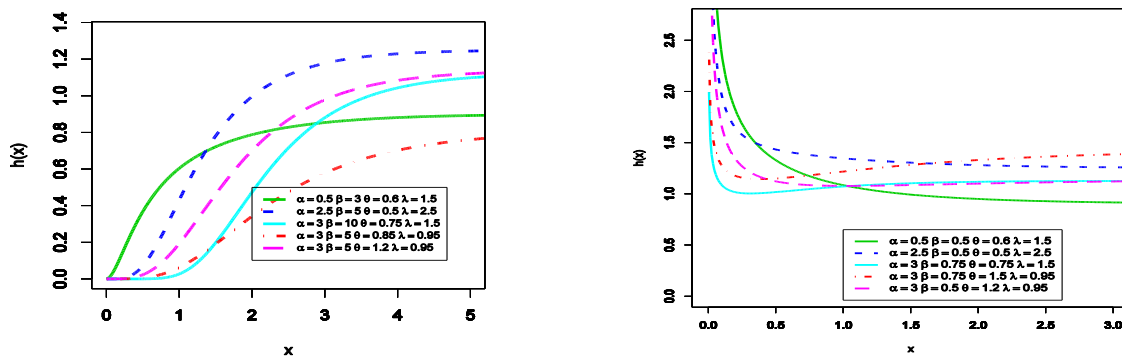


Figure 3: Plots of the EGAPEx hazard rate for some values of the parameters  $\alpha, \beta, \theta, \lambda$

Figure 3 illustrates that the EGAPEx HRF can be L-shaped, increasing, decreasing and upside-down bathtub shapes. These features make the EGAPEx distribution very competitive over exponential, generalize exponential and alpha power exponential distributions as these distributions can model phenomena showing only monotonic failure rates which is a major weakness because most empirical life systems have a bathtub or unimodal shapes for their HRF and therefore EGAPEx distribution seems to be more flexible for analyzing lifetime, economic data, etc.

### 3. Statistical Properties

In this section, we study some statistical properties of the EGAPEx distribution such as quantile function, moments, moment generating function and stress-strength reliability.

#### 3.1. Quantile function

If  $X$  has  $EGAPEx(\Omega)$  distribution, then the quantile of a random variable  $X$ , is given by

$$x_u = \frac{-1}{\lambda} \ln \left[ 1 - \frac{1}{\ln(\alpha)} \ln \left( 1 + (\alpha - 1) \left[ 1 - \left( 1 - u^{1/\beta} \right)^{1/\theta} \right] \right) \right]. \tag{11}$$

#### 3.2. Moments, Moment Generating Functions and Related Measures

In this subsection, we will present the  $r^{th}$  moments of EGAPEx distribution. Moments are important in any statistical analysis. Moments are useful to determine the properties of distribution such as measures of central tendency, variance, skewness, and kurtosis. Let  $X$  be a random variable having the exponential distribution with rate parameter  $\lambda$ . The ordinary moments of  $X$  is  $E(X^r) = \frac{\Gamma(r+1)}{\lambda^r}$ , the moment generating functions of  $X$  is  $M_X(t) = \frac{\lambda}{(\lambda-t)}$ ;  $t < \lambda$ , and the characteristic function of  $X$  is  $\varphi_X(t) = \frac{\lambda}{(\lambda-it)}$ .

Let  $X$  be a random variable having the EGAPEx distribution with vector parameter  $\Omega$ , and by using Eq. (8) and the ordinary moments of the exponential distribution, the  $r^{th}$  ordinary moment of EGAPEx distribution is as follows

$$E(x^n) = \mathfrak{P}(\alpha, \beta, \theta) \frac{\Gamma(r + 1)}{\lambda(p + 1)r}$$

By using the moment generating functions of the exponential distribution, the moment generating functions of EGAPEx is as follows

$$M_X(t) = \mathfrak{P}(\alpha, \beta, \theta) \frac{\lambda(p + 1)}{(\lambda(p + 1) - t)}; t < \lambda(p + 1)$$

By using the characteristic function of exponential distribution, the characteristic function of EGAPEx is as follows

$$\mathfrak{F}_X(t) = \mathfrak{P}(\alpha, \beta, \theta) \frac{\lambda(p + 1)}{(\lambda(p + 1) - it)}$$

The cumulant-generating function of EGAPEx distribution is as follows

$$K(t) = \log(\mathfrak{F}_X(t)) = \log\left(\mathfrak{P}(\alpha, \beta, \theta) \frac{\lambda(p + 1)}{(\lambda(p + 1) - it)}\right)$$

### 3.3. Survival and Hazard Rate Function

The survival function (reliability function) of EGAPEx distribution is given by

$$S(x; \Omega) = 1 - \left[1 - \left(1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1}\right)^\theta\right]^\beta$$

The hazard rate function (failure rate) of a lifetime random variable  $X$  with EGAPEx distribution is given by

$$h(x; \Omega) = \frac{\theta\beta \left[1 - \left(1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1}\right)^\theta\right]^{\beta-1} \left(1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1}\right)^{\theta-1} \frac{\ln(\alpha) \lambda e^{-\lambda x} \alpha^{1-e^{-\lambda x}}}{\alpha - 1}}{1 - \left[1 - \left(1 - \frac{\alpha^{1-e^{-\lambda x}} - 1}{\alpha - 1}\right)^\theta\right]^\beta}$$

### 3.4. Stress-Strength Reliability

Let  $X$  and  $Y$  are the independent strength and stress random variables observed from EGAPEx. Then, the stress-strength reliability  $R$  is defined as;

$$\begin{aligned} R = P(Y < X) &= \int_{x=0}^{\infty} \left\{ \int_{y=0}^x f(y; \Omega_2) dy \right\} f(x; \Omega_1) dx. \\ &= \int_0^{\infty} f(x; \Omega_1) F(x; \Omega_2) dx = \frac{\mathfrak{G}(\alpha, \beta, \theta)}{(p + 1)} \end{aligned} \tag{12}$$

where  $\mathfrak{G}(\alpha, \beta, \theta) = \frac{\theta\beta \ln(\alpha)}{(\alpha - 1)} \sum_{k=0}^{\infty} \binom{2\beta-1}{k} \sum_{j=0}^{\infty} \binom{\theta(k+1)-1}{j} (1 - \alpha)^j \sum_{h=0}^{\infty} \binom{j}{h} \sum_{h=0}^{\infty} \frac{((h+1)\ln(\alpha))^q}{q!} \sum_{p=0}^{\infty} \binom{q}{p} (-1)^{k+j+h+p}$

## 4. Parameter Estimation

In this section, we adopt the maximum likelihood estimation (MLE) method, maximum product spacing estimation (MPS) method and Bayesian estimation method to estimate the model parameters. Many papers used these methods based on complete samples as Almetwally and Almongy (2019b), Basheer et al. (2021), Almongy et al. (2021) Almetwally (2022a,b), while based on censored samples see Almetwally et al. (2018,2019), Almetwally and Almongy (2019b), Alshenawy et al. (2021), and Almongy et al. (2021), while in the case of estimate fuzzy reliability model see Sabry et al. (2021), and Riad et al. (2022).

### 4.1. Maximum Likelihood Estimation

The likelihood function is the most frequently used method of parameter estimation. The likelihood function of the EGAPEx distribution is given by

$$L(\Omega) = \prod_{i=1}^n \left\{ \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^{\beta-1} (1 - \varphi(x_i, \alpha, \lambda))^{\theta-1} \alpha^{1-e^{-\lambda x_i}} \right\} \left( \frac{\theta \beta \ln(\alpha) \lambda}{\alpha - 1} \right)^n e^{-\lambda \sum_{i=1}^n x_i}, \tag{13}$$

where  $\varphi(x_i, \alpha, \lambda) = \frac{\alpha^{1-e^{-\lambda x_i}} - 1}{\alpha - 1}$ , and the log likelihood function is given as

$$l(\Omega) = n \ln \left( \frac{\theta \beta \ln(\alpha) \lambda}{\alpha - 1} \right) - \lambda \sum_{i=1}^n x_i + \ln(\alpha) \sum_{i=1}^n (1 - e^{-\lambda x_i}) + (\theta - 1) \sum_{i=1}^n \ln(1 - \varphi(x_i, \alpha, \lambda)) + (\beta - 1) \sum_{i=1}^n \ln \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]. \tag{14}$$

The likelihood equations for the model parameters are

$$\frac{\partial \ln l(\Omega)}{\partial \alpha} = \frac{n}{\hat{\alpha} \ln(\hat{\alpha})} - \frac{n}{\hat{\alpha} - 1} + \frac{1}{\hat{\alpha}} \sum_{i=1}^n (1 - e^{-\hat{\lambda} x_i}) + (\hat{\theta} - 1) \sum_{i=1}^n \varrho(x_i, \hat{\alpha}, \hat{\lambda}) - (\hat{\beta} - 1) \sum_{i=1}^n \varsigma(x_i, \hat{\alpha}, \hat{\theta}, \hat{\lambda}) = 0,$$

$$\frac{\partial \ln l(\Omega)}{\partial \beta} = \frac{n}{\hat{\beta}} + \sum_{i=1}^n \ln \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^{\hat{\theta}} \right] = 0,$$

$$\frac{\partial \ln l(\Omega)}{\partial \theta} = \frac{n}{\hat{\theta}} + \sum_{i=1}^n \ln(1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda})) - (\hat{\beta} - 1) \sum_{i=1}^n \frac{\ln(1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda})) (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^{\hat{\theta}}}{1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^{\hat{\theta}}} = 0,$$

and

$$\frac{\partial \ln l(\Omega)}{\partial \lambda} = \frac{n}{\hat{\lambda}} - \sum_{i=1}^n x_i + \ln(\hat{\alpha}) \sum_{i=1}^n x_i e^{-\hat{\lambda} x_i} - \frac{(\hat{\theta} - 1) \ln(\hat{\alpha})}{(\hat{\alpha} - 1)} \sum_{i=1}^n \frac{\hat{\alpha}^{1-e^{-\hat{\lambda} x_i}} x_i e^{-\hat{\lambda} x_i}}{1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda})} + (\hat{\beta} - 1) \frac{\hat{\theta} \ln(\hat{\alpha})}{(\hat{\alpha} - 1)} \sum_{i=1}^n \frac{\hat{\alpha}^{1-e^{-\hat{\lambda} x_i}} x_i e^{-\hat{\lambda} x_i} (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^{\hat{\theta}-1}}{1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^{\hat{\theta}}} = 0,$$

where

$$\varrho(x_i, \alpha, \lambda) = \frac{\frac{\varphi(x_i, \alpha, \lambda)}{\alpha - 1} + \frac{e^{-\lambda x_i} - 1}{(\alpha - 1)\alpha e^{-\lambda x_i}}}{1 - \varphi(x_i, \alpha, \lambda)},$$

$$\varsigma(x_i, \alpha, \vartheta, \lambda) = \frac{\theta(1 - \varphi(x_i, \alpha, \lambda))^{\theta-1} \varrho(x_i, \alpha, \lambda)}{1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta}.$$

These equations cannot be solved explicitly, so numerical methods like the Conjugate Gradients algorithms can be used to calculate the MLE of  $\Omega$ .

### 4.2. Maximum Product Spacing

The maximum product spacing (MPS) method was proposed by Cheng and Amin (1983). The geometric mean of the differences is given as

$$G = \left( \prod_{i=1}^{n+1} D_i \right)^{\frac{1}{n+1}}, \tag{15}$$

where  $G$  is defined as the geometric mean of the product spacing function. Also, the difference  $D_i$  is defined as

$$D_i = \begin{cases} D_1 = F(x_1, \Omega) \\ D_i = F(x_i, \Omega) - F(x_{i-1}, \Omega); i = 2 \dots n, \\ D_{n+1} = 1 - F(x_n, \Omega), \end{cases}$$

such that  $\sum D_i = 1$ , then the product spacing function for EGAPEx distribution is

$$G(\Omega) = \left( \left[ 1 - (1 - \varphi(x_1, \alpha, \lambda))^\theta \right]^\beta \left( 1 - \left[ 1 - (1 - \varphi(x_n, \alpha, \lambda))^\theta \right]^\beta \right) \right)^{\frac{1}{n+1}} \tag{16}$$

$$\left( \prod_{i=2}^n \left( \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^\beta - \left[ 1 - (1 - \varphi(x_{i-1}, \alpha, \lambda))^\theta \right]^\beta \right) \right)^{\frac{1}{n+1}}$$

The natural logarithm of the product spacing function is

$$\ln G(\Omega) = \frac{1}{n+1} \left[ \beta \ln \left( 1 - (1 - \varphi(x_1, \alpha, \lambda))^\theta \right) + \ln \left( 1 - \left[ 1 - (1 - \varphi(x_n, \alpha, \lambda))^\theta \right]^\beta \right) + \sum_{i=2}^n \ln \left( \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^\beta - \left[ 1 - (1 - \varphi(x_{i-1}, \alpha, \lambda))^\theta \right]^\beta \right) \right]. \tag{17}$$

To estimate of the unknown parameters, we differentiate Equation (17) partially with respect to the parameter  $\Omega$  and equate them to zero. The estimators  $\hat{\Omega}$  of  $\Omega$  can be obtained as the solution of the following equations.

$$\begin{aligned} \frac{\partial \ln G(\Omega)}{\partial \alpha} &= \frac{1}{n+1} \left[ \beta \zeta(x_1, \hat{\theta}, \hat{\alpha}, \hat{\lambda}) \left[ 1 - (1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1} + \frac{\beta \zeta(x_n, \hat{\theta}, \hat{\alpha}, \hat{\lambda}) \left( 1 - (1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}))^\theta \right) \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1}}{1 - \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta} \right. \\ &\quad \left. + \beta \sum_{i=2}^n \frac{\zeta(x_i, \hat{\theta}, \hat{\alpha}, \hat{\lambda}) \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1} - \zeta(x_{i-1}, \hat{\theta}, \hat{\alpha}, \hat{\lambda}) \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1}}{\left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta - \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta} \right] = 0, \\ \frac{\partial \ln G(\Omega)}{\partial \beta} &= \frac{1}{n+1} \left[ \ln \left[ 1 - (1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}))^\theta \right] - \frac{\ln \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right] \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}}}{1 - \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta} \right. \\ &\quad \left. + \sum_{i=2}^n \frac{\ln \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right] \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta - \ln \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right] \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta}{\left( \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta - \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta \right)} \right] = 0, \\ \frac{\partial \ln G(\Omega)}{\partial \theta} &= \frac{1}{n+1} \left[ \frac{-\beta \ln \left( 1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}) \right) \left( 1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}) \right)^\theta}{\left( 1 - (1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}))^\theta \right)} + \frac{\beta \ln \left( 1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}) \right) \left( 1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}) \right)^\theta \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1}}{1 - \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta} \right. \\ &\quad \left. + \beta \sum_{i=2}^n \frac{\ln \left( 1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}) \right) \left( 1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}) \right)^\theta \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1} - \ln \left( 1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}) \right) \left( 1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}) \right)^\theta \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1}}{\left( \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta - \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta \right)} \right] = 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \ln G(\Omega)}{\partial \lambda} &= \frac{1}{n+1} \left[ \frac{\hat{\theta} \ln(\hat{\alpha})}{(\hat{\alpha}-1) \sum_{i=1}^n \frac{\hat{\alpha}^{1-e^{-\lambda x_i}} x_i e^{-\lambda x_i} \left( 1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda}) \right)^{\hat{\theta}-1}}{1 - \varphi(x_1, \hat{\alpha}, \hat{\lambda})}} - \frac{\hat{\beta} \hat{\theta} \ln(\hat{\alpha})}{(\hat{\alpha}-1) \sum_{i=1}^n \frac{\hat{\alpha}^{1-e^{-\lambda x_n}} x_n e^{-\lambda x_n} \left( 1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}) \right)^{\hat{\theta}-1} \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1}}{1 - \left[ 1 - (1 - \varphi(x_n, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta}} \right. \\ &\quad \left. + \frac{\hat{\beta} \hat{\theta} \ln(\hat{\alpha})}{(\hat{\alpha}-1) \sum_{i=2}^n \frac{\hat{\alpha}^{1-e^{-\lambda x_i}} x_i e^{-\lambda x_i} \left( 1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}) \right)^{\hat{\theta}-1} \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1} - \hat{\alpha}^{1-e^{-\lambda x_{i-1}}} x_{i-1} e^{-\lambda x_{i-1}} \left( 1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}) \right)^{\hat{\theta}-1} \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^{\hat{\theta}-1}}{\left( \left[ 1 - (1 - \varphi(x_i, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta - \left[ 1 - (1 - \varphi(x_{i-1}, \hat{\alpha}, \hat{\lambda}))^\theta \right]^\beta \right)}} \right] = 0. \end{aligned}$$

These equations cannot be solved explicitly, so numerical methods like the conjugate gradients algorithms can be used to calculate the MPS of  $\Omega$ .



### 4.3. Bayesian Estimation

In this section, we present a Bayesian estimation of the unknown parameters  $\alpha, \beta, \theta$  and  $\lambda$  of EGAPEx distribution. As a powerful and valid alternative to classical estimation, Bayesian analysis is a natural way to combine the observed information with the prior information which makes it very valuable in reliability, lifetime study and other associated fields where one of the major challenges is the limited availability of data. It is assumed that  $\Omega = (\alpha, \beta, \theta, \lambda)$  is independently distributed as gamma distribution which is a conjugate prior to the EGAPEx distribution. Assumed that  $\Omega_j \sim \text{Gamma}(a_j, b_j); j = 1, 2, 3, 4$ , then the joint prior density of  $\Omega$  can be written as

$$g(\Omega) \propto \prod_{j=1}^4 \Omega_j^{a_j-1} e^{-\frac{\Omega_j}{b_j}}; \quad a_j, b_j > 0, \tag{18}$$

here all the hyper parameters  $a_j, b_j$  are known and non-negative.

Based on the likelihood function (5) and the joint prior density (18), the joint posterior of  $\Omega$  is

$$g(\Omega|x) \propto e^{-\frac{\alpha}{b_1}} e^{-\frac{\beta}{b_2}} e^{-\frac{\theta}{b_3}} \left( \frac{\theta^{\alpha_3-1} \beta^{\alpha_2-1} \ln(\alpha) \lambda^{\alpha_4-1}}{\alpha - 1} \right)^n e^{-\lambda(\sum_{i=1}^n x_i + \frac{1}{b_4})} \prod_{i=1}^n \left\{ \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^{\beta-1} (1 - \varphi(x_i, \alpha, \lambda))^{\theta-1} \alpha^{a_1-e^{-\lambda x_i}} \right\}, \tag{19}$$

The Markov chain Monte Carlo (MCMC) can be used to generate posterior samples from the posterior density function (19) which in turn computes the Bayes estimates of the unknown parameters. To generate posterior samples from (19), we can use the fully conditional posterior distributions of the parameters and are given by

$$g_1(\alpha|\beta, \theta, \lambda, x) \propto e^{-\frac{\alpha}{b_1}} \left( \frac{\ln(\alpha)}{\alpha - 1} \right)^n \prod_{i=1}^n \left\{ \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^{\beta-1} (1 - \varphi(x_i, \alpha, \lambda))^{\theta-1} \alpha^{a_1-e^{-\lambda x_i}} \right\},$$

$$g_2(\beta|\alpha, \theta, \lambda, x) \propto e^{-\frac{\beta}{b_2}} (\beta^{\alpha_2-1})^n \prod_{i=1}^n \left\{ \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^{\beta-1} \right\},$$

$$g_3(\theta|\alpha, \beta, \lambda, x) \propto e^{-\frac{\theta}{b_3}} (\theta^{\alpha_3-1})^n \prod_{i=1}^n \left\{ \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^{\beta-1} (1 - \varphi(x_i, \alpha, \lambda))^{\theta-1} \right\},$$

and

$$g_4(\lambda|\alpha, \beta, \theta, x) \propto (\lambda^{\alpha_4-1})^n e^{-\lambda(\sum_{i=1}^n x_i + \frac{1}{b_4})} \prod_{i=1}^n \left\{ \left[ 1 - (1 - \varphi(x_i, \alpha, \lambda))^\theta \right]^{\beta-1} (1 - \varphi(x_i, \alpha, \lambda))^{\theta-1} \alpha^{a_1-e^{-\lambda x_i}} \right\}.$$

Since the full conditional posterior distributions do not have simple forms, we adopt a Metropolis–Hastings within Gibbs sampling approach to generate random samples from the conditional densities of the parameters and use them to obtain the Bayes credible intervals and point Bayes point estimates based on the squared error loss function.

The Metropolis-Hastings algorithm generates samples to estimate the parameters of EGAPEx distribution as follows:

1. By using initial values  $\Omega_l^{(0)}; l = 1, \dots, 4$ , where satisfying  $\pi(\Omega_l^{(0)}) > 0$ .
2. After using the initial values, to get candidate point  $\Omega^*$  from proposal  $q(\Omega^*)$ , here we use the normal distribution as proposal distribution.
3. When  $t = 0$  to  $N$ , given the candidate point  $\Omega^*$ , we calculate acceptance probability as shown

$$\forall_l = \min \left( 1, \frac{L(\Omega|x)\pi(\Omega_l^*)q(\Omega_l)}{L_1(\Omega|x)\pi(\Omega_l)q(\Omega_l^*)} \right)$$

4. Simulate a number  $u$  from the uniform (0,1) distribution,  $\forall_l^{(t+1)} = \begin{cases} \Omega_l^* & \text{if } u \leq \forall_l \\ \Omega_l^{(t)} & \text{otherwise} \end{cases}$ .

5. Repeat steps 2 - 4 ( $t + 1$ ) times until we get  $N$  iteration of the MCMC process.
6. Repeat these steps  $l$  times to get Bayes estimate of  $\Omega_l$ .

By using the generated random samples from the above Gibbs sampling technique, the approximate Bayes estimate of the parameters under the squared error loss function can be obtained as  $\tilde{\Omega}_l = \sum_{i=1}^N \frac{\Omega_l^{(i)}}{N}$ , where  $N$  is the number of iterations in the MCMC process, which equals 10,000.

### 5. Confidence Interval (CI)

In this section, we discuss the asymptotic confidence intervals, Bayes credible intervals, and bootstrap confidence intervals of the parameters of EGAPEx distribution.

#### 5.1. Asymptotic Confidence Intervals

Since the MLEs of the unknown parameters of the model cannot be derived in closed form, it is not easy to derive the exact distributions of the MLEs. Using the asymptotic distribution of the MLEs, the approximate confidence intervals for unknown parameters can be obtained by inverting the Fisher matrix  $F$ , which consists of the negative second derivatives of the natural logarithm of the likelihood function evaluated at the MLEs of the parameters.

Assume some regularity conditions to be satisfied, a  $100(1 - \gamma) \%$  approximate confidence intervals for EGAPEx distribution parameters are given respectively, as follows

$$\hat{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{v_{11}}, \hat{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{v_{22}}, \hat{\theta} \pm z_{\frac{\gamma}{2}} \sqrt{v_{33}} \text{ and } \hat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{v_{44}}$$

where  $v_{11}, v_{22}, v_{33}$ , and  $v_{44}$  are the elements on the main diagonal of the variance - covariance matrix of EGAPEx distribution and  $z_{\frac{\gamma}{2}}$  is the percentile of the standard normal distribution with right tail  $\frac{\gamma}{2}$ .

#### 5.2. Bayes Credible Intervals

Using Chen and Shao (1999) procedure, we obtain Bayes credible intervals of the parameters of EGAPEx distribution  $\alpha, \beta, \theta$  and  $\lambda$  as follow:

- Arrange  $\tilde{\alpha}^{(j)}$  as  $(\tilde{\alpha}^{(1)} < \tilde{\alpha}^{(2)} < \dots < \tilde{\alpha}^{(N)})$ ,  $\tilde{\beta}^{(j)}$  as  $(\tilde{\beta}^{(1)} < \tilde{\beta}^{(2)} < \dots < \tilde{\beta}^{(N)})$ ,  $\tilde{\theta}^{(j)}$  as  $(\tilde{\theta}^{(1)} < \tilde{\theta}^{(2)} < \dots < \tilde{\theta}^{(N)})$ , and  $\tilde{\lambda}^{(j)}$  as  $(\tilde{\lambda}^{(1)} < \tilde{\lambda}^{(2)} < \dots < \tilde{\lambda}^{(N)})$ , where  $N$  is the number of periods in the MCMC process.
- The  $100(1 - \gamma)\%$  symmetric credible intervals of  $\Omega$  become  $(\tilde{\alpha}^{(N\frac{\gamma}{2})}, \tilde{\alpha}^{(N(1-\frac{\gamma}{2})})$ ,  $(\tilde{\beta}^{(N\frac{\gamma}{2})}, \tilde{\beta}^{(N(1-\frac{\gamma}{2})})$ ,  $(\tilde{\theta}^{(N\frac{\gamma}{2})}, \tilde{\theta}^{(N(1-\frac{\gamma}{2})})$ , and  $(\tilde{\lambda}^{(N\frac{\gamma}{2})}, \tilde{\lambda}^{(N(1-\frac{\gamma}{2})})$ .

#### 5.3. Bootstrap Method

Here, we construct percentile bootstrap and bootstrap-t- bootstrap confidence interval of the parameters of EGAPEx distribution as follows where

the estimators of MLE  $\Psi^1 = \hat{\Omega} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$ ,

the estimators of MPS  $\Psi^2 = \tilde{\Omega} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{\lambda})$  and

the estimators of Bayesian  $\Psi^3 = \tilde{\Omega} = (\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}, \tilde{\lambda})$ .

##### 1) Percentile Bootstrap Confidence Interval (PB)

- i. Compute the estimators for parameters of EGAPEx distribution  $\Psi^l = (\alpha^l, \beta^l, \theta^l, \lambda^l); l= 1,2,3$ .
- ii. Generate a bootstrap sample using  $\Psi^l$  to obtain the bootstrap estimate of  $\alpha$ , say  $\alpha^{lb}$ ,  $\beta$ , say  $\beta^{lb}$ ,  $\theta$ , say  $\theta^{lb}$ , and  $\lambda$ , say  $\lambda^{lb}$  using the bootstrap sample.
- iii. Repeat step (ii)  $B$  times to have  $(\alpha^{lb(1)}, \alpha^{lb(2)}, \dots, \alpha^{lb(B)})$ ,  $(\beta^{lb(1)}, \beta^{lb(2)}, \dots, \beta^{lb(B)})$ ,  $(\theta^{lb(1)}, \theta^{lb(2)}, \dots, \theta^{lb(B)})$  and  $(\lambda^{lb(1)}, \lambda^{lb(2)}, \dots, \lambda^{lb(B)})$ .

iv. Arrange  $(\alpha^{lb(1)}, \alpha^{lb(2)}, \dots, \alpha^{lb(B)}), (\beta^{lb(1)}, \beta^{lb(2)}, \dots, \beta^{lb(B)}), (\theta^{lb(1)}, \theta^{lb(2)}, \dots, \theta^{lb(B)})$  and  $(\lambda^{lb(1)}, \lambda^{lb(2)}, \dots, \lambda^{lb(B)})$  in ascending order as  $(\alpha^{lb[1]}, \alpha^{lb[2]}, \dots, \alpha^{lb[B]}), (\beta^{lb[1]}, \beta^{lb[2]}, \dots, \beta^{lb[B]}), (\theta^{lb[1]}, \theta^{lb[2]}, \dots, \theta^{lb[B]})$  and  $(\lambda^{lb[1]}, \lambda^{lb[2]}, \dots, \lambda^{lb[B]})$ .

v. A two sided 100  $(1 - \gamma)\%$  percentile bootstrap confidence interval for the unknown parameters  $\alpha^l, \beta^l, \theta^l$  and  $\lambda^l$  are given by  $\{\alpha^{lb[B\gamma/2]}, \alpha^{lb[B(1-\gamma/2)]}\}, \{\beta^{lb[B\gamma/2]}, \beta^{lb[B(1-\gamma/2)]}\}, \{\theta^{lb[B\gamma/2]}, \theta^{lb[B(1-\gamma/2)]}\}$  and  $\{\lambda^{lb[B\gamma/2]}, \lambda^{lb[B(1-\gamma/2)]}\}$ .

**2) Bootstrap-t Confidence Interval (BT)**

- i. Same as the steps (i-ii) in Boot-p
- ii. Compute the t-statistic of  $\Psi^l$  as  $T = (\Psi^{lb} - \Psi^l) / \sqrt{V(\Psi^{lb})}$ , where  $V(\Psi^{lb})$  is a variance of  $\Psi^{lb}$ .
- iii. Repeat step (ii) B times and obtain  $T^{(1)}, T^{(2)}, \dots, T^{(B)}$ .
- iv. Arrange  $T^{(1)}, T^{(2)}, \dots, T^{(B)}$  in ascending order as  $T[1], T[2], \dots, T[B]$ .
- v. A two sided 100  $(1 - \gamma)\%$  bootstrap-t confidence interval for the unknown parameters  $\alpha, \beta, \theta$ , and  $\lambda$  are given by

$$\{\alpha^l + T_1[B \gamma/2] \sqrt{V(\alpha^l)}, \alpha^l + T_1[B(1 - \gamma/2)] \sqrt{V(\alpha^l)}\}, \{\beta^l + T_2[B \gamma/2] \sqrt{V(\beta^l)}, \beta^l + T_2[B(1 - \gamma/2)] \sqrt{V(\beta^l)}\}, \{\theta^l + T_3[B \gamma/2] \sqrt{V(\theta^l)}, \theta^l + T_3[B(1 - \gamma/2)] \sqrt{V(\theta^l)}\}$$

$$\text{ and } \{\lambda^l + T_4[B \gamma/2] \sqrt{V(\lambda^l)}, \lambda^l + T_4[B(1 - \gamma/2)] \sqrt{V(\lambda^l)}\}.$$

**6. Simulation Study**

In this section, we conduct a Monte Carlo simulation analysis to decide the best estimation method that can be used to estimate the EGAPEx distribution parameters because it is not possible to compare the output of the different estimators derived in the previous sections theoretically. Using the EGAPEx distribution and different actual parameters, we produce 10000 random samples of sizes 50, 100, and 200. The bias of estimates, as well as bias, the mean square error (MSE), Relative efficacy (RE), and the length of the confidence interval, are calculated for each combination of different parameter combinations as follows:

For MLE: Bias =  $\widehat{\Omega} - \Omega$ ,  $MSE = Mean(\widehat{\Omega} - \Omega)^2$ , and L.CI=upper CI( $\widehat{\Omega}$ ) - lower CI( $\widehat{\Omega}$ ).

For MPS: Bias =  $\widetilde{\Omega} - \Omega$ ,  $MSE = Mean(\widetilde{\Omega} - \Omega)^2$ , and L.CI=upper CI( $\widetilde{\Omega}$ ) - lower CI( $\widetilde{\Omega}$ ).

For Bayesian: Bias =  $\widetilde{\Omega} - \Omega$ ,  $MSE = Mean(\widetilde{\Omega} - \Omega)^2$ , and L.CCI=upper credible CI - lower credible CI.

$$RE_1 = \frac{MSE(\widehat{\Omega})_{MLE}}{MSE(\widetilde{\Omega})_{MPS}}, RE_2 = \frac{MSE(\widehat{\Omega})_{MLE}}{MSE(\widetilde{\Omega})_{Bayesian}}$$

Solving four nonlinear equations yields all of the estimators. This is a well-known problem in the literature, and estimates can be obtained using numerical techniques. We first used the maximum likelihood approach to estimate the parameters, maximizing the log-likelihood function using the conjugate-gradient method. The average value of maximum likelihood estimates was used as the initial value for MPS and Bayesian estimation methods. In Tables 2-6, these values are shown for the various estimation methods.

R-program version 3.5.3 was used to conduct the simulation analysis. As we can see from these tables, the bias, MSE, and length of CI decrease as the sample size grow, implying that the various methods have consistent estimators. In most cases, Bayesian estimates outperform other estimates in terms of minimum MSE and CI length, followed by MPS and MLE.

Table 2: Bias, MSE, and RE for MLE, MPS and Bayesian of EGAPEx parameters with different Actuals Values when  $n = 50$

| Actuals values |         |          |           | MLE             |         | MPS    |         | Bayesian |         |        |        |       |
|----------------|---------|----------|-----------|-----------------|---------|--------|---------|----------|---------|--------|--------|-------|
| $\alpha$       | $\beta$ | $\theta$ | $\lambda$ | Bias1           | MSE     | Bias2  | MSE     | Bias     | MSE     | $RE_1$ | $RE_2$ |       |
| 0.75           | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 0.6705  | 2.9029 | 0.6335  | 2.8836   | 0.1052  | 0.2283 | 101%   | 1272% |
|                |         |          |           | $\hat{\beta}$   | 0.1228  | 0.2131 | -0.0413 | 0.1400   | 0.0400  | 0.0680 | 152%   | 313%  |
|                |         |          |           | $\hat{\theta}$  | 0.1016  | 0.3557 | -0.0617 | 0.2146   | 0.1040  | 0.2063 | 166%   | 172%  |
|                |         |          |           | $\hat{\lambda}$ | 0.1149  | 0.3150 | -0.0316 | 0.3629   | 0.0520  | 0.2953 | 87%    | 107%  |
| 1.5            | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 0.6083  | 3.8214 | 0.6853  | 4.5316   | -0.1484 | 0.4569 | 84%    | 836%  |
|                |         |          |           | $\hat{\beta}$   | 0.2101  | 0.2927 | 0.0242  | 0.1820   | 0.0844  | 0.0840 | 161%   | 348%  |
|                |         |          |           | $\hat{\theta}$  | 0.2607  | 0.5033 | -0.0289 | 0.2459   | 0.1131  | 0.1903 | 205%   | 264%  |
|                |         |          |           | $\hat{\lambda}$ | -0.0460 | 0.2556 | 0.0137  | 0.3304   | 0.0013  | 0.2371 | 77%    | 108%  |
| 1.5            | 1.25    | 0.5      | 1.5       | $\hat{\alpha}$  | 0.0503  | 1.0917 | 0.3323  | 1.1970   | -0.1529 | 0.3373 | 91%    | 324%  |
|                |         |          |           | $\hat{\beta}$   | 0.2225  | 0.2461 | -0.0237 | 0.1170   | 0.1052  | 0.0933 | 210%   | 264%  |
|                |         |          |           | $\hat{\theta}$  | 0.2012  | 0.2173 | 0.0720  | 0.2361   | 0.2031  | 0.1228 | 92%    | 177%  |
|                |         |          |           | $\hat{\lambda}$ | -0.1813 | 0.2085 | -0.1250 | 0.2015   | -0.1548 | 0.1961 | 103%   | 106%  |
| 1.5            | 1.5     | 2.5      | 1.5       | $\hat{\alpha}$  | 0.8470  | 5.0995 | 0.7003  | 5.1997   | -0.2628 | 0.5336 | 98%    | 956%  |
|                |         |          |           | $\hat{\beta}$   | 0.1379  | 0.2565 | -0.0407 | 0.1815   | 0.1654  | 0.1876 | 141%   | 137%  |
|                |         |          |           | $\hat{\theta}$  | -0.0055 | 0.3588 | -0.3387 | 0.3880   | -0.0300 | 0.2632 | 92%    | 136%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0783  | 0.2792 | 0.0622  | 0.5998   | 0.0526  | 0.2447 | 47%    | 114%  |
| 2.5            | 1.5     | 2.5      | 0.75      | $\hat{\alpha}$  | 0.1873  | 2.8673 | 0.1096  | 2.5720   | -0.1594 | 1.1990 | 111%   | 239%  |
|                |         |          |           | $\hat{\beta}$   | 0.1670  | 0.2308 | -0.0232 | 0.1555   | 0.1854  | 0.1859 | 148%   | 124%  |
|                |         |          |           | $\hat{\theta}$  | 0.0440  | 0.1777 | -0.2283 | 0.2545   | -0.0554 | 0.1833 | 70%    | 97%   |
|                |         |          |           | $\hat{\lambda}$ | 0.0061  | 0.0584 | 0.0235  | 0.1091   | 0.0323  | 0.0293 | 54%    | 199%  |
| 1.5            | 1.25    | 1.2      | 0.75      | $\hat{\alpha}$  | 0.1755  | 1.5901 | 0.2140  | 1.6898   | -0.1641 | 0.3931 | 94%    | 404%  |
|                |         |          |           | $\hat{\beta}$   | 0.2121  | 0.2663 | 0.0254  | 0.1550   | 0.1216  | 0.0987 | 172%   | 270%  |
|                |         |          |           | $\hat{\theta}$  | 0.0872  | 0.1747 | -0.0871 | 0.1153   | -0.0577 | 0.1628 | 152%   | 107%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0126  | 0.0724 | 0.0132  | 0.1032   | 0.0217  | 0.0614 | 70%    | 118%  |
| 1.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 0.0947  | 1.0227 | 0.1269  | 1.0537   | -0.2042 | 0.3211 | 97%    | 319%  |
|                |         |          |           | $\hat{\beta}$   | 0.1448  | 0.1496 | -0.0216 | 0.0933   | 0.1299  | 0.1135 | 160%   | 132%  |
|                |         |          |           | $\hat{\theta}$  | 0.0596  | 0.0859 | -0.0723 | 0.0750   | -0.0767 | 0.0722 | 115%   | 119%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0041  | 0.0172 | 0.0092  | 0.0310   | 0.0052  | 0.0126 | 55%    | 136%  |
| 0.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 0.3224  | 0.8970 | 0.2705  | 0.9285   | 0.1737  | 0.1739 | 97%    | 516%  |
|                |         |          |           | $\hat{\beta}$   | 0.1642  | 0.1906 | -0.0299 | 0.1041   | 0.0564  | 0.0933 | 183%   | 204%  |
|                |         |          |           | $\hat{\theta}$  | -0.1514 | 0.1543 | -0.0783 | 0.0972   | -0.1475 | 0.1495 | 159%   | 103%  |
|                |         |          |           | $\hat{\lambda}$ | 0.1639  | 0.1158 | 0.0063  | 0.0577   | 0.1252  | 0.1065 | 201%   | 109%  |
| 1.5            | 2.5     | 1.2      | 1.5       | $\hat{\alpha}$  | 0.3319  | 2.3942 | 0.5047  | 2.9124   | -0.0384 | 0.4910 | 82%    | 488%  |
|                |         |          |           | $\hat{\beta}$   | 0.4766  | 1.1620 | 0.0365  | 0.7660   | 0.1073  | 0.3592 | 152%   | 324%  |
|                |         |          |           | $\hat{\theta}$  | 0.3350  | 0.5057 | 0.1249  | 0.2666   | 0.1882  | 0.2793 | 190%   | 181%  |
|                |         |          |           | $\hat{\lambda}$ | -0.1611 | 0.1858 | -0.1304 | 0.2437   | 0.0247  | 0.1312 | 76%    | 142%  |

Table 3: Bias, MSE and RE for MLE, MPS and Bayesian of EGAPEx parameters with different Actuals Values when  $n = 100$

| Actuals values |         |          |           | MLE             |         | MPS    |         | Bayesian |         |        |        |      |
|----------------|---------|----------|-----------|-----------------|---------|--------|---------|----------|---------|--------|--------|------|
| $\alpha$       | $\beta$ | $\theta$ | $\lambda$ | Bias            | MSE     | Bias   | MSE     | Bias     | MSE     | $RE_1$ | $RE_2$ |      |
| 0.75           | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 0.3513  | 1.2206 | 0.2951  | 1.1492   | 0.0100  | 0.1462 | 106%   | 835% |
|                |         |          |           | $\hat{\beta}$   | 0.0703  | 0.1019 | -0.0189 | 0.0790   | 0.0376  | 0.0413 | 129%   | 247% |
|                |         |          |           | $\hat{\theta}$  | 0.0236  | 0.1343 | -0.0224 | 0.1016   | 0.0752  | 0.1141 | 132%   | 118% |
|                |         |          |           | $\hat{\lambda}$ | 0.0779  | 0.1944 | -0.0720 | 0.2386   | -0.0238 | 0.1863 | 81%    | 104% |
| 1.5            | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 0.2230  | 1.4221 | 0.2874  | 1.6845   | -0.1510 | 0.2531 | 84%    | 562% |
|                |         |          |           | $\hat{\beta}$   | 0.1067  | 0.1290 | 0.0003  | 0.0921   | 0.0518  | 0.0463 | 140%   | 279% |
|                |         |          |           | $\hat{\theta}$  | 0.1007  | 0.1570 | -0.0471 | 0.1009   | 0.0895  | 0.1266 | 156%   | 124% |
|                |         |          |           | $\hat{\lambda}$ | -0.0359 | 0.1206 | 0.0150  | 0.1679   | -0.0257 | 0.1138 | 72%    | 106% |
| 1.5            | 1.25    | 0.5      | 1.5       | $\hat{\alpha}$  | 0.0524  | 0.7136 | 0.2766  | 0.8378   | -0.1471 | 0.2626 | 85%    | 272% |
|                |         |          |           | $\hat{\beta}$   | 0.1043  | 0.1019 | -0.0322 | 0.0693   | 0.0657  | 0.0453 | 147%   | 225% |
|                |         |          |           | $\hat{\theta}$  | 0.1315  | 0.1133 | 0.0630  | 0.0752   | 0.1746  | 0.0951 | 151%   | 119% |
|                |         |          |           | $\hat{\lambda}$ | -0.1130 | 0.1523 | -0.0505 | 0.1488   | -0.1068 | 0.1225 | 102%   | 124% |
| 1.5            | 1.5     | 2.5      | 1.5       | $\hat{\alpha}$  | 0.5087  | 2.8735 | 0.3627  | 2.6425   | -0.1891 | 0.3818 | 109%   | 753% |
|                |         |          |           | $\hat{\beta}$   | 0.0619  | 0.1263 | -0.0331 | 0.1066   | 0.0770  | 0.0667 | 118%   | 189% |
|                |         |          |           | $\hat{\theta}$  | -0.0088 | 0.1990 | -0.2272 | 0.2079   | -0.0115 | 0.1365 | 96%    | 146% |
|                |         |          |           | $\hat{\lambda}$ | 0.0272  | 0.1497 | 0.0176  | 0.3444   | 0.0174  | 0.1091 | 43%    | 137% |
| 2.5            | 1.5     | 2.5      | 0.75      | $\hat{\alpha}$  | 0.0912  | 1.4561 | 0.1118  | 1.6823   | -0.2710 | 0.5444 | 87%    | 267% |
|                |         |          |           | $\hat{\beta}$   | 0.0776  | 0.1028 | -0.0351 | 0.0819   | 0.0701  | 0.0532 | 126%   | 193% |
|                |         |          |           | $\hat{\theta}$  | 0.0226  | 0.1294 | -0.2069 | 0.2511   | -0.0349 | 0.1146 | 52%    | 113% |
|                |         |          |           | $\hat{\lambda}$ | 0.0044  | 0.0332 | 0.0419  | 0.0671   | 0.0108  | 0.0108 | 49%    | 308% |
| 1.5            | 1.25    | 1.2      | 0.75      | $\hat{\alpha}$  | 0.1541  | 1.1054 | 0.2314  | 1.3205   | -0.1032 | 0.2479 | 84%    | 446% |
|                |         |          |           | $\hat{\beta}$   | 0.0987  | 0.1129 | -0.0088 | 0.0813   | 0.0573  | 0.0408 | 139%   | 277% |
|                |         |          |           | $\hat{\theta}$  | 0.0734  | 0.1386 | -0.0747 | 0.1154   | -0.0528 | 0.1275 | 120%   | 109% |
|                |         |          |           | $\hat{\lambda}$ | 0.0047  | 0.0576 | 0.0485  | 0.0998   | 0.0133  | 0.0502 | 58%    | 115% |
| 1.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 0.0495  | 0.5572 | 0.0591  | 0.5649   | -0.1580 | 0.1996 | 99%    | 279% |
|                |         |          |           | $\hat{\beta}$   | 0.0681  | 0.0694 | -0.0232 | 0.0550   | 0.0709  | 0.0407 | 126%   | 170% |
|                |         |          |           | $\hat{\theta}$  | 0.0357  | 0.0508 | -0.0718 | 0.0458   | -0.0134 | 0.0452 | 111%   | 112% |
|                |         |          |           | $\hat{\lambda}$ | -0.0013 | 0.0084 | 0.0136  | 0.0185   | 0.0014  | 0.0058 | 45%    | 144% |
| 0.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 0.2042  | 0.4829 | 0.1458  | 0.4827   | 0.1172  | 0.1234 | 100%   | 391% |
|                |         |          |           | $\hat{\beta}$   | 0.0832  | 0.1025 | -0.0264 | 0.0630   | 0.0301  | 0.0478 | 163%   | 214% |
|                |         |          |           | $\hat{\theta}$  | -0.1307 | 0.1124 | -0.0387 | 0.0697   | -0.1106 | 0.1037 | 161%   | 108% |
|                |         |          |           | $\hat{\lambda}$ | 0.1153  | 0.0706 | -0.0044 | 0.0429   | 0.0914  | 0.0677 | 164%   | 104% |
| 1.5            | 2.5     | 1.2      | 1.5       | $\hat{\alpha}$  | 0.1920  | 1.2139 | 0.3375  | 1.4913   | -0.0836 | 0.2863 | 81%    | 424% |
|                |         |          |           | $\hat{\beta}$   | 0.2286  | 0.5152 | -0.0451 | 0.3865   | 0.0678  | 0.1626 | 133%   | 317% |
|                |         |          |           | $\hat{\theta}$  | 0.1794  | 0.2082 | 0.0419  | 0.1273   | 0.1027  | 0.1613 | 164%   | 129% |
|                |         |          |           | $\hat{\lambda}$ | -0.1066 | 0.0975 | -0.0459 | 0.1139   | 0.0099  | 0.0820 | 86%    | 119% |

Table 4: Bias, MSE and RE for MLE, MPS and Bayesian of EGAPEx parameters with different Actuals Values when  $n = 200$

| Actuals values |         |          |           | MLE             |         | MPS    |         | Bayesian |         |        |        |       |
|----------------|---------|----------|-----------|-----------------|---------|--------|---------|----------|---------|--------|--------|-------|
| $\alpha$       | $\beta$ | $\theta$ | $\lambda$ | Bias1           | MSE     | Bias2  | MSE     | Bias     | MSE     | $RE_1$ | $RE_2$ |       |
| 0.75           | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 0.2011  | 0.6064 | 0.1798  | 0.6032   | -0.0028 | 0.0751 | 101%   | 807%  |
|                |         |          |           | $\hat{\beta}$   | 0.0367  | 0.0493 | -0.0129 | 0.0431   | 0.0191  | 0.0202 | 114%   | 244%  |
|                |         |          |           | $\hat{\theta}$  | 0.0262  | 0.0861 | 0.0249  | 0.0674   | 0.0083  | 0.0416 | 128%   | 207%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0328  | 0.1130 | -0.0644 | 0.1670   | 0.0064  | 0.0697 | 68%    | 162%  |
| 1.5            | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 0.2388  | 1.1194 | 0.2979  | 1.2094   | -0.0336 | 0.1006 | 93%    | 1113% |
|                |         |          |           | $\hat{\beta}$   | 0.0489  | 0.0509 | -0.0133 | 0.0433   | 0.0100  | 0.0162 | 117%   | 314%  |
|                |         |          |           | $\hat{\theta}$  | 0.0905  | 0.1303 | -0.0340 | 0.0919   | 0.0082  | 0.0438 | 142%   | 297%  |
|                |         |          |           | $\hat{\lambda}$ | -0.0239 | 0.1032 | 0.0624  | 0.1529   | 0.0185  | 0.0528 | 67%    | 195%  |
| 1.5            | 1.25    | 0.5      | 1.5       | $\hat{\alpha}$  | 0.0003  | 0.3797 | 0.1529  | 0.3450   | -0.0905 | 0.1640 | 110%   | 232%  |
|                |         |          |           | $\hat{\beta}$   | 0.0637  | 0.0484 | -0.0237 | 0.0327   | 0.0458  | 0.0239 | 148%   | 203%  |
|                |         |          |           | $\hat{\theta}$  | 0.0751  | 0.0476 | 0.0315  | 0.0303   | 0.0944  | 0.0429 | 157%   | 111%  |
|                |         |          |           | $\hat{\lambda}$ | -0.0878 | 0.0889 | -0.0371 | 0.0814   | -0.0816 | 0.0712 | 109%   | 125%  |
| 1.5            | 1.5     | 2.5      | 1.5       | $\hat{\alpha}$  | 0.3104  | 1.3989 | 0.2360  | 1.5026   | -0.0178 | 0.0338 | 93%    | 4139% |
|                |         |          |           | $\hat{\beta}$   | 0.0298  | 0.0544 | -0.0205 | 0.0534   | 0.0003  | 0.0128 | 102%   | 425%  |
|                |         |          |           | $\hat{\theta}$  | -0.0130 | 0.0748 | -0.1554 | 0.1087   | -0.0138 | 0.0260 | 69%    | 288%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0157  | 0.0793 | 0.0202  | 0.1778   | 0.0013  | 0.0167 | 45%    | 474%  |
| 2.5            | 1.5     | 2.5      | 0.75      | $\hat{\alpha}$  | 0.1835  | 1.4148 | 0.1649  | 1.5365   | -0.0163 | 0.0302 | 92%    | 4679% |
|                |         |          |           | $\hat{\beta}$   | 0.0387  | 0.0466 | -0.0247 | 0.0458   | 0.0042  | 0.0143 | 102%   | 327%  |
|                |         |          |           | $\hat{\theta}$  | 0.0365  | 0.1399 | -0.2154 | 0.2205   | -0.0195 | 0.0253 | 63%    | 553%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0027  | 0.0214 | 0.0624  | 0.0582   | 0.0081  | 0.0046 | 37%    | 460%  |
| 1.5            | 1.25    | 1.2      | 0.75      | $\hat{\alpha}$  | 0.1412  | 0.7696 | 0.1979  | 0.8841   | -0.0517 | 0.0927 | 87%    | 830%  |
|                |         |          |           | $\hat{\beta}$   | 0.0503  | 0.0471 | -0.0108 | 0.0403   | 0.0192  | 0.0179 | 117%   | 263%  |
|                |         |          |           | $\hat{\theta}$  | 0.0397  | 0.0823 | -0.0773 | 0.0795   | -0.0300 | 0.0626 | 104%   | 131%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0109  | 0.0393 | 0.0648  | 0.0687   | 0.0627  | 0.0346 | 57%    | 113%  |
| 1.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 0.0514  | 0.3562 | 0.0774  | 0.3544   | -0.0478 | 0.0934 | 100%   | 381%  |
|                |         |          |           | $\hat{\beta}$   | 0.0354  | 0.0276 | -0.0203 | 0.0231   | 0.0244  | 0.0152 | 119%   | 181%  |
|                |         |          |           | $\hat{\theta}$  | 0.0362  | 0.0394 | -0.0443 | 0.0356   | -0.0282 | 0.0180 | 111%   | 219%  |
|                |         |          |           | $\hat{\lambda}$ | -0.0022 | 0.0051 | 0.0119  | 0.0080   | 0.0167  | 0.0021 | 64%    | 247%  |
| 0.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 0.1168  | 0.2352 | 0.0725  | 0.2382   | 0.0148  | 0.0459 | 99%    | 513%  |
|                |         |          |           | $\hat{\beta}$   | 0.0432  | 0.0445 | -0.0150 | 0.0314   | 0.0252  | 0.0221 | 141%   | 202%  |
|                |         |          |           | $\hat{\theta}$  | -0.0783 | 0.0684 | 0.0075  | 0.0501   | -0.0794 | 0.0681 | 137%   | 100%  |
|                |         |          |           | $\hat{\lambda}$ | 0.0647  | 0.0374 | -0.0171 | 0.0259   | 0.0663  | 0.0248 | 144%   | 151%  |
| 1.5            | 2.5     | 1.2      | 1.5       | $\hat{\alpha}$  | 0.0632  | 0.3830 | 0.1259  | 0.4513   | -0.0067 | 0.0276 | 85%    | 1386% |
|                |         |          |           | $\hat{\beta}$   | 0.0974  | 0.1683 | -0.0442 | 0.1558   | -0.0080 | 0.0238 | 108%   | 707%  |
|                |         |          |           | $\hat{\theta}$  | 0.0691  | 0.0401 | 0.0088  | 0.0299   | 0.0026  | 0.0134 | 134%   | 299%  |
|                |         |          |           | $\hat{\lambda}$ | -0.0578 | 0.0241 | -0.0311 | 0.0270   | 0.0050  | 0.0183 | 89%    | 132%  |

**Table 5:** Asymptotic and bootstrap Confidence Intervals for MLE, MPS, and Bayesian of EGAPEx parameters with different Actuals Values when  $n = 50$

| Actuals values |         |          |           | MLE             |        |        | MPS    |        |        | Bayesian |        |        |        |
|----------------|---------|----------|-----------|-----------------|--------|--------|--------|--------|--------|----------|--------|--------|--------|
| $\alpha$       | $\beta$ | $\theta$ | $\lambda$ | L.CI            | LBP    | LBT    | L.CI   | LBP    | LBT    | L.CCI    | LBP    | LBT    |        |
| 0.75           | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 7.0741 | 0.8237 | 0.8077 | 6.9017 | 0.7957 | 0.7825   | 1.3044 | 0.3969 | 0.3782 |
|                |         |          |           | $\hat{\beta}$   | 1.7482 | 0.2047 | 0.1975 | 1.6097 | 0.2001 | 0.1997   | 0.4330 | 0.1520 | 0.1435 |
|                |         |          |           | $\hat{\theta}$  | 2.5375 | 0.3721 | 0.3607 | 2.4399 | 0.2240 | 0.2236   | 1.0937 | 0.3984 | 0.3969 |
|                |         |          |           | $\hat{\lambda}$ | 2.2802 | 0.3197 | 0.3079 | 2.1093 | 0.3130 | 0.3021   | 1.6174 | 0.2891 | 0.2847 |
| 1.5            | 1.25    | 1.2      | 1.5       | $\hat{\alpha}$  | 7.2979 | 0.9854 | 1.0372 | 7.0985 | 0.9236 | 0.9026   | 1.7521 | 3.4223 | 2.7759 |
|                |         |          |           | $\hat{\beta}$   | 2.2814 | 0.2959 | 0.2958 | 2.0235 | 0.2527 | 0.2490   | 1.2641 | 1.8334 | 1.6004 |
|                |         |          |           | $\hat{\theta}$  | 2.1419 | 0.2843 | 0.2793 | 2.0221 | 0.2697 | 0.2517   | 1.2170 | 0.3510 | 0.3512 |
|                |         |          |           | $\hat{\lambda}$ | 1.8509 | 0.2657 | 0.2662 | 1.7039 | 0.2566 | 0.2480   | 1.1746 | 0.1736 | 0.1774 |
| 1.5            | 1.25    | 0.5      | 1.5       | $\hat{\alpha}$  | 3.7469 | 0.5100 | 0.4875 | 3.6396 | 0.4576 | 0.4558   | 1.4222 | 0.2143 | 0.2182 |
|                |         |          |           | $\hat{\beta}$   | 1.4507 | 0.2081 | 0.2070 | 1.3451 | 0.1701 | 0.1743   | 0.6002 | 1.2652 | 1.0062 |
|                |         |          |           | $\hat{\theta}$  | 1.2188 | 0.2044 | 0.2030 | 1.2188 | 0.1018 | 0.1032   | 0.8868 | 0.1825 | 0.1885 |
|                |         |          |           | $\hat{\lambda}$ | 0.4984 | 0.0808 | 0.0826 | 0.4863 | 0.0806 | 0.0802   | 0.6946 | 0.3637 | 0.3467 |
| 1.5            | 1.5     | 2.5      | 1.5       | $\hat{\alpha}$  | 7.2979 | 0.9854 | 1.0372 | 7.3597 | 1.0359 | 1.0262   | 1.7521 | 3.4223 | 2.7759 |
|                |         |          |           | $\hat{\beta}$   | 2.2814 | 0.2959 | 0.2958 | 2.2694 | 0.2527 | 0.2590   | 1.2641 | 1.8334 | 1.6004 |
|                |         |          |           | $\hat{\theta}$  | 2.1419 | 0.2843 | 0.2793 | 2.0986 | 0.2697 | 0.2517   | 1.2170 | 0.3510 | 0.3512 |
|                |         |          |           | $\hat{\lambda}$ | 1.8509 | 0.2657 | 0.2662 | 1.9855 | 0.3656 | 0.3680   | 1.1746 | 0.1736 | 0.1774 |
| 2.5            | 1.5     | 2.5      | 0.75      | $\hat{\alpha}$  | 5.7927 | 0.9186 | 0.9151 | 5.6524 | 0.7230 | 0.7237   | 2.4159 | 0.5116 | 0.5164 |
|                |         |          |           | $\hat{\beta}$   | 1.8068 | 0.2816 | 0.2843 | 1.7653 | 0.1800 | 0.1813   | 0.6045 | 0.1069 | 0.1093 |
|                |         |          |           | $\hat{\theta}$  | 1.5421 | 0.2143 | 0.2206 | 1.6136 | 0.2290 | 0.2376   | 2.0197 | 0.3330 | 0.3214 |
|                |         |          |           | $\hat{\lambda}$ | 0.9033 | 0.1638 | 0.1626 | 0.8759 | 0.1460 | 0.1474   | 0.6732 | 0.1984 | 0.2000 |
| 1.5            | 1.25    | 1.2      | 0.75      | $\hat{\alpha}$  | 4.7354 | 0.6417 | 0.6361 | 4.6085 | 0.6951 | 0.6850   | 1.4971 | 0.2911 | 0.2815 |
|                |         |          |           | $\hat{\beta}$   | 1.6913 | 0.2259 | 0.2300 | 1.4954 | 0.2135 | 0.2183   | 0.3405 | 4.2267 | 3.7733 |
|                |         |          |           | $\hat{\theta}$  | 1.5168 | 0.2157 | 0.2202 | 1.3854 | 0.1666 | 0.1639   | 1.1046 | 0.2664 | 0.2679 |
|                |         |          |           | $\hat{\lambda}$ | 1.1334 | 0.1094 | 0.1097 | 1.3896 | 0.1320 | 0.1295   | 1.0304 | 0.1548 | 0.1563 |
| 1.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 3.7469 | 0.5100 | 0.4875 | 3.9517 | 0.5764 | 0.5583   | 1.4222 | 0.2143 | 0.2182 |
|                |         |          |           | $\hat{\beta}$   | 1.4507 | 0.2081 | 0.2070 | 1.2988 | 0.1701 | 0.1743   | 0.6002 | 1.2652 | 1.0062 |
|                |         |          |           | $\hat{\theta}$  | 1.2188 | 0.2044 | 0.2030 | 1.0192 | 0.1018 | 0.1032   | 0.8868 | 0.1825 | 0.1885 |
|                |         |          |           | $\hat{\lambda}$ | 0.4984 | 0.0808 | 0.0826 | 0.4095 | 0.0860 | 0.0824   | 0.6946 | 0.3637 | 0.3467 |
| 0.5            | 1.25    | 1.2      | 0.4       | $\hat{\alpha}$  | 3.2667 | 0.5483 | 0.5377 | 3.4585 | 0.6014 | 0.5898   | 0.7100 | 0.0812 | 0.0815 |
|                |         |          |           | $\hat{\beta}$   | 1.5788 | 0.2337 | 0.2313 | 1.2065 | 0.2029 | 0.2021   | 0.3644 | 0.1394 | 0.1396 |
|                |         |          |           | $\hat{\theta}$  | 1.4662 | 0.2102 | 0.2072 | 1.2114 | 0.1478 | 0.1476   | 0.7932 | 0.1356 | 0.1340 |
|                |         |          |           | $\hat{\lambda}$ | 1.2474 | 0.1845 | 0.1783 | 0.9579 | 0.1006 | 0.1005   | 0.6913 | 0.3306 | 0.3258 |
| 1.5            | 2.5     | 1.2      | 1.5       | $\hat{\alpha}$  | 5.2380 | 0.7025 | 0.7126 | 6.7710 | 0.8382 | 0.8434   | 1.6644 | 0.2194 | 0.2118 |
|                |         |          |           | $\hat{\beta}$   | 3.8567 | 0.5442 | 0.5390 | 3.4287 | 0.4258 | 0.4214   | 1.0752 | 0.2319 | 0.2455 |
|                |         |          |           | $\hat{\theta}$  | 2.1754 | 0.3597 | 0.3572 | 2.0534 | 0.3252 | 0.3216   | 0.9780 | 0.2127 | 0.2055 |
|                |         |          |           | $\hat{\lambda}$ | 1.5140 | 0.2202 | 0.2252 | 1.9995 | 0.2481 | 0.2470   | 1.3391 | 0.2319 | 0.2308 |

Table 6: Asymptotic and bootstrap Confidence Intervals for MLE, MPS, and Bayesian of EGAPEx parameters with different Actuals Values when  $n = 200$

| Actuals values |         |          |         | MLE             |        |        | MPS    |        |        | Bayesian |        |        |        |
|----------------|---------|----------|---------|-----------------|--------|--------|--------|--------|--------|----------|--------|--------|--------|
| $\alpha$       | $\beta$ | $\alpha$ | $\beta$ | L.CI            | LBP    | LBT    | L.CI   | LBP    | LBT    | L.CCI    | LBP    | LBT    |        |
| 0.75           | 1.25    | 1.2      | 1.5     | $\hat{\alpha}$  | 3.7610 | 0.4975 | 0.4903 | 3.5239 | 0.4194 | 0.4069   | 1.5484 | 0.2354 | 0.2477 |
|                |         |          |         | $\hat{\beta}$   | 1.8985 | 0.2721 | 0.2796 | 1.7032 | 0.2090 | 0.2113   | 0.9078 | 0.1915 | 0.1882 |
|                |         |          |         | $\hat{\theta}$  | 1.7976 | 0.2286 | 0.2344 | 1.6804 | 0.1858 | 0.1902   | 0.9978 | 1.4857 | 1.1814 |
|                |         |          |         | $\hat{\lambda}$ | 1.6229 | 0.2425 | 0.2430 | 1.5986 | 0.2346 | 0.2359   | 1.2127 | 0.2510 | 0.2471 |
| 1.5            | 1.25    | 1.2      | 1.5     | $\hat{\alpha}$  | 4.7354 | 0.6417 | 0.6361 | 4.5735 | 0.6395 | 0.6385   | 1.4971 | 0.2911 | 0.2815 |
|                |         |          |         | $\hat{\beta}$   | 1.6913 | 0.2259 | 0.2300 | 1.5639 | 0.2135 | 0.2183   | 0.3405 | 4.2267 | 3.7733 |
|                |         |          |         | $\hat{\theta}$  | 1.5168 | 0.2157 | 0.2202 | 1.4895 | 0.1666 | 0.1639   | 1.1046 | 0.2664 | 0.2679 |
|                |         |          |         | $\hat{\lambda}$ | 1.1334 | 0.1094 | 0.1097 | 1.0270 | 0.0913 | 0.0930   | 1.0304 | 0.1548 | 0.1563 |
| 1.5            | 1.25    | 0.5      | 1.5     | $\hat{\alpha}$  | 5.2380 | 0.7025 | 0.7126 | 5.2965 | 0.6838 | 0.6843   | 1.6644 | 0.2194 | 0.2118 |
|                |         |          |         | $\hat{\beta}$   | 3.8567 | 0.5442 | 0.5390 | 3.6599 | 0.4258 | 0.4214   | 1.0752 | 0.2319 | 0.2455 |
|                |         |          |         | $\hat{\theta}$  | 2.1754 | 0.3597 | 0.3572 | 2.0318 | 0.3252 | 0.3216   | 0.9780 | 0.2127 | 0.2055 |
|                |         |          |         | $\hat{\lambda}$ | 1.5140 | 0.2202 | 0.2252 | 1.4951 | 0.2148 | 0.2147   | 1.3391 | 0.1923 | 0.1920 |
| 1.5            | 1.5     | 2.5      | 1.5     | $\hat{\alpha}$  | 4.5075 | 0.3252 | 0.3224 | 4.5953 | 0.3271 | 0.3220   | 0.4045 | 0.0365 | 0.0370 |
|                |         |          |         | $\hat{\beta}$   | 0.9934 | 0.0709 | 0.0687 | 0.9899 | 0.0721 | 0.0714   | 0.0925 | 0.0534 | 0.0529 |
|                |         |          |         | $\hat{\theta}$  | 1.0863 | 0.0785 | 0.0799 | 1.1720 | 0.0659 | 0.0670   | 0.3491 | 0.0296 | 0.0292 |
|                |         |          |         | $\hat{\lambda}$ | 1.0698 | 0.0749 | 0.0753 | 1.0716 | 0.1011 | 0.0998   | 0.2017 | 0.0211 | 0.0203 |
| 2.5            | 1.5     | 2.5      | 0.75    | $\hat{\alpha}$  | 4.8702 | 0.3439 | 0.3541 | 4.7330 | 0.2873 | 0.2905   | 0.4071 | 0.0439 | 0.0437 |
|                |         |          |         | $\hat{\beta}$   | 0.9081 | 0.0676 | 0.0689 | 0.8912 | 0.0614 | 0.0622   | 0.0902 | 0.0239 | 0.0235 |
|                |         |          |         | $\hat{\theta}$  | 1.5203 | 0.1020 | 0.0989 | 1.6099 | 0.1125 | 0.1119   | 0.3375 | 0.0382 | 0.0382 |
|                |         |          |         | $\hat{\lambda}$ | 0.6356 | 0.0404 | 0.0423 | 0.6599 | 0.0617 | 0.0623   | 0.0456 | 0.0125 | 0.0119 |
| 1.5            | 1.25    | 1.2      | 0.75    | $\hat{\alpha}$  | 3.2678 | 0.2357 | 0.2307 | 3.4098 | 0.2460 | 0.2435   | 0.4159 | 0.2984 | 0.2679 |
|                |         |          |         | $\hat{\beta}$   | 0.7801 | 0.0566 | 0.0572 | 0.7295 | 0.0546 | 0.0544   | 0.0976 | 0.4522 | 0.4418 |
|                |         |          |         | $\hat{\theta}$  | 1.1645 | 0.0797 | 0.0806 | 1.0987 | 0.0763 | 0.0774   | 0.2779 | 0.0227 | 0.0222 |
|                |         |          |         | $\hat{\lambda}$ | 0.8608 | 0.0588 | 0.0598 | 0.8608 | 0.0783 | 0.0821   | 0.1580 | 0.0128 | 0.0131 |
| 1.5            | 1.25    | 1.2      | 0.4     | $\hat{\alpha}$  | 2.2994 | 0.1645 | 0.1698 | 2.3970 | 0.1740 | 0.1790   | 0.4264 | 0.1224 | 0.1126 |
|                |         |          |         | $\hat{\beta}$   | 0.6695 | 0.0490 | 0.0490 | 0.6209 | 0.0444 | 0.0447   | 0.0674 | 0.3282 | 0.2574 |
|                |         |          |         | $\hat{\theta}$  | 0.7986 | 0.0521 | 0.0516 | 0.7871 | 0.0518 | 0.0527   | 0.2900 | 0.0398 | 0.0395 |
|                |         |          |         | $\hat{\lambda}$ | 0.2801 | 0.0208 | 0.0212 | 0.2924 | 0.0283 | 0.0291   | 0.1161 | 0.0141 | 0.0140 |
| 0.5            | 1.25    | 1.2      | 0.4     | $\hat{\alpha}$  | 1.8910 | 0.1211 | 0.1180 | 1.9648 | 0.1390 | 0.1385   | 0.1890 | 0.0175 | 0.0172 |
|                |         |          |         | $\hat{\beta}$   | 0.8395 | 0.0583 | 0.0576 | 0.7346 | 0.0527 | 0.0529   | 0.0653 | 0.0781 | 0.0755 |
|                |         |          |         | $\hat{\theta}$  | 0.9427 | 0.0620 | 0.0632 | 0.8080 | 0.0565 | 0.0558   | 0.2458 | 0.0267 | 0.0274 |
|                |         |          |         | $\hat{\lambda}$ | 0.6061 | 0.0439 | 0.0437 | 0.5618 | 0.0416 | 0.0416   | 0.1134 | 0.0137 | 0.0138 |
| 1.5            | 2.5     | 1.2      | 1.5     | $\hat{\alpha}$  | 2.9053 | 0.2036 | 0.2088 | 3.1092 | 0.2116 | 0.2179   | 0.4102 | 0.0403 | 0.0399 |
|                |         |          |         | $\hat{\beta}$   | 1.7501 | 0.1186 | 0.1203 | 1.6794 | 0.1117 | 0.1128   | 0.2994 | 0.0293 | 0.0294 |
|                |         |          |         | $\hat{\theta}$  | 0.8366 | 0.0636 | 0.0633 | 0.7120 | 0.0507 | 0.0495   | 0.2013 | 0.0188 | 0.0191 |
|                |         |          |         | $\hat{\lambda}$ | 0.6388 | 0.0467 | 0.0467 | 0.7207 | 0.0488 | 0.0484   | 0.3162 | 0.0299 | 0.0294 |

### 7. Two Applications of Real Data

This section is presented to illustrate the potentiality of the EGAPEx distribution for two real data sets and compared with other competitive models, namely: the exponentiated generalized exponential (EGEx), Alpha Power Generalized Exponential (APGEx), Alpha Power Exponential (APEX), Generalized Exponential (GEX) and exponential (Ex)



distributions. To compare the fits of all models, we consider the Kolmogorov Smirnov (K-S) statistic and its p-value, and some different criteria measures as Akaike information criterion (AIC), corrected AIC (CAIC), Hannan-Quinn information criterion (HQIC), Cramér-von Mises (CVM) and Anderson-Darling (AD).

The first data: Almetwally et al. (2019) introduced the real data set consists of economic data set consists of 31 observations subjected to a GDP growth of Egypt. The data are given below: Data1: 10.01, 3.76, 9.91, 7.40, 6.09, 6.60, 2.65, 2.52, 7.93, 4.97, 5.70, 1.08, 4.43, 2.90, 3.97, 4.64, 4.99, 5.49, 4.03, 6.11, 5.37, 3.54, 2.37, 3.19, 4.09, 4.48, 6.85, 7.09, 7.16, 4.67, 5.15.

The second data: Badar and Priest (1982) discussed the real data set of sample size 69 observed failure times, the data set is represented the data measured in GPA, for single carbon fibers and impregnated 1000 carbon fiber tows. Data 2: 0.562, 0.564, 0.729, 1.216, 1.474, 1.632, 1.816, 2.020, 2.317, 1.247, 1.490, 1.676, 1.824, 2.023, 2.334, 1.256, 1.503, 1.684, 1.836, 2.050, 2.340, 0.802, 1.271, 1.520, 1.685, 1.879, 2.059, 2.346, 0.950, 1.277, 1.522, 1.728, 1.883, 2.068, 2.378, 1.053, 1.305, 1.524, 1.740, 1.892, 2.071, 2.483, 1.111, 1.348, 1.551, 1.764, 1.934, 2.130, 2.835, 1.115, 1.313, 1.551, 1.761, 1.898, 2.098, 2.683, 1.194, 1.390, 1.609, 1.785, 1.947, 2.204, 2.835, 1.208, 1.429, 1.632, 1.804, 1.976, 2.262.

Table 7: MLE, K-S and p-values for data 1

| Model  | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\lambda}$ | K-S    | P-Value |
|--------|----------------|---------------|----------------|-----------------|--------|---------|
| EGAPEX | 38.8065        | 2.6267        | 7.1558         | 0.1695          | 0.0486 | 0.99999 |
| APEX   | 142.8432       | -             | -              | 0.4311          | 0.1460 | 0.4794  |
| APGEX  | 170.8721       | -             | 3.0280         | 0.2548          | 0.1288 | 0.6362  |
| Ex     | -              | -             | -              | 0.1948          | 0.3375 | 0.0012  |
| GEX    | -              | -             | 8.3654         | 0.0233          | 0.3377 | 0.0012  |
| EGEX   | -              | 7.4348        | 14.2389        | 0.0358          | 0.0906 | 0.9411  |

Table 8: AIC, BIC, CAIC, HQIC, CVM and AD for data 1

| Model  | AIC      | BIC      | CAIC     | HQIC     | CVM    | AD     |
|--------|----------|----------|----------|----------|--------|--------|
| EGAPEX | 138.9972 | 143.5331 | 140.1356 | 140.5670 | 0.0143 | 0.1320 |
| APEX   | 140.8450 | 143.7129 | 141.2735 | 141.7799 | 0.0153 | 0.1357 |
| APGEX  | 140.5094 | 144.8114 | 141.3983 | 141.9118 | 0.0150 | 0.1349 |
| Ex     | 165.4194 | 166.8534 | 165.5574 | 165.8869 | 0.0216 | 0.1838 |
| GEX    | 167.4194 | 170.2874 | 167.8480 | 168.3543 | 0.0216 | 0.1838 |
| EGEX   | 139.3125 | 143.6145 | 140.2014 | 140.7149 | 0.0362 | 0.2770 |

Table 9: MLE, K-S and p-values for data 2

| Model  | $\hat{\alpha}$ | $\hat{\beta}$ | $\hat{\theta}$ | $\hat{\lambda}$ | K-S    | P-Value |
|--------|----------------|---------------|----------------|-----------------|--------|---------|
| EGAPEX | 134.8556       | 4.8093        | 56.8063        | 0.2587          | 0.0516 | 0.9929  |
| APEX   | 193.9359       | -             | -              | 1.3186          | 0.2165 | 0.0031  |
| APGEX  | 247.4593       | -             | 3.6256         | 0.7610          | 0.2147 | 0.0034  |
| Ex     | -              | -             | -              | 0.5878          | 0.3926 | 0.0000  |
| GEX    | -              | -             | 124.2706       | 0.0047          | 0.3926 | 0.0000  |
| EGEX   | -              | 16.6852       | 4.3895         | 0.4498          | 0.1030 | 0.4570  |

Table 10: AIC, BIC, CAIC, HQIC, CVM and AD for data 2

| Model  | AIC      | BIC      | CAIC     | HQIC     | CVM    | AD     |
|--------|----------|----------|----------|----------|--------|--------|
| EGAPEX | 107.8578 | 116.7942 | 108.4828 | 111.4032 | 0.0419 | 0.3165 |
| APEX   | 140.6218 | 145.0900 | 140.8036 | 142.3945 | 0.0710 | 0.5052 |
| APGEX  | 130.8831 | 137.5854 | 131.2523 | 133.5421 | 0.0510 | 0.3399 |
| Ex     | 213.3396 | 215.5737 | 213.3993 | 214.2259 | 0.0849 | 0.6067 |
| GEX    | 215.3396 | 219.8078 | 215.5214 | 217.1123 | 0.0849 | 0.6067 |
| EGEX   | 117.0317 | 123.7340 | 117.4009 | 119.6907 | 0.1750 | 1.1913 |

The MLE of the parameters for EGAPEx distribution, with the values of the K-S statistic with p-values are reported in Tables 7-10. First, we check whether the considered data sets actually come EGAPEx distribution or not by goodness-of-fit test and compare the fits with the exponentiated generalized exponential (EGEx), alpha power generalized exponential (APGEx), alpha power exponential (APEX), generalized exponential (GEx), exponential (Ex) distributions. The EGAPEx distribution gives the lowest value for the K-S, AIC, BIC, CAIC, and HAIC statistics among all the fitted models to these data sets, also The EGAPEx distribution gives the highest value for the P-Value comparing another distribution.

## 8. Conclusion

In this paper, we introduced a new three-parameter model, called the exponentiated generalized alpha power family. We used the new family to discuss a new four-parameter model, called the EGAPEx distribution which extends the exponential distribution. The EGAPEx distribution is motivated by the wide utilization of the exponential distribution in life testing and provides more flexibility to analyze lifetime data. We obtain some mathematical properties such as the quantile function, moments, generating function, characteristic function, cumulant-generating function, hazard rate function, and stress–strength reliability. We provide some applications of EGAPEx distribution in the context of statistics. The parameter estimation of EGAPEx distribution is derived by MLE, MPS, and Bayesian. The methods of estimation are employed to estimate the model parameters and simulation results are provided to assess the model performance. Two real-life data proposed model provides a consistently better fit than the APEX, APGEx, EGEx, GEx and Ex distributions.

## References

- Almetwally, E. M. (2022a). The odd Weibull inverse topp–leone distribution with applications to COVID-19 data. *Annals of Data Science*, 9(1), 121-140.
- Almetwally, E. M. (2022b). Marshall olkin alpha power extended Weibull distribution: different methods of estimation based on type I and type II censoring. *Gazi University Journal of Science*, 35(1), 293-312.
- Almetwally, E. M., & Almongy, H. M. (2019a). Estimation Methods for the New Weibull-Pareto Distribution: Simulation and Application. *Journal of Data Science*, 17(3), 610-630.
- Almetwally, E. M., & Almongy, H. M. (2019b). Maximum Product Spacing and Bayesian Method for Parameter Estimation for Generalized Power Weibull Distribution under Censoring Scheme. *Journal of Data Science*, 17(2), 407-444.
- Almetwally, E. M., Almongy, H. M., & El sayed Mubarak, A. (2018). Bayesian and Maximum Likelihood Estimation for the Weibull Generalized Exponential Distribution Parameters Using Progressive Censoring Schemes. *Pakistan Journal of Statistics and Operation Research*, 14(4), 853-868.
- Almetwally, E. M., Almongy, H. M., & El-Sherpieny, E. A. (2019). Adaptive Type-II Progressive Censoring Schemes based on Maximum Product Spacing with Application of Generalized Rayleigh Distribution. *Journal of Data Science*, 17(4), 767 - 793.
- Almongy, H. M., Almetwally, E. M., & Mubarak, A. E. (2021). Marshall–Olkin alpha power lomax distribution: Estimation methods, applications on physics and economics. *Pakistan Journal of Statistics and Operation Research*, 17(1), 137-153.
- Almongy, H. M., Alshenawy, F. Y., Almetwally, E. M., & Abdo, D. A. (2021). Applying transformer insulation using Weibull extended distribution based on progressive censoring scheme. *Axioms*, 10(2), 100.
- Alshenawy, R., Sabry, M. A., Almetwally, E. M., & Almongy, H. M. (2021). Product spacing of stress–strength under progressive hybrid censored for exponentiated-gumbel distribution. *Computers, Materials & Continua*, 66(3), 2973-2995.
- Andrade, T., Rodrigues, H., Bourguignon, M., & Cordeiro, G. (2015). The exponentiated generalized Gumbel distribution. *Revista Colombiana de Estadística*, 38(1), 123-143.
- Aryal, G., & Elbatal, I. (2015). On the exponentiated generalized modified Weibull distribution. *Communications for Statistical Applications and Methods*, 22(4), 333-348.
- Bader, M. G., & Priest, A. M. (1982). Statistical aspects of fibre and bundle strength in hybrid composites. *Progress in science and engineering of composites*, 1129-1136.
- Basheer, A. M. (2019). Alpha power inverse Weibull distribution with reliability application. *Journal of Taibah University for Science*, 13(1), 423-432.
- Basheer, A. M., Almetwally, E. M., & Okasha, H. M. (2021). Marshall-olkin alpha power inverse Weibull distribution: non bayesian and bayesian estimations. *Journal of Statistics Applications & Probability*, 10(2), 327-345.

- Cordeiro, G. M., & Lemonte, A. J. (2014). The exponentiated generalized Birnbaum–Saunders distribution. *Applied Mathematics and Computation*, 247, 762-779.
- Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11(1), 1-27.
- De Andrade, T. A., Bourguignon, M., & Cordeiro, G. M. (2016). The exponentiated generalized extended exponential distribution. *Journal of Data Science*, 14(3), 393-413.
- Dey, S., Ghosh, I., & Kumar, D. (2018). Alpha-power transformed lindley distribution: properties and associated inference with application to earthquake data. *Annals of Data Science*, 1-28.
- Dey, S., Nassar, M., & Kumar, D. (2019). Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function. *Journal of Computational and Applied Mathematics*, 348, 130-145.
- Eghwerido, J. T., Utoyo-Amrevughere, O. T. and Efe-Eyefia, E. (2020). The alpha power Weibull Frechet distribution: properties and applications. *Turkish Journal of Science*. 5(3), 170–185.
- Elbatal, I., Ahmad, Z., Elgarhy, B. M., & Almarashi, A. M. (2018). A New Alpha Power Transformed Family of Distributions: Properties and Applications to the Weibull Model. *Journal of Nonlinear Science and Applications*, 12(1), 1-20.
- Elgarhy, M., ulHaq, M. A., & ulAin, Q. (2018). Exponentiated Generalized Kumaraswamy Distribution with Applications. *Annals of Data Science*, 5(2), 273-292.
- Handique, L., Chakraborty, S., & de Andrade, T. A. (2018). The Exponentiated Generalized Marshall–Olkin Family of Distribution: Its Properties and Applications. *Annals of Data Science*, 1-21.
- Hassan, A. S., & Abd-Allah, M. (2019). On the Inverse Power Lomax Distribution. *Annals of Data Science*, 6(2), 259-278.
- Mahdavi, A., & Kundu, D. (2017). A new method for generating distributions with an application to exponential distribution. *Communications in Statistics-Theory and Methods*, 46(13), 6543-6557.
- Nasiru, S., Mwita, P. N., & Ngesa, O. (2018). Exponentiated generalized power series family of distributions. *Annals of Data Science*, 1-27.
- Nassar, M., Alzaatreh, A., Mead, M., & Abo-Kasem, O. (2017). Alpha power Weibull distribution: Properties and applications. *Communications in Statistics-Theory and Methods*, 46(20), 10236-10252.
- Reyad, H. M., Alizadeh, M., Jamal, F., Othman, S., & Hamedani, G. G. (2019). The Exponentiated Generalized Topp Leone-G Family of Distributions: Properties and Applications. *Pakistan Journal of Statistics and Operation Research*, 15(1), 1-24.
- Riad, F. H., Alruwaili, B., Almetwally, E. M., & Hussam, E. (2022). Fuzzy Reliability Analysis of the COVID-19 Mortality Rate Using a New Modified Kies Kumaraswamy Model. *Journal of Mathematics*, 2022. <https://doi.org/10.1155/2022/3427521>
- Sabry, M. A., Almetwally, E. M., Alamri, O. A., Yusuf, M., Almongy, H. M., & Eldeeb, A. S. (2021). Inference of fuzzy reliability model for inverse Rayleigh distribution. *Aims Math*, 6(9), 9770-9785.
- Smith, R. L., & Naylor, J. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 36(3), 358-369.