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A new bivariate distribution with Gompertz marginals

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Abstract

In this paper, we propose a new bivariate distribution with the Gompertz marginals. Some properties of this new bivariate distribution have been investigated. Several properties of this distribution have been discussed. Parameters estimation using moments and maximum likelihood methods are obtained. A numerical illustration experiments have been performed to see the behavior of the MLEs. One data set has been analyzed for illustrative purpose.

Keywords: *Bivariate model; Gompertz distribution function; maximum likelihood estimators; moment estimators; fisher information matrix.*

1 Introduction

The Gompertz distribution was originally introduced by Gompertz [8].This distribution is widely used to describe human mortality and establish actuarial tables. It has been used as a growth model and also used to fit the tumor growth. The Gompertz distribution is related by a simple transformation to certain distribution in the family of distributions obtained by Pearson. Applications and more recent survey of the Gompertz distribution can be introduced by Ahuja and Nash [2].
In many practical problems, multivariate lifetime data arise frequently, and in these situations it is important to consider different multivariate models that could be used to model such multivariate lifetime data. For an encyclopedia treatment on various multivariate models and their properties and applications, one may refer to the book by Kotz et al. [9].
In fact, shock models are used in reliability to describe different applications. Shocks can refer for example to damage caused to biological organs by illness or environmental causes of damage acting on a technical system, El-Gohary and Sarhan [6], and A-hameed and Proschan [1]. Also Al-Ruzaiza and El-Gohary have obtained a new class of bivariate distribution with Pareto of Marshall-Olkin type [4].
The objective of this paper is to introduce a new bivariate Gompertz distribution of Marsall-Olkin type. It is considered as a distribution of the life times of two dependent components each has a Gompertz distribution. Also discuss about the computation of the maximum likelihood estimators and moment generating function.
The paper is organized as follows. Section 2 presents the shock model yielding the new bivariate Gompertz distribution. The joint survival and probability density function of bivariate Gompertz distribution is obtained. Section 3 presents the joint moment generating function of this bivariate distribution and its marginal moment generating functions. Section 4 discusses the maximum likelihood estimation of proposed new bivariate Gompertz distributions. Section 5 presents the simulation and one data analysis results. Finally we conclude the paper in section 6.

2 The new bivariate Gompertz distribution

We define a new bivariate Gompertz distribution (NBG), shortly denoted by NBG distribution. We start with the joint survival function of the distribution and then derive the corresponding joint probability density function.

2.1 The Joint Survival Function

It is assumed that the univariate Gompertz distribution with the shape parameter α>0 and the scale parameter λ>0 has the following probability density function, cumulative distribution function and survival function for x>0;

$$f(x;\alpha,\lambda) = \alpha e^{\lambda x} e^{\frac{-\alpha}{\lambda}(e^{\lambda x}-1)}, \tag{1}$$

$$F(x;\alpha,\lambda) = e^{\frac{-\alpha}{\lambda}\left(e^{\lambda x}-1\right)} \tag{2}$$

respectively. Suppose U_1 follows$(\sim)G(\alpha_1,\lambda)$, $U_2\sim G(\alpha_2,\lambda)$, $U_3\sim G(\alpha_3,\lambda)$ and they are mutually independent. Now define $X_1 = min(U_1,U_3)$ and $X_2 = min(U_2,U_3)$ then the bivariate vector (X_1, X_2) has the new bivariate Gompertz distribution with the parameters $(\alpha_1,\alpha_2,\alpha_3,\lambda)$, and it will be denoted by $NBG(\alpha_1,\alpha_2,\alpha_3,\lambda)$ distribution.

We now study the joint survival distribution of the random variables X_1 and X_2. The following lemma gives the joint survival function of X_1 and X_2, which is the survival function of the NBG distribution.

Lemma2.1: *The joint survival functions of X_1 and X_2 is*

$$S_{NBG}(x_1,x_2) = \left(e^{\frac{-\alpha_1}{\lambda}\left(e^{\lambda x_1}-1\right)}\right)\left(e^{\frac{-\alpha_2}{\lambda}\left(e^{\lambda x_2}-1\right)}\right)\left(e^{\frac{-\alpha_3}{\lambda}\left(e^{\lambda z}-1\right)}\right) \tag{3}$$

where $z = max(x_1,x_2)$

Proof: *Since*

$$S_{NBG}(x_1,x_2) = P(X_1 > x_1, X_2 > x_2)$$

we have

$$\begin{aligned} S_{NBG}(x_1,x_2) &= P(min(U_1,U_3) > x_1, min(U_2,U_3) > x_2) \\ &= P(U_1 > x_1, U_3 > x_1, U_2 > x_2, U_3 > x_2) \\ &= P(U_1 > x_1, U_2 > x_2, U_3 > max(x_1,x_2)) \end{aligned}$$

as $U_i(i = 1,2,3)$ are mutually independent, we readily obtain

$$S_{NBG}(x_1,x_2) = P(U_1 > x_1)P(U_2 > x_2)P\big(U_3 > max(x_1,x_2)\big)$$

$$S_{NBG}(x_1,x_2) = S_G(x_1;\alpha_1,\lambda)S_G(x_2;\alpha_2,\lambda)S_G(z;\alpha_3,\lambda) \tag{4}$$

Substituting from (2) into (4), we obtain (3), which completes the proof of the lemma.

Corollary: *The joint survival function of the $NBG(\alpha_1,\alpha_2,\alpha_3,\lambda)$ can also be written as*

$$S_{X_1,X_2}(x_1,x_2) = S_G(x_1;\alpha_1,\lambda)S_G(x_2;\alpha_2,\lambda)S_G(z;\alpha_3,\lambda)$$

$$S_{X_1,X_2}(x_1,x_2) = \begin{cases} S_G(x_1;(\alpha_1+\alpha_3),\lambda)S_G(x_2;\alpha_2,\lambda) & \text{if } x_2 < x_1 \\ S_G(x_1;\alpha_1,\lambda)S_G(x_2;(\alpha_2+\alpha_3),\lambda) & \text{if } x_1 < x_2 \\ S_G\big((x;(\alpha_1+\alpha_2+\alpha_3),\lambda)\big) & \text{if } x_1 = x_2 = x \end{cases}$$

2.2 The joint probability density function

The following theorem gives the joint probability density function of the NBG distribution.

Theorem 2.1: *If the joint survival function of (X_1, X_2) is as in (3), the joint probability density function of (X_1, X_2) is given by*

$$f_{NBG}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & \text{if } x_2 < x_1 \\ f_2(x_1,x_2) & \text{if } x_1 < x_2 \\ f_3(x,x) & \text{if } x_1 = x_2 = x \end{cases}$$

where

$$f_1(x_1,x_2) = (\alpha_1+\alpha_3)\alpha_2 e^{\lambda x_1}e^{\lambda x_2}\left(e^{\frac{-(\alpha_1+\alpha_3)}{\lambda}\left(e^{\lambda x_1}-1\right)}\right)\left(e^{\frac{-\alpha_2}{\lambda}\left(e^{\lambda x_2}-1\right)}\right) \tag{5}$$

$$f_2(x_1,x_2) = \alpha_1(\alpha_2+\alpha_3)e^{\lambda x_1}e^{\lambda x_2}\left(e^{\frac{-\alpha_1}{\lambda}\left(e^{\lambda x_1}-1\right)}\right)\left(e^{\frac{-(\alpha_2+\alpha_3)}{\lambda}\left(e^{\lambda x_2}-1\right)}\right) \tag{6}$$

$$f_3(x,x) = \alpha_3 e^{\lambda x}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}\left(e^{\lambda x}-1\right)}\right) \tag{7}$$

Proof: *Let us first assume that $x_2 < x_1$ In this case, $S_{NBG}(x_1,x_2)$ in (3) becomes*

$$S_1(x_1,x_2) = \left(e^{\frac{-(\alpha_1+\alpha_3)}{\lambda}\left(e^{\lambda x_1}-1\right)}\right)\left(e^{\frac{-\alpha_2}{\lambda}\left(e^{\lambda x_2}-1\right)}\right).$$

Then, upon differentiation, we obtain the expression of $f_{NBG}(x_1,x_2) = \frac{\partial^2 S_{NBG}(x_1,x_2)}{\partial x_1 \partial x_2}$ distribution to be $f_1(x_1,x_2)$ given in (5). Similarly, we find the expression of $f_{NBG}(x_1,x_2)$ distribution to be $f_2(x_1,x_2)$ when $x_1 < x_2$ but, $f_3(x,x)$ cannot be derived in a similar way. For this reason, we use the identity

$$\int_0^\infty \int_{x_2}^\infty f_1(x_1,x_2)\,dx_1 dx_2 + \int_0^\infty \int_{x_1}^\infty f_2(x_1,x_2)\,dx_2 dx_1 + \int_0^\infty f_3(x,x)\,dx = 1 \quad (8)$$

One can verify that

$$I_1 = \int_0^\infty \alpha_2 e^{\lambda x_2}\left(e^{\frac{-\alpha_2}{\lambda}(e^{\lambda x_2}-1)}\right)\int_{x_2}^\infty (\alpha_1+\alpha_3)e^{\lambda x_1}\left(e^{\frac{-(\alpha_1+\alpha_3)}{\lambda}(e^{\lambda x_1}-1)}\right)dx_1 dx_2$$

$$I_1 = \int_0^\infty \alpha_2 e^{\lambda x_2}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x_2}-1)}\right)dx_2 \quad (9)$$

and

$$I_2 = \int_0^\infty \alpha_1 e^{\lambda x_1}\left(e^{\frac{-\alpha_1}{\lambda}(e^{\lambda x_1}-1)}\right)\int_{x_1}^\infty (\alpha_2+\alpha_3)e^{\lambda x_2}\left(e^{\frac{-(\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x_2}-1)}\right)dx_2 dx_1$$

$$I_2 = \int_0^\infty \alpha_1 e^{\lambda x_1}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x_1}-1)}\right)dx_1 \quad (11)$$

From (10) and (11), we then get

$$I_3 = \int_0^\infty (\alpha_1+\alpha_2+\alpha_3)e^{\lambda x}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x}-1)}\right)dx - \int_0^\infty \alpha_2 e^{\lambda x}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x}-1)}\right)dx$$

$$-\int_0^\infty \alpha_1 e^{\lambda x}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x}-1)}\right)dx$$

$$\int_0^\infty f_3(x,x)\,dx = \int_0^\infty \alpha_3 e^{\lambda x}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x}-1)}\right)dx$$

$$f_3(x,x) = \alpha_3 e^{\lambda x}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}(e^{\lambda x}-1)}\right) \quad (12)$$

This completes the proof of the theorem.

2.3 Marginal and conditional probability density functions

In this section, we derive the marginal density functions of X_i and the conditional density functions of $X_i|X_j$, $i \neq j = 1,2$. We also present the joint moment generating function of X_1 and X_2.

Theorem 2.2: *The marginal pdf of* $X_i (i = 1,2)$ *is given by*

$$f_{X_i(x_i)} = (\alpha_i+\alpha_3)e^{\lambda x_i}\left(e^{\frac{-(\alpha_i+\alpha_3)}{\lambda}(e^{\lambda x_i}-1)}\right), \quad x_i > 0, i = 1,2. \quad (13)$$

Proof: *The marginal pdf of* X_i *can be derived from the marginal survival function of* X_i *say* $S(x_i)$, *as follows*:

$$S_{X_i}(x_i) = P(X_i > x_i) = P(min(U_i, U_3) > x_i) = P(U_i > x_i, U_3 > x_i)$$

And since U_i is independent of U_3, we simply have

$$S_{X_i}(x_i) = \left(e^{\frac{-(\alpha_i+\alpha_3)}{\lambda}(e^{\lambda x_i}-1)}\right)$$

From which we readily derive the pdf of X_i, $S_{X_i}(x_i) = -\frac{\partial(x_i)}{\partial x_i}$, as in (13).

2.4 Conditional probability density functions

Having obtained the marginal probability density functions of X_1 and X_2, we can now derive the conditional probability density functions as presented in the following theorem.

Theorem 2.3: *The conditional pdf of* X_i*, given* $X_j = x_j$*, denoted by* $f_{i|j}(x_i|x_j)$. $(i \neq j = 1,2)$*, is given by*

$$f_{X_i|X_j}(x_i|x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & \text{if } x_j < x_i, \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & \text{if } x_i < x_j, \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & \text{if } x_1 = x_2 = x, \end{cases} \tag{14}$$

Where

$$f^{(1)}_{X_i|X_j}(x_i|x_j) = \frac{(\alpha_1 + \alpha_3)\alpha_2 e^{\lambda x_i}\left(e^{\frac{-(\alpha_i+\alpha_3)}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)}{(\alpha_2 + \alpha_3)\left(e^{\frac{-\alpha_3}{\lambda}\left(e^{\lambda x_j}-1\right)}\right)}$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j) = \alpha_1 e^{\lambda x_i}\left(e^{\frac{-\alpha_1}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)$$

$$f^{(3)}_{X_i|X_j}(x_i|x_j) = \frac{\alpha_3\left(e^{\frac{-\alpha_1}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)}{(\alpha_2 + \alpha_3)}$$

Proof: *The theorem follows readily upon substituting for the joint pdf of* (X_1, X_2) *in (6), (7) and (8) and the marginal pdf of* X_i $(i = 1,2)$ *in (13), in the relation*

$$f_{X_i|X_j}(x_i|x_j) = \frac{f_{X_i,X_j}(x_i, x_j)}{f_{X_j}(x_j)}, \quad (i = 1,2)$$

3 Moment generating functions

In this subsection, we present the joint moment generating function of (X_1, X_2) and the marginal moment generating function of X_i $(i = 1,2)$.

Lemma 3.1 *The moment generating function of* X_i $(i = 1,2)$ *is given by*

$$M_{X_i}(t_i) = (\alpha_i + \alpha_3)e^{\alpha_i+\alpha_3}\sum_{j=0}^{\infty}\frac{(-1)^j(\alpha_i + \alpha_3)^j}{j!}\frac{1}{(t_i - 1 - j)} \tag{15}$$

Proof: *Sarhan and Balakrishnan [10] introduced the definition of the moment generating function of* (X_i) *as follows*

$$M_{X_i}(t_i) = E(e^{-t_i x_i}) = \int_0^{\infty} e^{-t_i x_i} f(x_i)dx_i$$

And substituting for $f_{X_i}(x_i)$ from (13) when $\lambda = 1$, we get

$$M_{X_i}(t_i) = (\alpha_i + \alpha_3)e^{\alpha_i+\alpha_3}\int_0^{\infty} e^{-t_i x_i}e^{x_i}\left(e^{-(\alpha_i+\alpha_3)(e^{x_i})}\right)dx_i$$

Using the Taylor series expansion of $e^{-(\alpha_i+\alpha_3)(e^{x_i})}$ we get

$$e^{-(\alpha_i+\alpha_3)(e^{x_i})} = \sum_{j=0}^{\infty}\frac{(-1)^j(\alpha_i + \alpha_3)^j e^{(jx_i)}}{j!} \tag{16}$$

We can express

$$M_{X_i}(t_i) = (\alpha_i + \alpha_3)e^{\alpha_i+\alpha_3} \sum_{j=0}^{\infty} \frac{(-1)^j(\alpha_i + \alpha_3)^j}{j!} \int_0^{\infty} e^{-(t_i-1-j)x_i}dx_i$$

From which we readily derive the expression of $M_{X_i}(t_i)$ given in (15).

Note that the moment generating function $M_{X_i}(t_i)$ can be used, instead of the marginal pdf $f_{X_i}(x_i)$ to derive the marginal expectation of X_i as

$$E(X_i) = -\frac{d}{dt_i}M_{X_i}(t_i)\Big|_{t_i=0}$$

From (15), we obtain

$$-\frac{d}{dt_i}M_{X_i}(t_i) = (\alpha_i + \alpha_3)e^{\alpha_i+\alpha_3} \sum_{j=0}^{\infty} \frac{(-1)^j(\alpha_i + \alpha_3)^j}{j!} \frac{1}{(t_i - 1 - j)^2}$$

In which if we set $t_i = 0$, we obtain $E(X_i)$.

Similarly, the second moment of X_i, can be derived from $M_{X_i}(t_i)$ as its second derivative at $t_i = 0$.The expression for the function $M_{X_i}(t_i)$ in (15) can be used to derive the r^{th} moment of X_i as given below

$$E(X_i^r) = -\frac{d^{(r)}}{dt_i^r}M_{X_i}(t_i) = (\alpha_i + \alpha_3)e^{\alpha_i+\alpha_3} \sum_{j=0}^{\infty} \frac{(-1)^j(\alpha_i + \alpha_3)^j}{j!} \frac{r!}{(t_i - 1 - j)^{r+1}}$$

The following theorem gives the joint moment generating function of (X_1, X_2).

Theorem 3.2: *The joint moment generating function of (X_1, X_2) is given by*

$$\begin{aligned} M(t_1,t_2) = {} & e^{\alpha_2}e^{(\alpha_1+\alpha_3)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_1+\alpha_3)^{j+1}\alpha_2^{i+1}}{j!\,i!\,(t_2-1-i)(t_1-1-j)} \\ & -e^{\alpha_2}e^{(\alpha_1+\alpha_3)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_1+\alpha_3)^{j+1}\alpha_2^{i+1}}{j!\,i!\,(t_2-1-i)(t_1+t_2-2-i-j)} \\ & +e^{\alpha_1}e^{(\alpha_2+\alpha_3)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_2+\alpha_3)^{j+1}\alpha_1^{i+1}}{j!\,i!\,(t_1-1-i)(t_2-1-j)} \\ & -e^{\alpha_1}e^{(\alpha_2+\alpha_3)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_2+\alpha_3)^{j+1}\alpha_1^{i+1}}{j!\,i!\,(t_1-1-i)(t_1+t_2-2-i-j)} \\ & +\alpha_3 e^{(\alpha_1+\alpha_2+\alpha_3)} \sum_{i=0}^{\infty} \frac{(-1)^i(\alpha_1+\alpha_2+\alpha_3)^i}{i!} \frac{1}{(t_1+t_2-1-i)}. \end{aligned} \tag{17}$$

Proof: *The joint moment generating function of (X_1, X_2) is given by*

$$M(t_1,t_2) = E\left(e^{-(t_1x_1+t_2x_2)}\right) = \int_0^{\infty}\int_0^{\infty} e^{-(t_1x_1+t_2x_2)} f(x_1,x_2)dx_1dx_2$$

$$\begin{aligned} M(t_1,t_2) = {} & \int_0^{\infty}\int_0^{x_1} e^{-(t_1x_1+t_2x_2)} f_1(x_1,x_2)dx_2dx_1 + \int_0^{\infty}\int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_2(x_1,x_2)dx_1dx_2 \\ & + \int_0^{\infty} e^{-(t_1+t_2)x} f_3(x,x)dx \end{aligned} \tag{18}$$

Let

$$I_1 = \int_0^{\infty}\int_0^{x_1} e^{-(t_1x_1+t_2x_2)} f_1(x_1,x_2)dx_2dx_1$$

$$I_2 = \int_0^{\infty}\int_0^{x_2} e^{-(t_1x_1+t_2x_2)} f_2(x_1,x_2)dx_1dx_2$$

$$I_3 = \int_0^\infty e^{-(t_1+t_2)x} f_3(x,x)dx$$

Substituting from (6) into I_1

$$f_1(x_1,x_2) = (\alpha_1+\alpha_3)\alpha_2 e^{\lambda x_1} e^{\lambda x_2} \left(e^{\frac{-(\alpha_1+\alpha_3)}{\lambda}\left(e^{\lambda x_1}-1\right)}\right)\left(e^{\frac{-\alpha_2}{\lambda}\left(e^{\lambda x_2}-1\right)}\right)$$

$$I_1 = \int_0^\infty \int_0^{x_1} e^{-(t_1x_1+t_2x_2)} f_1(x_1,x_2)dx_2dx_1$$

$$I_1 = (\alpha_1+\alpha_3)\alpha_2 e^{\alpha_2} e^{(\alpha_1+\alpha_3)} \int_0^\infty e^{-(t_1x_1)} e^{x_1}\left(e^{-(\alpha_1+\alpha_3)e^{x_1}}\right) \int_0^{x_1} e^{-(t_2x_2)} e^{x_2}\left(e^{-(\alpha_2)e^{x_2}}\right)dx_2dx_1$$

Let

$$I_{11} = \int_0^{x_1} e^{-(t_2x_2)} e^{x_2}\left(e^{-(\alpha_2)e^{x_2}}\right)dx_2$$

Using the relation in (16) I_{11} becomes as following

$$I_{11} = \sum_{i=0}^\infty \frac{(-1)^i \alpha_2^i}{i!} \int_0^{x_1} e^{-(t_2-1-i)x_2}\, dx_2$$

We can express

$$I_{11} = \sum_{i=0}^\infty \frac{(-1)^i \alpha_2^i}{i!} \frac{\left(1-e^{-(t_2-1-i)x_1}\right)}{(t_2-1-i)}$$

Substituting from I_{11} into I_1

$$I_1 = (\alpha_1+\alpha_3)\alpha_2 e^{\alpha_2} e^{(\alpha_1+\alpha_3)} \sum_{i=0}^\infty \frac{(-1)^i \alpha_2^i}{i!\,(t_2-1-i)}$$

$$\times \int_0^\infty e^{-(t_1x_1)} e^{x_1}\left(e^{-(\alpha_1+\alpha_3)e^{x_1}}\right)\left(1-e^{-(t_2-1-i)x_1}\right)dx_1$$

Using the relation in (16) I_1 becomes as following

$$I_1 = e^{\alpha_2} e^{(\alpha_1+\alpha_3)} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j}(\alpha_1+\alpha_3)^{j+1}\alpha_2^{i+1}}{j!\,i!\,(t_2-1-i)} \int_0^\infty e^{-(t_1-1-j)x_1}\left(1-e^{-(t_2-1-i)x_1}\right)dx_1$$

Let

$$A = e^{\alpha_2} e^{(\alpha_1+\alpha_3)} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j}(\alpha_1+\alpha_3)^{j+1}\alpha_2^{i+1}}{j!\,i!\,(t_2-1-i)}$$

$$I_1 = A\int_0^\infty e^{-(t_1-1-j)x_1}\, dx_1 - A\int_0^\infty e^{-(t_1+t_2-2-i-j)x_1}\, dx_1$$

$$I_1 = \frac{A}{(t_1-1-j)} - \frac{A}{(t_1+t_2-2-i-j)}. \tag{19}$$

Similarly we can obtain I_2 as follows

$$I_2 = e^{\alpha_1} e^{(\alpha_2+\alpha_3)} \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{i+j}(\alpha_2+\alpha_3)^{j+1}\alpha_1^{i+1}}{j!\,i!\,(t_1-1-i)} \int_0^\infty e^{-(t_2-1-j)x_2}\left(1-e^{-(t_1-1-i)x_2}\right)dx_2$$

Let

$$B = e^{\alpha_1} e^{(\alpha_2+\alpha_3)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(\alpha_2+\alpha_3)^{j+1}\alpha_1^{i+1}}{j!\,i!\,(t_1-1-i)}$$

$$I_2 = B\int_0^{\infty} e^{-(t_2-1-j)x_2}\,dx_2 - B\int_0^{\infty} e^{-(t_1+t_2-2-i-j)x_2}\,dx_2$$

$$I_2 = \frac{B}{(t_2-1-j)} - \frac{B}{(t_1+t_2-2-i-j)}. \tag{20}$$

And we can obtain I_3 as follows

$$I_3 = \alpha_3 e^{(\alpha_1+\alpha_2+\alpha_3)} \sum_{i=0}^{\infty} \frac{(-1)^i(\alpha_1+\alpha_2+\alpha_3)^i}{i!}\frac{1}{(t_1+t_2-1-i)}. \tag{21}$$

Upon substituting from (19), (20) and (21) we can derive the expression for $M(t_1,t_2)$ given in (17)

4 Maximum likelihood estimation

Suppose $((x_1,x_2),\dots,(x_n,x_n))$ is a random sample from $NBG(\alpha_1,\alpha_2,\alpha_3,\lambda)$ distribution.
Consider the following notation

$$n_1 = (i; X_{1i} < X_{2i}),\ \ n_2 = (i; X_{1i} > X_{2i}),\ \ n_3 = (i; X_{1i} = X_{2i} = X_i),\ \ n = n_1+n_2+n_3$$

The likelihood of the sample of size n given by:

$$l(\alpha_1,\alpha_2,\alpha_3,\lambda) = \prod_{i=1}^{n_1} f_1(x_{1i},x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i},x_{2i}) \prod_{i=1}^{n_3} f_3(x_i,x_i)$$

Based on the observations, and using the density functions (6), (7) and (8) the likelihood function becomes:

$$\begin{aligned} l(\alpha_1,\alpha_2,\alpha_3,\lambda) &= \left((\alpha_1+\alpha_3)\alpha_2\right)^{n_1} \prod_{i=1}^{n_1} e^{\lambda x_{1i}} e^{\lambda x_{2i}} \left(e^{\frac{-(\alpha_1+\alpha_3)}{\lambda}\left(e^{\lambda x_{1i}}-1\right)}\right)\left(e^{\frac{-\alpha_2}{\lambda}\left(e^{\lambda x_{2i}}-1\right)}\right) \\ &\times \left(\alpha_1(\alpha_2+\alpha_3)\right)^{n_2} \prod_{i=1}^{n_2} e^{\lambda x_{1i}} e^{\lambda x_{2i}} \left(e^{\frac{-\alpha_1}{\lambda}\left(e^{\lambda x_{1i}}-1\right)}\right)\left(e^{\frac{-(\alpha_2+\alpha_3)}{\lambda}\left(e^{\lambda x_{2i}}-1\right)}\right) \\ &\times (\alpha_3)^{n_3} \prod_{i=1}^{n_3} e^{\lambda x_i}\left(e^{\frac{-(\alpha_1+\alpha_2+\alpha_3)}{\lambda}\left(e^{\lambda x_i}-1\right)}\right) \end{aligned}$$

The log-likelihood function can be written as

$$\begin{aligned} L(\alpha_1,\alpha_2,\alpha_3,\lambda) &= n_1 ln\left((\alpha_1+\alpha_3)\alpha_2\right) - \frac{(\alpha_1+\alpha_3)}{\lambda}\sum_{i=1}^{n_1}\left(e^{\lambda x_{1i}}-1\right) - \frac{\alpha_2}{\lambda}\sum_{i=1}^{n_1}\left(e^{\lambda x_{2i}}-1\right) + \lambda\sum_{i=1}^{n_1} x_{1i} \\ &+\lambda\sum_{i=1}^{n_1} x_{2i} + n_2 ln\left(\alpha_1(\alpha_2+\alpha_3)\right) - \frac{(\alpha_2+\alpha_3)}{\lambda}\sum_{i=1}^{n_2}\left(e^{\lambda x_{2i}}-1\right) + \lambda\sum_{i=1}^{n_2} x_{1i} + \lambda\sum_{i=1}^{n_2} x_{2i} \\ &- \frac{\alpha_1}{\lambda}\sum_{i=1}^{n_2}\left(e^{\lambda x_{1i}}-1\right) + n_3 ln(\alpha_3) + \lambda\sum_{i=1}^{n_3} x_i - \frac{(\alpha_1+\alpha_2+\alpha_3)}{\lambda}\sum_{i=1}^{n_3}\left(e^{\lambda x_i}-1\right) \end{aligned} \tag{22}$$

Computing the first partial derivatives of (22) with respect to α_1, α_2, α_3 and λ, and setting the results equal zeros, we get the likelihood equations as in the following form

$$\frac{\partial L}{\partial \alpha_1} = \frac{n_1}{(\alpha_1+\alpha_3)} - \frac{1}{\lambda}\sum_{i=1}^{n_1}\left(e^{\lambda x_{1i}}-1\right) + \frac{n_2}{\alpha_1} - \frac{1}{\lambda}\sum_{i=1}^{n_2}\left(e^{\lambda x_{1i}}-1\right) - \frac{1}{\lambda}\sum_{i=1}^{n_3}\left(e^{\lambda x_i}-1\right) \tag{23}$$

$$\frac{\partial L}{\partial \alpha_2} = \frac{n_1}{\alpha_2} - \frac{1}{\lambda}\sum_{i=1}^{n_1}\left(e^{\lambda x_{2i}}-1\right) + \frac{n_2}{(\alpha_2+\alpha_3)} - \frac{1}{\lambda}\sum_{i=1}^{n_2}\left(e^{\lambda x_{2i}}-1\right) - \frac{1}{\lambda}\sum_{i=1}^{n_3}\left(e^{\lambda x_i}-1\right) \tag{24}$$

$$\frac{\partial L}{\partial \alpha_3} = \frac{n_1}{(\alpha_1+\alpha_3)} - \frac{1}{\lambda}\sum_{i=1}^{n_1}\left(e^{\lambda x_{1i}}-1\right) + \frac{n_2}{(\alpha_2+\alpha_3)} - \frac{1}{\lambda}\sum_{i=1}^{n_2}\left(e^{\lambda x_{2i}}-1\right) + \frac{n_3}{\alpha_3} - \frac{1}{\lambda}\sum_{i=1}^{n_3}\left(e^{\lambda x_i}-1\right) \tag{25}$$

$$\frac{\partial L}{\partial \lambda} = -(\alpha_1+\alpha_3)\sum_{i=1}^{n_1}\frac{\lambda x_{1i}e^{\lambda x_{1i}}-\left(e^{\lambda x_{1i}}-1\right)}{\lambda^2} - \alpha_2\sum_{i=1}^{n_1}\frac{\lambda x_{2i}e^{\lambda x_{2i}}-\left(e^{\lambda x_{2i}}-1\right)}{\lambda^2} + \sum_{i=1}^{n_1} x_{1i} + \sum_{i=1}^{n_1} x_{2i} - (\alpha_2+\alpha_3)\sum_{i=1}^{n_2}\frac{\lambda x_{2i}e^{\lambda x_{2i}}-\left(e^{\lambda x_{2i}}-1\right)}{\lambda^2} + \sum_{i=1}^{n_2} x_{1i} + \sum_{i=1}^{n_2} x_{2i} - \alpha_1\sum_{i=1}^{n_2}\frac{\lambda x_{1i}e^{\lambda x_{1i}}-\left(e^{\lambda x_{1i}}-1\right)}{\lambda^2} + \sum_{i=1}^{n_3} x_i - (\alpha_1+\alpha_2+\alpha_3)\sum_{i=1}^{n_3}\frac{\lambda x_i e^{\lambda x_i}-\left(e^{\lambda x_i}-1\right)}{\lambda^2} \quad (26)$$

To get the MLEs of the parameters $\alpha_1, \alpha_2, \alpha_3$ and λ, we have to solve the above system of four non-linear equations with respect to $\alpha_1, \alpha_2, \alpha_3$ and λ. The solution of equations (21), (22), (23) and (24) is not possible in closed form, so numerical technique is needed to get the MLEs.

The approximate confidence intervals of the parameters based on the asymptotic distributions of their MLEs are derived. For the observed information matrix of $\alpha_1, \alpha_2, \alpha_3$ and λ, we find the second partial derivatives as follows

$$\frac{\partial^2 L}{\partial \alpha_1^2} = \frac{-n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{(\alpha_1)^2}, \qquad \frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2} = 0, \qquad \frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_3} = \frac{-n_1}{(\alpha_1+\alpha_3)^2}$$

$$\frac{\partial^2 L}{\partial \alpha_1 \partial \lambda} = -\sum_{i=1}^{n_1}\frac{\lambda x_{1i}e^{\lambda x_{1i}}-\left(e^{\lambda x_{1i}}-1\right)}{\lambda^2} - \sum_{i=1}^{n_2}\frac{\lambda x_{1i}e^{\lambda x_{1i}}-\left(e^{\lambda x_{1i}}-1\right)}{\lambda^2} - \sum_{i=1}^{n_3}\frac{\lambda x_i e^{\lambda x_i}-\left(e^{\lambda x_i}-1\right)}{\lambda^2}$$

$$\frac{\partial^2 L}{\partial \alpha_2^2} = \frac{-n_1}{(\alpha_2)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2}, \qquad \frac{\partial^2 L}{\partial \alpha_2 \partial \alpha_3} = \frac{-n_2}{(\alpha_2+\alpha_3)^2}$$

$$\frac{\partial^2 L}{\partial \alpha_2 \partial \lambda} = -\sum_{i=1}^{n_1}\frac{\lambda x_{2i}e^{\lambda x_{2i}}-\left(e^{\lambda x_{2i}}-1\right)}{\lambda^2} - \sum_{i=1}^{n_2}\frac{\lambda x_{2i}e^{\lambda x_{2i}}-\left(e^{\lambda x_{2i}}-1\right)}{\lambda^2} - \sum_{i=1}^{n_3}\frac{\lambda x_i e^{\lambda x_i}-\left(e^{\lambda x_i}-1\right)}{\lambda^2}$$

$$\frac{\partial^2 L}{\partial \alpha_3^2} = \frac{-n_1}{(\alpha_1+\alpha_3)^2} - \frac{n_2}{(\alpha_2+\alpha_3)^2} - \frac{n_3}{(\alpha_3)^2}$$

$$\frac{\partial^2 L}{\partial \alpha_3 \partial \lambda} = -\sum_{i=1}^{n_1}\frac{\lambda x_{1i}e^{\lambda x_{1i}}-\left(e^{\lambda x_{1i}}-1\right)}{\lambda^2} - \sum_{i=1}^{n_2}\frac{\lambda x_{2i}e^{\lambda x_{2i}}-\left(e^{\lambda x_{2i}}-1\right)}{\lambda^2} - \sum_{i=1}^{n_3}\frac{\lambda x_i e^{\lambda x_i}-\left(e^{\lambda x_i}-1\right)}{\lambda^2}$$

$$\frac{\partial L}{\partial \lambda^2} = -(\alpha_1+\alpha_3)\sum_{i=1}^{n_1}\frac{(\lambda x_{1i})^2 e^{\lambda x_{1i}} - 2\lambda x_{1i}e^{\lambda x_{1i}} + 2\left(e^{\lambda x_{1i}}-1\right)}{\lambda^3} - \alpha_2\sum_{i=1}^{n_1}\frac{\left(\lambda x_{2i}e^{\lambda x_{2i}}\right) - 2\lambda x_{2i}e^{\lambda x_{2i}} + 2\left(e^{\lambda x_{2i}}-1\right)}{\lambda^3} - \alpha_1\sum_{i=1}^{n_2}\frac{(\lambda x_{1i})^2 e^{\lambda x_{1i}} - 2\lambda x_{1i}e^{\lambda x_{1i}} + 2\left(e^{\lambda x_{1i}}-1\right)}{\lambda^3} - (\alpha_2+\alpha_3)\sum_{i=1}^{n_2}\frac{(\lambda x_{2i})^2 e^{\lambda x_{2i}} - 2\lambda x_{2i}e^{\lambda x_{2i}} - \left(e^{\lambda x_{2i}}-1\right)}{\lambda^3} - (\alpha_1+\alpha_2+\alpha_3)\sum_{i=1}^{n_3}\frac{(\lambda x_i)^2 e^{\lambda x_i} - 2\lambda x_i e^{\lambda x_i} - \left(e^{\lambda x_i}-1\right)}{\lambda^3}$$

Then the observed information matrix is given by

$$I = -\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}$$

So the variance-covariance matrix may be approximated as

$$V = -\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}^{-1} = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{pmatrix}$$

5 Simulation and data analysis

In this section first we present Monte Carlo simulation results to study the behavior of the MLEs and then present one data analysis results mainly for illustrative purpose.

5.1 Simulation Results

In this subsection we present some simulation results to see how the MLEs behave for different sample sizes and for different initial parameter values. We have used different sample sizes namely $n = 20$, 40, 60, 80 and 100 and two different sets of parameter values: Set 1: $\alpha_1 = 1.1, \alpha_2 = 0.9, \alpha_3 = \lambda = 1$ and Set 2: $\alpha_1 = 1.2, \alpha_2 = 1.1, \alpha_3 = \lambda = 1$. In each case we have computed the MLEs of the unknown parameters by maximizing the log-likelihood function (36). We compute the average estimates and mean square error over 1000 replications and the results are reported in Tables 1. Some of the points are quite clear from Tables 1. In all the cases the performances of the maximum likelihood estimate are quite satisfactory. It is observed that as sample size increases the average estimates and the mean squared error decrease for all the parameters, as expected.

5.2 Data Analysis

The following data represent the American Football (National Football League) League data and they are obtained from the matches played on three consecutive weekends in 1986. The data were first published in 'Washington Post' and they are also available in Csorgo and Welsh [5].

It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the 'game time' to the first points scored by kicking the ball between goal posts, and X_2 represents the 'game time' to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game.

The data (scoring times in minutes and seconds) are represented in Table 2. The data set was first analyzed by Csorgo and Welsh [5], by converting the seconds to the decimal minutes, i.e. 2:03 has been converted to 2.05, 3:59 to 3.98 and so on. We have also adopted the same procedure. Here also all the data points are divided by 100 just for computational purposes.

Table 1: The average of MLEs and the associated mean square errors (within brackets below)

	Set 1				Set 2			
Model n	$\alpha_1 = 1.1$	$\alpha_2 = 0.9$	$\alpha_3 = 1$	$\lambda = 1$	$\alpha_1 = 1.2$	$\alpha_2 = 1.1$	$\alpha_3 = 1$	$\lambda = 1$
n=20	0.466	0.464	1.841	2.669	0.538	0.54	2.035	2.685
	(0.352)	(2.592)	(1.166)	(3.974)	(0.293)	(2.207)	(1.728)	(3.955)
n=40	0.504	0.501	1.864	2.305	0.578	0.581	2.057	2.339
	(0.277)	(2.276)	(0.94)	(2.166)	(0.217)	(2.054)	(1.365)	(2.3)
n=60	0.517	0.513	1.896	2.204	0.592	0.587	2.076	2.257
	(0.254)	(2.234)	(0.932)	(1.74)	(0.192)	(2.025)	(1.335)	(1.886)
n=80	0.525	0.516	1.904	2.151	0.594	0.584	2.078	2.22
	(0.241)	(2.217)	(0.927)	(1.524)	(0.183)	(2.024)	(1.294)	(1.712)
n=100	0.532	0.529	1.913	2.096	0.604	0.599	2.099	2.152
	(0.231)	(2.176)	(0.921)	(1.367)	(0.173)	(1.979)	(1.108)	(1.516)

Table 2: American football league (N F L) data

X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	5.78	25.98	10.40	10.25
9.05	9.05	13.80	49.75	2.98	2.98
0.85	0.85	7.25	7.25	3.88	6.43
3.43	3.43	4.25	4.25	0.75	0.75
7.78	7.78	1.65	1.65	11.63	17.37
10.57	14.28	6.42	15.08	1.38	1.38
7.05	7.05	4.22	9.48	10.53	10.53
2.58	2.58	15.53	15.53	12.13	12.13
7.23	9.68	2.90	2.90	14.58	14.58
6.85	34.58	7.02	7.02	11.82	11.82
32.45	42.35	6.42	6.42	5.52	11.27
8.53	14.57	8.98	8.98	19.65	10.70
31.13	49.88	10.15	10.15	17.83	17.83
14.58	20.57	8.87	8.87	10.85	38.07

The variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$ means that the first score is a field goal (ii) $X_1 = X_2$ means the first score is a converted touchdown, (iii) $X_1 > X_2$ means the first score is an unconverted touchdown or safety. In this case the ties are exact because no 'game time' elapses between a touchdown and a point-after conversion attempt. Therefore, here ties occur quite naturally and they cannot be ignored. It should be noted that the possible scoring times are restricted by the duration of the game but it has been ignored similarly as in Csorgo and Welsh [5].

If we define the following random variables:

U_1= time to first field goal

U_2 = time to first safety or unconverted touchdown

U_3 = time to first converted touchdown,

Then $X_1 = min(U_1, U_3)$, and $X_2 = min(U_2, U_3)$ Therefore, (X_1, X_2) has a similar structure as the Marshall-Olkin bivariate exponential ($MOBE$) model or the proposed NBG model.

We analyze the data using the NBG model. We have taken the initial guesses of $\alpha_1, \alpha_2, \alpha_3$ and λ are all equal to 1. The estimate of $\alpha_1, \alpha_2, \alpha_3$ and λ become 0.288, 1.412, 7.565 and 4.698 respectively. The corresponding log-likelihood value is 66.559. The 95% confidence intervals of $\alpha_1, \alpha_2, \alpha_3$ and λ are (0, 0.859), (0.512, 2.312), (5.121, 10.01), (2.696, 6.7) respectively.

6 Conclusion

In this paper we have proposed bivariate Gompertz distribution function of Marshall-Olkin type whose marginals are Gompertz distributions. The moment generating function of proposed distribution is derived. The generation of random samples from proposed bivariate distribution is very simple, and therefore Monte Carlo simulation can be performed very easily for different statistical inference purpose. It is observed that the MLEs of the unknown parameters can be obtained by solving four non-linear equations and Monte Carlo simulation indicate that the performance of the MLEs are quite satisfactory. Analysis of one real data indicates that the performance of the confidence intervals based on asymptotic distribution.

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