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Optimum Group Limits for Maximum Likelihood Estimation of the Exponentiated Fréchet Distribution Based on Grouped Data

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Authors' contributions

This work was carried out in collaboration between all authors. Author AAM managed the literature search, designed the study, performed the statistical analysis, and wrote the first draft of the manuscript. Authors HZ and EAE designed and supervised the study, all authors read and approved the final manuscript.

Research Article

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ABSTRACT

In many situations, instead of a complete sample, data are available only in grouped form. In this situation the values of individual observations are not known, but the number of observations that fall in each group is only known. Here the model under consideration is the exponentiated Fréchet distribution. The aim of this paper is finding the MLE's for the parameters of the exponentiated Fréchet distribution based on grouped data. The asymptotic variance-covariance matrix has been derived and computed numerically. Optimal group limits in the case unequi-spaced groupings so as to have a maximum asymptotic relative efficiency are worked out.

Keywords: Maximum likelihood estimation; exponentiated fréchet distribution; grouped data; optimum grouping.

1. INTRODUCTION

Estimation of the unknown parameters of statistical distributions is one of the most important problems facing statisticians. This paper presents estimation of the unknown parameters of

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the exponentiated Fréchet distribution model based on grouped data. The Fréchet distribution has over fifty applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds and track race records. [1] gave some applications in their book.

In recent years, several standard life time distributions have been generalized via exponentiation. Examples of such exponentiated distributions are the exponentiated Weibull family, the exponentiated exponential, the exponentiated Rayleigh, the exponentiated Fréchet and the exponentiated Pareto family of distributions.

Let X be a random variable with probability density function (p.d.f) $f(x)$ and cumulative function (c.d.f) $F(x)$, $x \in R$. Consider a random variable Z with cdf given by

$$G_{\alpha}(z) = [F(z)]^{\alpha}, z \in R, \alpha > 0. \qquad (1)$$

then Z is said to have an exponentiated distribution, see [2]

The pdf of Z is given by

$$g_{\alpha}(z) = \alpha [F(z)]^{\alpha-1} f(z). \qquad (2)$$

The baseline distribution $F(z)$ can be obtained as a special case of the exponentiated distribution when $\alpha = 1$. The advantage of this approach for modeling failure time data lies in its flexibility to model both monotonic as well as non-monotonic failure rates even though the baseline failure rate may be monotonic.

Following this idea, [3] introduced the exponentiated exponential (EE) distribution as a generalization of the exponential distribution. In the same way, [4] proposed four more exponentiated distributions that generalize the gamma, Weibull, Gumbel and Fréchet distributions and provided some mathematical properties for each distribution. Several other authors have considered exponentiated distributions, for example, [5,6,7,8,9,10,11,12].

[4] introduced introduced a new lifetime model named the Exponentiated Fréchet distribution (EF). It is a generalization of the standard Fréchet distribution (known as the extreme value distribution of type II). The EF distribution is referred to in the literature as the inverse of exponentiated Weibull distribution. The cumulative distribution function c.d.f. of the EF can be written as:

$$F(x;\alpha,\lambda,\sigma) = 1 - \left[1 - \exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]^{\alpha}, x > 0 \qquad (3)$$

where, $\alpha > 0$, $\lambda > 0$ are shape parameters, and $\sigma > 0$ is the scale parameter, and the corresponding probability density function is:

$$f(x;\alpha,\lambda,\sigma)=\alpha\,\lambda\,\sigma^{\lambda}\left[1-\exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\}\right]^{\alpha-1}x^{-(1+\lambda)}\exp\left\{-\left(\frac{\sigma}{x}\right)^{\lambda}\right\} \tag{4}$$

There are several types of data that arise in every day life. Among these, there are simple data, grouped data, truncated data, censored data, progressively censored data, and so on.

It is very common in practice that a raw sample is grouped into a frequency distribution with equi-spaced or unequi-spaced intervals, for example in life testing experiments, we observe the failure time of a component to the nearest hour, day or month. In this situation the values of individual observations are not known, but the number of observations that fall in each group is only known. Exact measurements often require costly skilled, personal and sophisticated instruments; whereas grouped data is usually quicker, easier and therefore cheaper.

In general grouped data can be formulated as follows, suppose that n independent observations are made on a random variable X with density and cumulative functions $f(x;\theta)$ and $F(x;\theta)$ respectively, where θ is the unknown parameter. Let x_0 be the lower bound of the distribution, such that $F(x_o;\theta)=0$, and let x_k, analogously, be the upper bound of the distribution. It will be assumed that the range of variation (x_0, x_k) is divided (partitioned) into k intervals, and let $x_1, x_2, ..., x_{k-1}$ be the dividing points, such that $x_o < x_1 < x_2 < ... < x_{k-1} < x_k$. The interval (x_{i-1}, x_i) will be referred to as the ith interval or the ith group, and the points $x_i (i=0,1,...,k)$ will be called group limits, the number of observations falling into one of the k adjacent groups, say i, will be denoted by n_i such that, $\sum_{i=1}^{k} n_i = n$.

[13] defined the likelihood function for estimating the unknown parameter in case of grouped samples as follows:

$$L(x;\theta) = a\prod_{i=1}^{k} p_i^{n_i}\ , \tag{5}$$

where $p_i = p_i(\theta)$ is the probability of an observation falling in the ith group, n_i is the number of observations falling in group number i, $\sum_{i=1}^{k} n_i = n$, and a is a constant not involving the parameters of the function. Using the cumulative functions where: $p_i = \int_{x_{i-1}}^{x_i} f(x;\theta)dx = F(x_i;\theta) - F(x_{i-1};\theta)$, $F(x_o;\theta)=0$, and $F(x_k;\theta)=1$, so the likelihood function (5) may be written as follows:

$$L(x;\theta) = a.\prod_{i=1}^{k}(F(x_i;\theta) - F(x_{i-1};\theta))^{n_i}, \tag{6}$$

Many authors have considered the estimation problem of the unknown parameters of different distributions based on grouped samples. Some early works on grouped samples can be found in [13], who obtained the maximum likelihood estimate (MLE) of the exponential distribution parameter when the data are grouped, other work can be found in [14,15,16,17,18,19,20,21,22,23,24,25,26].

This paper is organized as follows. In Section 2, the likelihood density for the grouped data has been derived as well as the MLE (Maximum Likelihood Estimators) for the exponentiated Fréchet distribution's parameters with known shape parameter λ from grouped data. In Section 3, the Fisher's information matrix based on the grouped data has been derived and used to obtain the variance covariance matrix. In Section 4, the optimum group limits for the exponentiated Fréchet distribution has been discussed. In Section 5, a numerical example has been given to determine the maximum likelihood estimates of the parameters of the exponentiated Fréchet distribution, as well as the asymptotically optimum group limits and the asymptotic relative efficiencies of the maximum likelihood estimates of the parameters with optimum grouping. Finally the conclusions were stated in Section 6.

2. PARAMETERS ESTIMATION

Suppose that a random sample of size n is given from the exponentiated Fréchet distribution with shape parameters α and λ, (λ known), and scale parameter σ which has a probability density function $f(x;\alpha,\sigma)$ and cumulative distribution function $F(x;\alpha,\sigma)$ as defined in equations (4) and (3) respectively.

According to [13] and [4], the likelihood function of the grouped exponentiated Fréchet distribution may be written as follows:

$$L(x;\alpha,\sigma) = a.\prod_{i=1}^{k}\left\{\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]^{\alpha} - \left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]^{\alpha}\right\}^{n_i} \tag{7}$$

Taking the logarithm for both sides, then the log-likelihood function will be:

$$\ln L(x;\alpha,\sigma) = \ln a + \sum_{i=1}^{k} n_i \ln\left\{\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]^{\alpha} - \left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]^{\alpha}\right\}. \tag{8}$$

To obtain the log-likelihood equations for the unknown parameters α and σ, the log-likelihood function in equation (8) will be differentiated partially with respect to α, and σ respectively, as follows:

$$\frac{\partial \ln L}{\partial \alpha}=\sum_{i=1}^{k} n_i \left\{ \frac{\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]^{\alpha} .\ln\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]-\left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]^{\alpha} .\ln\left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]}{\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]^{\alpha}-\left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]^{\alpha}} \right\}$$

(9a)

$$\frac{\partial \ln L}{\partial \sigma}=\frac{\alpha}{\sigma}\sum_{i=1}^{k} n_i \left\{ \frac{\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda} .e^{-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}} .\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]^{\alpha-1}-\left(\frac{\sigma}{x_i}\right)^{\lambda} .e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}} .\left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]^{\alpha-1}}{\left[1-\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]^{\alpha}-\left[1-\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]^{\alpha}} \right\}$$

(9b)

The simultaneous solution of equations (9) when they are set equal to zero provides the maximum likelihood estimates $\hat{\alpha}$, and $\hat{\sigma}$. The equations (9) has no solution explicitly for α , and σ . Hence, a numerical iterative technique using the package (MathCAD 14) can be used to obtain the maximum likelihood estimates of these parameters. The results will be listed in Table 1.

3. FISHER'S INFORMATION AND VARIANCE COVARIANCE MATRIX

Denote the Fisher information matrix associated with α and σ by $I(\alpha,\sigma)$, so:

$$I(\alpha,\sigma)=-E\begin{bmatrix} \frac{\partial^2 \ln L(\alpha,\sigma)}{\partial \alpha^2} & \frac{\partial^2 \ln L(\alpha,\sigma)}{\partial \alpha \, \partial \sigma} \\ \frac{\partial^2 \ln L(\alpha,\sigma)}{\partial \alpha \, \partial \sigma} & \frac{\partial^2 \ln L(\alpha,\sigma)}{\partial \sigma^2} \end{bmatrix} \quad (10)$$

See [27].

The elements of the sample information matrix, for the unequi-spaced grouped sample will be:

$$\frac{\partial^2 \ln L}{\partial \alpha^2}=\sum_{i=1}^{k} n_i \left\{ \frac{\left[(A_{i-1})^{\alpha}-(A_i)^{\alpha}\right]\left[(A_{i-1})^{\alpha}.(\ln(A_{i-1}))^2-(A_i)^{\alpha}.(\ln(A_i))^2\right]-\left[(A_{i-1})^{\alpha}.\ln(A_{i-1})-(A_i)^{\alpha}.\ln(A_i)\right]^2}{\left[(A_{i-1})^{\alpha}-(A_i)^{\alpha}\right]^2} \right\}$$

(11)

$$\frac{\partial^2 \ln L(X;\alpha,\lambda,\sigma)}{\partial\sigma^2} = $$

$$= \frac{\alpha\lambda}{\sigma}.\sum_{i=1}^{k} n_i \left\{ \frac{\left\{\left[(A_{i-1})^{\alpha} - (A_i)^{\alpha}\right].\left[N_1 - N_2\right]\right\} - \left[\left(\frac{\alpha.\lambda}{x_{i-1}}\right).\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda-1}.e^{-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}}.\left[A_{i-1}\right]^{\alpha-1} - \left(\frac{\alpha.\lambda}{x_i}\right).\left(\frac{\sigma}{x_i}\right)^{\lambda-1}.e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}.\left[A_i\right]^{\alpha-1}\right]^2}{\left[(A_{i-1})^{\alpha} - (A_i)^{\alpha}\right]^2} \right\} \tag{12}$$

$$\frac{\partial^2 \ln L}{\partial\alpha\,\partial\sigma} = \frac{\partial}{\partial\sigma}\left\{\frac{\partial \ln L}{\partial\alpha}\right\}$$

$$= \sum_{i=1}^{k} n_i \left[\frac{\left\{\left[(A_{i-1})^{\alpha} - (A_i)^{\alpha}\right].\left[C_1 - C_2\right]\right\} - \left\{\left[(A_{i-1})^{\alpha}.\ln(A_{i-1}) - (A_i)^{\alpha}.\ln(A_i)\right].C_3\right\}}{\left[(A_{i-1})^{\alpha} - (A_i)^{\alpha}\right]^2} \right] \tag{13}$$

where:

$$A_{i-1} = \left[1 - \exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right)\right]$$

$$A_i = \left[1 - \exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right)\right]$$

$$N_1 = \alpha.\lambda.\sigma^{\lambda-2}.\left(\frac{1}{x_{i-1}}\right)^{\lambda}.e^{-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}}.(A_{i-1})^{\alpha-1}.\left[(\lambda-1) - \lambda\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda} + \lambda(\alpha-1).\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda} e^{-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}}.(A_{i-1})^{-1}\right]$$

$$N_2 = \alpha.\lambda.\sigma^{\lambda-2}.\left(\frac{1}{x_i}\right)^{\lambda}.e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}.(A_i)^{\alpha-1}.\left[(\lambda-1) - \lambda\left(\frac{\sigma}{x_i}\right)^{\lambda} + \lambda(\alpha-1).\left(\frac{\sigma}{x_i}\right)^{\lambda} e^{-\left(\frac{\sigma}{x_i}\right)^{\lambda}}.(A_i)^{-1}\right]$$

$$C_1 = \lambda.\left[(A_{i-1})^{\alpha-1}.\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right).\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda-1}.\left(\frac{1}{x_{i-1}}\right).\left[1 + \alpha.\ln(A_{i-1})\right]\right]$$

$$C_2 = \lambda.\left[(A_i)^{\alpha-1}.\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right).\left(\frac{\sigma}{x_i}\right)^{\lambda-1}.\left(\frac{1}{x_i}\right).\left[1 + \alpha.\ln(A_i)\right]\right]$$

$$C_3 = \frac{\alpha.\lambda}{\sigma}.\left[(A_{i-1})^{\alpha-1}.\exp\left(-\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda}\right).\left(\frac{\sigma}{x_{i-1}}\right)^{\lambda} - (A_i)^{\alpha-1}.\exp\left(-\left(\frac{\sigma}{x_i}\right)^{\lambda}\right).\left(\frac{\sigma}{x_i}\right)^{\lambda}\right]$$

The asymptotic variance-covariance matrix denoted by $AsVar(\alpha,\sigma)$ for the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\sigma}$ is obtained by inverting the information matrix (10) through the following relation:

$$AsVar(\alpha,\sigma) = \left[I\,(\alpha,\sigma)\right]^{-1}, \qquad (14)$$

A computer program was used to calculate the values of the asymptotic variance-covariance matrix for the maximum likelihood estimates $\hat{\alpha}$ and $\hat{\sigma}$ in the case of unequi-spaced grouped sample and the results presented in Table 1.

4. ASYMPTOTIC OPTIMUM GROUP LIMITS

Our problem is to determine the group limits $x_1, x_2, ..., x_{k-1}$ optimally so as to make the asymptotic variance of the estimates a minimum for a given values of "n" and "k", where n is the sample size and k is the number of groups . And this can be achieved by minimizing the asymptotic generalized variance (the determinant of the asymptotic variance covariance matrix) of the MLE of α and σ or equivalently maximize the determinant of the corresponding information matrix with respect to $x_i^{'}s$.

The asymptotic optimum group limits $x_i's$ have been computed by maximizing the determinant of the information matrix given in (10) for k =3(1)10, α = 0.5, 1, 1.5, and σ = 1, 2, 3, and also we obtained the asymptotic relative efficiencies of the MLE's of α and σ from the optimally grouped sample in relation to the corresponding MLEs from ungrouped sample. These results were presented in Table 2.

Because of the effect of the known shape parameter (λ) on the behavior of the exponentiated Fréchet distribution, different values for the shape parameter λ (λ =0.5, 1, 1.5, 2, 2.5) were considered and the results were summarized in Table 3, which represents the minimum and maximum value of the asymptotic variances for α and σ, also the minimum and maximum value of the asymptotic relative efficiencies for both α and σ considering different values of λ .

Table 1. MLE of the parameters α and σ of the exponentiated fréchet distribution and the asymptotic variances and covariance of $\hat{\alpha}$ and $\hat{\sigma}$ for grouped samples

			Grouped data		Optimum grouping													
Groups	**α**	**σ**	**α^**	**σ^**	**Asvar (α)**	**Asvar (σ)**	**Cov (α,σ)**	**Relative (α)**	**Relative (σ)**	**x1**	**x2**	**x3**	**x4**	**x5**	**x6**	**x7**	**x8**	**x9**
3	0.5	1	0.470	1.203	5.720E-03	1.786E-02	1.950E-02	0.617	0.633	0.569	17.04							
4	0.5	1	0.465	0.767	4.716E-03	1.258E-02	1.371E-02	0.762	0.862	0.17	1.19	7.783						
5	0.5	1	0.519	1.235	4.050E-03	1.207E-02	1.211E-02	0.851	0.888	0.243	1.389	157.31	237.3					
6	0.5	1	0.482	1.128	4.108E-03	1.163E-01	1.211E-02	0.858	0.923	0.093	0.298	1.456	75.624	519.301				
7	0.5	1	0.606	1.533	3.771E-03	1.133E-01	1.094E-02	0.917	0.928	0.121	0.296	0.906	6.325	125.772	319.203			
8	0.5	1	0.522	1.260	4.727E-03	1.171E-01	1.375E-02	0.918	0.956	0.067	0.143	0.329	0.95	4.743	58.375	186.79		
9	0.5	1	0.526	1.004	3.794E-03	1.091E-01	1.103E-02	0.947	0.991	0.066	0.141	0.307	0.788	3.172	40.091	159.522	299.741	
10	0.5	1	0.554	1.233	3.743E-03	1.018E-01	1.039E-02	0.986	0.992	0.114	0.218	0.463	1.226	5.997	15.013	48.558	134.373	350.595
3	0.5	2	0.562	2.080	5.650E-03	1.698E-02	3.767E-02	0.625	0.674	1.108	35.88							
4	0.5	2	0.655	2.809	4.066E-03	4.839E-01	2.345E-02	0.851	0.869	0.424	2.605	17.602						
5	0.5	2	0.592	2.861	3.797E-03	3.977E-01	1.997E-02	0.864	0.889	0.772	4.571	248.529	1925					
6	0.5	2	0.592	2.257	4.004E-03	4.242E-01	2.237E-02	0.886	0.928	0.392	1.442	16.477	48.963	319.604				
7	0.5	2	0.631	2.655	3.769E-03	4.536E-01	2.188E-02	0.913	0.936	0.244	0.601	1.883	13.608	261.592	675.731			
8	0.5	2	0.623	1.701	4.443E-03	4.627E-01	2.601E-02	0.918	0.944	0.131	0.295	0.703	2.115	14.198	19.476	47.619		
9	0.5	2	0.631	2.728	3.716E-03	4.454E-01	2.144E-02	0.931	0.958	0.199	0.385	0.806	2.067	8.539	83.143	139.313	244.143	
10	0.5	2	0.543	2.752	3.604E-03	3.992E-01	1.969E-02	0.949	0.986	0.285	0.531	1.09	2.801	11.496	16.97	58.589	152.753	196.77
3	0.5	3	0.422	3.086	5.557E-03	1.519E-02	5.373E-02	0.578	0.622	1.166	39.86							
4	0.5	3	0.634	2.935	4.224E-03	5.760E-01	3.635E-02	0.814	0.881	0.872	8.456	30.451						
5	0.5	3	0.569	2.692	3.842E-03	9.746E-01	3.218E-02	0.893	0.892	0.694	3.623	26.829	272.284					
6	0.5	3	0.482	3.385	3.876E-03	9.529E-02	3.249E-02	0.893	0.903	0.642	2.772	17.423	24.861	180.021				
7	0.5	3	0.572	2.813	3.945E-03	1.24E-02	3.501E-02	0.897	0.913	0.25	0.688	2.399	24.684	74.015	419.246			
8	0.5	3	0.502	3.455	4.169E-03	1.180E-02	3.650E-02	0.905	0.915	0.191	0.413	0.964	2.868	18.86	21.075	41.16		
9	0.5	3	0.551	3.190	3.788E-03	9.900E-01	3.323E-02	0.923	0.952	0.188	0.392	0.85	2.158	8.712	16.068	21.913	57.454	
10	0.5	3	0.447	4.050	3.700E-03	9.254E-01	3.113E-02	0.933	0.995	0.306	0.576	1.146	2.714	9.782	17.176	49.866	53.241	2.16.452
3	1.5	1	1.291	1.060	7.535E-02	9.343E-02	6.343E-02	0.632	0.625	0.202	10.516							
4	1.5	1	1.814	1.308	5.298E-02	6.377E-02	4.150E-02	0.836	0.87	0.159	0.865	2.091						
5	1.5	1	1.781	1.515	5.314E-02	6.672E-02	4.294E-02	0.867	0.875	0.114	0.407	4.468	41.629					
6	1.5	1	1.900	1.510	4.746E-02	5.561E-02	3.572E-02	0.894	0.911	0.142	0.355	1.371	6.079	27.994				
7	1.5	1	1.673	1.526	4.849E-02	5.760E-02	3.746E-02	0.922	0.925	0.098	0.236	0.734	3.825	18.167	123.362			
8	1.5	1	1.690	1.630	5.119E-02	4.440E-02	3.606E-02	0.925	0.956	0.053	0.11	0.232	0.586	2.326	9.265	42.182		
9	1.5	1	1.803	1.145	4.745E-02	5.886E-02	3.758E-02	0.928	0.992	0.049	0.097	0.189	0.421	1.318	4.698	16.61	90.431	
10	1.5	1	1.979	1.437	4.636E-02	5.282E-02	3.452E-02	0.981	0.996	0.11	0.187	0.343	0.721	1.915	5.147	12.512	34.973	152.089
3	1.5	2	1.321	1.532	7.448E-02	3.698E-01	1.246E-01	0.611	0.63	0.392	22.091							
4	1.5	2	1.535	2.161	5.210E-02	2.531E-01	8.152E-02	0.834	0.894	0.438	5.119	11.645						
5	1.5	2	1.342	1.771	5.007E-02	2.665E-01	8.723E-02	0.849	0.902	0.291	1.023	9.319	78.184					

Table 1 continues……

6	1.5	2	1.435	2.084	4.943E-02	2.368E-01	7.683E-02	0.91	0.912	0.236	0.716	4.589	26.904	213.968				
7	1.5	2	1.370	2.024	4.714E-02	2.302E-01	7.312E-02	0.913	0.926	0.179	0.391	0.994	4.237	17.285	78.339			
8	1.5	2	1.690	2.259	5.188E-02	2.479E-01	8.233E-02	0.93	0.939	0.101	0.213	0.451	1.131	4.478	18.435	86.422		
9	1.5	2	2.030	2.654	4.677E-02	2.403E-01	7.380E-02	0.954	0.964	0.121	0.212	0.368	0.68	1.473	4.101	11.233	32.81	
10	1.5	2	1.363	1.189	4.647E-02	5.632E-02	3.607E-02	0.976	0.983	0.065	0.114	0.208	0.413	1.055	3.255	9.292	28.984	132.24
3	1.5	3	1.430	2.540	7.779E-02	8.599E-01	1.980E-01	0.637	0.674	0.577	24.297							
4	1.5	3	1.490	2.818	5.211E-02	5.695E-01	1.223E-01	0.881	0.906	0.655	7.649	12.961						
5	1.5	3	1.757	3.712	4.944E-02	5.309E-01	1.140E-01	0.894	0.92	0.409	1.376	11.762	88.323					
6	1.5	3	1.851	3.831	4.815E-02	5.275E-01	1.122E-01	0.908	0.947	0.306	0.772	2.765	15.858	87.512				
7	1.5	3	1.474	2.939	4.928E-02	5.547E-01	1.184E-01	0.912	0.951	0.187	0.473	1.371	8.189	43.637	306.344			
8	1.5	3	1.593	3.530	4.935E-02	5.182E-01	1.141E-01	0.92	0.957	0.251	0.506	1.129	3.52	13.29	49.98	250.107		
9	1.5	3	1.575	2.317	4.659E-02	5.263E-01	1.100E-01	0.96	0.986	0.172	0.316	0.58	1.168	3.015	9.721	30.854	109.744	
10	1.5	3	1.545	3.557	4.675E-02	5.317E-01	1.116E-01	0.989	0.993	0.121	0.227	0.41	0.775	1.7	4.713	13.992	40.477	156.386
3	2.5	1	2.368	0.873	2.770E-01	7.945E-02	1.224E-01	0.661	0.66	0.144	2.419							
4	2.5	1	2.582	1.075	1.810E-01	5.228E-02	7.561E-02	0.875	0.907	0.153	0.994	5.103						
5	2.5	1	2.971	1.200	1.692E-01	4.834E-02	6.953E-02	0.912	0.95	0.103	0.287	1.44	5.957					
6	2.5	1	2.576	1.022	1.639E-01	4.763E-02	6.782E-02	0.93	0.951	0.079	0.179	0.497	1.806	6.039				
7	2.5	1	2.444	0.976	1.606E-01	4.676E-02	6.638E-02	0.932	0.955	0.071	0.141	0.308	0.911	2.563	7.318			
8	2.5	1	2.689	1.152	1.678E-01	4.729E-02	6.920E-02	0.951	0.966	0.071	0.138	0.286	0.785	2.073	5.475	17.888		
9	2.5	1	2.560	0.789	1.574E-01	4.702E-02	6.579E-02	0.96	0.968	0.048	0.084	0.144	0.26	0.553	1.322	2.961	6.884	
10	2.5	1	1.612	2.475	4.622E-02	2.328E-01	7.236E-02	0.986	0.993	0.122	0.205	0.343	0.604	1.227	3.12	8.157	20.914	56.858
3	2.5	2	2.935	2.542	2.862E-01	3.397E-01	2.610E-01	0.643	0.635	0.305	6.707							
4	2.5	2	2.455	1.877	2.073E-01	2.559E-01	1.848E-01	0.879	0.891	0.229	1.723	8.511						
5	2.5	2	2.240	2.094	1.773E-01	1.993E-01	1.459E-01	0.903	0.914	0.204	0.616	3.666	18.188					
6	2.5	2	2.384	2.053	1.707E-01	1.950E-01	1.416E-01	0.915	0.927	0.182	0.475	2.013	7.464	31.274				
7	2.5	2	2.667	1.628	1.608E-01	1.925E-01	1.348E-01	0.916	0.952	0.114	0.23	0.482	1.308	3.849	11.058			
8	2.5	2	2.977	3.015	1.741E-01	2.007E-01	1.466E-01	0.94	0.965	0.086	0.177	0.358	0.838	2.658	7.593	24.614		
9	2.5	2	3.093	2.314	1.619E-01	1.914E-01	1.362E-01	0.977	0.986	0.08	0.152	0.275	0.527	1.219	3.175	7.948	23.634	
10	2.5	2	2.694	2.166	1.60E-01	1.90E-01	1.34E-01	0.994	0.995	0.073	0.132	0.228	0.402	0.787	1.882	4.344	9.876	24.969
3	2.5	3	2.367	2.543	2.811E-01	7.259E-01	3.746E-01	0.588	0.673	0.467	7.361							
4	2.5	3	3.109	3.825	1.801E-01	4.652E-01	2.242E-01	0.805	0.845	0.349	1.356	4.762						
5	2.5	3	2.851	3.282	1.795E-01	4.573E-01	2.230E-01	0.848	0.876	0.289	0.882	5.564	29.113					
6	2.5	3	2.423	3.079	1.747E-01	4.617E-01	2.220E-01	0.877	0.909	0.182	0.458	1.414	6.515	28.453				
7	2.5	3	2.269	2.995	1.668E-01	4.263E-01	2.064E-01	0.91	0.929	0.233	0.507	1.319	4.69	14.717	57.385			
8	2.5	3	3.068	3.663	1.739E-01	4.385E-01	2.153E-01	0.918	0.954	0.193	0.352	0.655	1.397	3.586	8.681	22.345		
9	2.5	3	2.839	3.350	1.629E-01	4.318E-01	2.055E-01	0.935	0.958	0.123	0.238	0.441	0.886	2.226	5.938	15.106	49.182	
10	2.5	3	2.390	3.104	1.596E-01	3.938E-01	1.920E-01	0.984	0.991	0.253	0.413	0.698	1.32	2.848	5.839	11.938	26.564	69.436

Table 2. The asymptotically optimal limits x_i's when estimating the two parameters α and σ and the corresponding asymptotic relative efficiencies (ares) as compared to ungrouped samples when λ =1

Number of groups	α	σ	Relative (α)	Relative (σ)	x1	x2	x3	x4	x5	x6	x7	x8	x9
3	0.5	1	0.554	0.59	0.657	29.483							
4	0.5	1	0.748	0.79	0.447	1.25	3.63						
5	0.5	1	0.836	0.856	0.697	2.155	19.881	289.73					
6	0.5	1	0.857	0.88	0.436	0.827	2.633	16.468	145.769				
7	0.5	1	0.887	0.896	0.3	0.524	1.087	5.654	33.792	320.939			
8	0.5	1	0.896	0.916	0.325	0.47	0.722	1.307	3.925	13.492	51.837		
9	0.5	1	0.935	0.937	0.258	0.384	0.595	1.007	2.543	11.494	48.587	157.126	
10	0.5	1	0.965	0.987	0.584	0.737	0.986	1.516	3.109	6.474	14.627	38.887	157.068
3	0.5	2	0.535	0.621	1.333	66.14							
4	0.5	2	0.824	0.854	1.624	3.356	21.367						
5	0.5	2	0.845	0.881	1.069	2.599	29.426	474.446					
6	0.5	2	0.855	0.882	0.618	1.14	2.758	27.617	386.664				
7	0.5	2	0.892	0.894	0.166	0.459	1.589	16.994	50.788	289.17			
8	0.5	2	0.902	0.903	0.656	0.955	1.471	2.679	8.099	27.897	123.271		
9	0.5	2	0.923	0.939	0.736	1.082	1.716	3.404	11.575	40.145	75.801	177.903	
10	0.5	2	0.928	0.985	0.862	1.099	1.43	1.984	3.431	7.845	17.873	51.477	240.311
3	0.5	3	0.502	0.568	2.555	68.113							
4	0.5	3	0.837	0.856	2.537	5.912	36.67						
5	0.5	3	0.86	0.867	1.639	4.029	46.282	823.681					
6	0.5	3	0.884	0.907	1.106	1.993	4.88	40.643	434.117				
7	0.5	3	0.907	0.929	0.894	1.549	3.166	15.289	88.333	763.795			
8	0.5	3	0.929	0.935	0.263	0.531	1.188	3.315	16.697	21.423	306.69		
9	0.5	3	0.938	0.945	1.406	2.026	3.206	6.441	18.282	56.805	100.648	168.073	
10	0.5	3	0.939	0.96	1.568	1.919	2.426	3.28	5.191	10.177	20.056	42.888	112.878
3	1.5	1	0.605	0.618	0.454	3.218							
4	1.5	1	0.811	0.831	0.514	1.939	4.998						
5	1.5	1	0.833	0.834	0.377	0.711	2.309	7.132					
6	1.5	1	0.886	0.893	0.355	0.59	1.341	3.096	7.983				
7	1.5	1	0.902	0.905	0.063	0.155	0.43	2.428	12.071	84.22			
8	1.5	1	0.907	0.93	0.238	0.349	0.516	0.896	1.918	3.856	9.222		
9	1.5	1	0.926	0.953	0.327	0.43	0.586	0.866	1.451	2.371	3.943	6.926	
10	1.5	1	0.967	0.996	0.337	0.427	0.531	0.671	0.913	1.382	2.047	3.119	5.342
3	1.5	2	0.562	0.542	0.877	4.963							
4	1.5	2	0.83	0.829	0.709	1.643	6.541						
5	1.5	2	0.848	0.849	0.638	1.185	3.667	10.501					

Table 2 continues……

6	1.5	2	0.858	0.865	0.586	1.044	2.615	6.898	21.483			
7	1.5	2	0.867	0.865	0.498	0.779	1.292	2.932	6.482	16.015		
8	1.5	2	0.89	0.902	0.131	0.244	0.461	0.979	2.781	8.801	27.025	
9	1.5	2	0.908	0.919	0.454	0.627	0.859	1.218	1.95	3.502	6.198	11.323	
10	1.5	2	0.972	0.967	0.783	0.948	1.156	1.445	1.955	2.872	4.128	6.092	9.866
3	1.5	3	0.568	0.611	1.498	9.221							
4	1.5	3	0.859	0.863	1.4	5.12	9.235						
5	1.5	3	0.88	0.874	1.244	2.351	7.058	20.564					
6	1.5	3	0.885	0.891	1.025	1.674	3.526	8.14	19.583				
7	1.5	3	0.91	0.927	0.296	0.708	2.18	11.305	51.565	351.26			
8	1.5	3	0.935	0.938	0.691	1.026	1.551	2.655	5.632	11.153	25.612		
9	1.5	3	0.936	0.956	0.825	1.119	1.545	2.288	4.006	6.967	12.123	23.129	
10	1.5	3	0.986	0.992	1.133	1.393	1.718	2.176	3.003	4.45	6.436	9.691	16.264
3	2.5	1	0.541	0.609	0.419	1.652							
4	2.5	1	0.841	0.843	0.353	1.986	18.961						
5	2.5	1	0.845	0.853	0.277	0.486	1.246	2.807					
6	2.5	1	0.863	0.869	0.297	0.452	0.795	1.495	2.768				
7	2.5	1	0.916	0.936	0.304	0.417	0.607	1.008	1.602	2.564			
8	2.5	1	0.954	0.957	0.231	0.307	0.406	0.557	0.835	1.273	1.905		
9	2.5	1	0.967	0.967	0.225	0.301	0.402	0.567	0.855	1.332	2.044	3.275	
10	2.5	1	0.981	0.984	0.418	0.476	0.544	0.63	0.746	0.913	1.115	1.352	1.639
3	2.5	2	0.551	0.59	0.8	3.427							
4	2.5	2	0.857	0.874	0.592	1.138	7.374						
5	2.5	2	0.886	0.901	0.542	0.931	2.165	4.586					
6	2.5	2	0.907	0.92	0.687	1.064	2.067	3.723	7.056				
7	2.5	2	0.912	0.925	0.153	0.33	0.852	3.033	9.519	38.097			
8	2.5	2	0.915	0.93	0.463	0.614	0.812	1.115	1.67	2.547	3.809		
9	2.5	2	0.935	0.954	0.467	0.632	0.86	1.244	2.035	3.214	5.08	8.832	
10	2.5	2	0.981	0.992	0.867	1.036	1.259	1.619	2.174	2.832	3.1714	5.095	7.859
3	2.5	3	0.541	0.573	1.16	4.632							
4	2.5	3	0.827	0.837	1.272	3.538	6.24						
5	2.5	3	0.841	0.842	1.049	1.802	4.204	9.047					
6	2.5	3	0.854	0.875	1.03	1.595	3.101	5.585	10.583				
7	2.5	3	0.887	0.896	0.798	1.13	1.672	2.899	4.852	8.202			
8	2.5	3	0.909	0.925	0.653	0.933	1.33	2.093	3.684	6.164	11.196		
9	2.5	3	0.933	0.937	0.797	1.057	1.42	2.052	3.23	4.89	7.474	12.268	
10	2.5	3	0.974	0.986	0.968	1.209	1.474	1.821	2.383	3.303	4.46	6.124	9.189

Table 3. The Asymptotically Minimum and Maximum variance values for the two unknown parameters α and σ and the Corresponding minimum and maximum (AREs) for different values of the known parameter λ

λ	α	σ	Min asvar(α)	Max var(α)	Min var(σ)	max var(σ)	min relative(α)	max relative(α)	min relative(σ)	max relative(σ)
0.5	0.5	1	3.743E-03	5.720E-03	1.786E-02	1.018E-01	0.617	0.986	0.633	0.992
0.5	0.5	2	3.604E-03	5.650E-03	1.698E-02	3.992E-01	0.625	0.949	0.674	0.986
0.5	0.5	3	3.700E-03	5.557E-03	1.519E-02	9.254E-01	0.578	0.993	0.622	0.995
0.5	1.5	1	4.636E-02	7.535E-02	5.282E-02	9.343E-02	0.632	0.981	0.625	0.984
0.5	1.5	2	4.647E-02	7.448E-02	5.632E-02	3.698E-01	0.611	0.983	0.63	0.976
0.5	1.5	3	4.675E-02	7.779E-02	5.317E-01	8.599E-01	0.637	0.989	0.674	0.993
0.5	2.5	1	4.622E-02	2.770E-01	7.945E-02	2.328E-01	0.661	0.986	0.66	0.993
0.5	2.5	2	1.595E-01	2.862E-01	1.897E-01	3.397E-01	0.643	0.994	0.635	0.995
0.5	2.5	3	1.596E-01	2.811E-01	3.938E-01	7.259E-01	0.588	0.984	0.673	0.991
1	0.5	1	3.748E-03	5.703E-03	1.358E-02	4.331E-02	0.554	0.965	0.59	0.987
1	0.5	2	3.701E-03	5.662E-03	4.942E-02	1.728E-01	0.535	0.928	0.621	0.985
1	0.5	3	3.768E-03	6.242E-03	1.173E-01	4.451E-01	0.502	0.939	0.568	0.96
1	1.5	1	4.691E-02	8.937E-02	6.528E-03	2.740E-02	0.605	0.967	0.618	0.996
1	1.5	2	4.715E-02	8.263E-02	2.627E-02	9.932E-02	0.562	0.972	0.542	0.967
1	1.5	3	4.707E-02	7.977E-02	5.896E-02	2.237E-01	0.568	0.986	0.611	0.992
1	2.5	1	1.579E-01	2.823E-01	5.185E-03	2.048E-02	0.541	0.981	0.609	0.984
1	2.5	2	1.791E-01	2.704E-01	2.354E-02	7.848E-02	0.551	0.981	0.59	0.992
1	2.5	3	1.625E-01	2.910E-01	4.801E-02	1.848E-01	0.541	0.974	0.573	0.986
1.5	0.5	1	3.672E-03	5.674E-03	1.224E-02	1.906E-02	0.527	0.913	0.553	0.926
1.5	0.5	2	3.708E-03	5.593E-03	5.009E-02	7.514E-02	0.45	0.907	0.527	0.927
1.5	0.5	3	3.680E-03	5.731E-03	1.101E-01	1.714E-01	0.488	0.933	0.528	0.944
1.5	1.5	1	4.869E-02	8.125E-02	6.795E-03	1.091E-02	0.57	0.938	0.608	0.94
1.5	1.5	2	4.978E-02	7.984E-02	2.806E-02	4.493E-02	0.515	0.934	0.506	0.962
1.5	1.5	3	4.734E-02	8.356E-02	5.951E-02	1.084E-01	0.562	0.958	0.583	0.961
1.5	2.5	1	1.643E-01	2.940E-01	5.382E-03	9.757E-03	0.522	0.935	0.541	0.94
1.5	2.5	2	1.704E-01	2.845E-01	2.242E-02	3.594E-02	0.517	0.962	0.528	0.964
1.5	2.5	3	1.762E-01	2.823E-01	5.225E-02	8.313E-02	0.507	0.927	0.509	0.935
2	0.5	1	3.712E-03	5.531E-03	7.055E-03	1.049E-02	0.527	0.922	0.575	0.927
2	0.5	2	3.767E-03	6.521E-03	2.939E-02	4.648E-02	0.554	0.91	0.62	0.931
2	0.5	3	3.917E-03	5.875E-03	7.223E-02	9.881E-02	0.554	0.941	0.582	0.954
2	1.5	1	5.154E-02	7.583E-02	4.139E-03	5.863E-03	0.611	0.95	0.664	0.952
2	1.5	2	5.240E-02	7.583E-02	1.687E-02	2.345E-02	0.527	0.96	0.557	0.967
2	1.5	3	4.710E-02	7.446E-02	3.315E-02	5.199E-02	0.596	0.924	0.616	0.96

Table 3 continues……

2	2.5	1	1.612E-01	3.227E-01	2.978E-03	5.469E-03	0.522	0.974	0.552	0.976
2	2.5	2	1.568E-01	2.589E-01	1.158E-02	1.885E-02	0.534	0.965	0.61	0.967
2	2.5	3	1.780E-01	2.780E-01	2.991E-02	4.566E-02	0.58	0.94	0.587	0.973
2.5	0.5	1	3.712E-03	6.521E-03	4.523E-03	7.436E-03	0.621	0.96	0.598	0.977
2.5	0.5	2	3.776E-03	5.875E-03	1.902E-02	2.810E-02	0.624	0.932	0.627	0.952
2.5	0.5	3	3.945E-03	5.552E-03	4.731E-02	6.055E-02	0.63	0.952	0.647	0.974
2.5	1.5	1	5.070E-02	7.445E-02	2.615E-03	3.697E-03	0.652	0.973	0.69	0.984
2.5	1.5	2	4.765E-02	7.593E-02	9.642E-03	1.515E-02	0.613	0.968	0.621	0.979
2.5	1.5	3	4.779E-02	8.682E-02	2.170E-02	3.889E-02	0.595	0.965	0.626	0.984
2.5	2.5	1	1.640E-01	2.921E-01	1.946E-03	3.337E-03	0.547	0.983	0.578	0.986
2.5	2.5	2	1.706E-01	2.847E-01	8.157E-03	1.319E-02	0.597	0.991	0.613	0.976
2.5	2.5	3	1.588E-01	2.644E-01	1.690E-02	2.772E-02	0.674	0.942	0.685	0.967

5. NUMERICAL ILLUSTRATION

From sections (3 and 4), it is noticed that there are no explicit forms for obtaining estimators for the exponentiated Fréchet distribution or the optimum group limits. Therefore, numerical solution and computer facilities are needed. The main objective of this section is to numerically illustrate an example to determine the maximum likelihood estimates of the parameters for the exponentiated Fréchet distribution, as well as the asymptotically optimum group limits and the asymptotic relative efficiencies of the maximum likelihood estimates of the parameters with optimum grouping using the following steps:

Step 1: A random sample of size 100 is generated from uniform (0,1) distribution. Then the usual transformation technique (inverse transformation) has been used to get the corresponding exponentiated Fréchet random sample.

Step 2: The shape parameters α and σ of the exponentiated Fréchet distribution take different values, α, take the values 0.5, 1, and 1.5, whereas σ take the values 1, 2, and 3. The number of groups "k" takes the values 3, 4, 5, 6, 7, 8, 9, and 10.

Step 3: Construct a frequency distribution (grouping) from the generated random numbers by enter a pre-specified group limits (not equal), and determine the number of observations falling in each group sample.

Step 4: The frequency distribution obtained in step (3) is used to determine the maximum likelihood estimators of α and σ. Also, the values of the information matrix elements for the grouped data are computed and displayed.

Step 5: Using the elements of the information matrix for the grouped data obtained in step (4), the optimum group limits can be evaluated, this can be done by maximizing the determinant of the information matrix, with respect to group limits x_i.

Step 6: The asymptotic variance-covariance matrix for the optimum group limits will be calculated depending on the optimum group limits obtained in step (5).

Step 7: The asymptotic variances of $\hat{\alpha}$ and $\hat{\sigma}$ from the raw sample (which has been generated in step (1)) will be evaluated and used to find the asymptotic relative efficiencies of $\hat{\alpha}$ and $\hat{\sigma}$ as in the following step.

Step 8: Evaluate the asymptotic relative efficiencies of the grouped sample estimates $\hat{\alpha}$ and $\hat{\sigma}$ using the optimum group limits by comparing its asymptotic variances obtained in step (6) with the corresponding asymptotic variances of $\hat{\alpha}$ and $\hat{\sigma}$ from the raw sample obtained in step (7).

Step 9: Repeat all the previous 8 steps for different values of λ (λ=0.5, 1, 1.5, 2, 2.5).

6. CONCLUSIONS

1. Table 3 represents the range of the relative efficiencies for the shape parameter α as well for the scale parameter σ over $k = 3,4,5,6,7,8,9,10$, supposing different values for α, σ and λ.
2. From Table 2 the efficiencies are greater than 50% for k=3, α = 0.5, 1.5, 2.5, and σ = 1, 2, 3 and increase rapidly with k and are about 83% when k is more than or equal to 5 groups. Thus the optimum group lengths suggested in Table 2 can be used to form a grouped sample with $k \geq 5$ unequi-spaced groups of a given raw sample without much loss in efficiency of the MLEs.
3. From Table 2, for a given number of group k, The asymptotic relative efficiency of the maximum likelihood estimate of the scale parameter σ is greater than the asymptotic relative efficiency of the maximum likelihood estimate of the shape parameter α in almost 94.5% of the all cases because, the restriction of unequi-spaced grouping causes less loss of information about σ than of α, and this can be noted from Table 2.
4. From Table 3 and for different values of λ in 93% of all cases the (AREs) of the MLE of the scale parameter σ is greater than the (AREs)of the MLE of the shape parameter α, because of the unequi-spaced grouping causes loss of information about σ less than it causes about α.
5. From Table 3 as λ increasing from 0.5 to 1.5 the minimum and maximum values of (AREs) of both the two estimates α and σ are decreasing, then as λ increasing from 1.5 up to 2.5 the minimum and maximum values of (AREs) of both the two estimates α and σ are increasing.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

1. Kotz S, Nadarajah S. Extreme Value Distributions: Theory and Applications. Imperial College Press: London; 2000.
2. Gupta RD. Exponentiated Exponential Family: An alternative to gamma and Weibull distribution. Biometrical Journal. 2001;43(1):117-130.
3. Gupta RC, and Kundu D. Generalized exponential distribution. Australian and New Zealand Journal of Statistics. 1999;41(2):173-188.
4. Nadarajah S, Kotz S. The Exponentiated type distributions. Acta Applicandae Mathematicae, 2006;92:97–111.
5. Mudholkar GS, Huston AD. The exponentiated Weibull family: some properties and a flood data application. Communications in Statistics-Theory and Methods. 1996;25:3059-3083.
6. Gupta RD. Kundu D. Generalized exponential distributions: different methods of estimation. Journal of Statistical Computation and Simulation. 2001;69(4):315-338.

7. Surles JG, Padgett W.J. Inference for reliability and stress-strength for a scaled Burr type X distribution. Lifetime Data Analysis. 2001;7:187-200.
8. Gupta RD, Kundu D. Generalized exponential distribution: existing results and some recent developments. Journal of Statistical Planning and Inference. 2007;137:3537–3547.
9. Kundu, D. and Gupta, R.D. Generalized exponential distribution; Bayesian inference. Computational Statistics and Data Analysis. 2008;52:1873-1883.
10. Ali M. M. and Woo, J. On the Ratio of Two Independent Exponentiated Pareto Variables. Australian Journal of Statistics. 2010;39(4):329-340.
11. Raja TA, Mir AH.. On Extension of some exponentiated distributions with application. Int. J. Contemp. Math. Sciences. 2011;6(8):393-400.
12. Jamjoom AA, Al-Saiary ZA. Computing the moments of order statistics from independent nonidentically distributed exponentiated Fréchet variables. Journal of Probability and Statistics; 2012. Article ID 248750.
13. Kulldorff G. Contribution to the Theory of Estimation from Grouped and Partially Grouped Samples. John Wiley & sons: New York; 1961.
14. Archer NP. Maximum likelihood estimation with Weibull models when the data are grouped. Communication in Statistics, Theory and Methods. 1982;11(2):199-207.
15. Cheng KF, Chen CH. Estimation of the Weibull parameters with grouped data, Communication in Statistics, Theory and Methods. 1988;17(2):325-341.
16. Rosaiah k, Kantam RRL, Narasimham VL. Optimum class limits for estimation in two parameter gamma distribution from a grouped data. Communication in Statistics, simulation. 1991;20(4):1173-1189.
17. Rao AV, Rao AVD, Narasimham VL. Asymptotically optimal grouping for maximum likelihood estimation of Weibull parameters. Communication in Statistics, simulation. 1994;4:1077-1096.
18. Chen Z, Mi J. Statistical estimation for the scale parameter of the gamma distribution based on grouped data. Communication in Statistics, Theory and Methods. 1998;27(12):3035-3045.
19. Komori Y, Hirose H. Parameter estimation based on grouped or continuous data for truncated exponential distribution. Communication in Statistics, Theory and Methods. (2002);6(6):889-900.
20. Kantam RRL, Roa, AV, Roa GS. Optimum group limits for estimation in scaled log-logistic distribution from grouped data, Statistical Papers. 2005;46(3):359-377.
21. Abuammoh A, Sarhan AM. Parameter estimation for the generalized exponential distribution. International Journal of Reliability and Applications. 2007;8(1):17-25.
22. Aludaat KM, Alodat MT, Alodat TT. Parameter estimation of Burr type X distribution for grouped data. Applied Mathematical Sciences. 2008;2(9):415 – 423.
23. Xia J, Mi J, Zhou YY. On the Existence and uniqueness of the maximum likelihood estimators of normal and lognormal population parameters with grouped data. Journal of Probability and Statistics; 2009; Article ID 310575, 16 pages. doi:10.1155/2009/310575.
24. Zaindin M. Parameter estimation of the modified Weibull model based on grouped and censored data. International Journal of Basic &Applied Sciences. 2010;10(2):122-132.
25. Jafari A, Makhdoom I. Statistical inference on the Burr type XII distribution using grouped and un-grouped data. Australian Journal of Basic and Applied Sciences. 2011;5(9):238-244.

26. Shadrokh A, Nasiri P. Estimation on the minimax distribution using grouped data. Journal of applied Science. 2011;11(18):3333-3337.
27. Lehmann E L. Theory of Point Estimation. Wiley: New York; 1991.

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