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Parameter Estimation for Kappa Distribution with Four-Parameter Under Type II Censored Samples

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Abstract: In this paper, maximum likelihood estimators (MLE's) for the unknown parameters and the corresponding asymptotic variance covariance matrix of the four-parameter kappa distribution are obtained under type II censored sample. Results obtained by Winchester (2000) in the complete case may be considered as a special case from present results. An illustrative example is carried out by using a simulated data.

Key words: Four-parameter kappa distribution, maximum likelihood estimators, type II censored sample, asymptotic variance covariance matrix.

INTRODUCTION

The four-parameter kappa distribution (abbreviated as K4D), introduced by Hosking (1994), is a very general distribution which includes a variety of distributions such as the generalized extreme-value distribution, the generalized logistic distribution, the generalized Pareto, etc. Though the K4D was invented for a modeling hydrologic data, its application may cover a wide spectrum of areas ranging from actuarial science, economics, finance to medicine.

Hosking (1994) and Parida (1999) suggest the K4D as an alternative to the generalized extreme value distribution among other common three parameter distributions and it is useful when generalized extreme value does not perform well.

Hosking (1994) introduced the probability density and the cumulative distribution functions of the four-parameter kappa distribution (K4D) given as, for $\alpha > 0$, $-\infty < x < \infty$,

$$f(x)=\alpha^{-1}[1-k(x-\xi)/\alpha]^{\frac{1}{k}-1}\{1-h[1-k(x-\xi)/\alpha]^{\frac{1}{k}}\}^{\frac{1}{h}-1}, \qquad if\ k\neq 0, h\neq 0 \qquad (1)$$

$$F(x)=\{1-h[1-k(x-\xi)/\alpha]^{\frac{1}{k}}\}^{\frac{1}{h}}, \qquad if\ k\neq 0, h\neq 0 \qquad (2)$$

where ξ is a location parameter, α is a scale parameter, and h and k are shape parameters, Which include the limits at k = 0 and h = 0. The distribution (1) includes many distributions as special cases: the generalized Pareto distribution for h=1, the generalized extreme value distribution for h=0, the generalized logistic distribution for h=-1, the Gumbel distribution for h=0, k=0, the exponential distribution for h=1, k=0, the uniform distribution for h=1, k=1, and the reverse exponential distribution (i.e. 1-F(-x) is exponential) for h=0, k=1. For $\xi = \beta$, $\alpha = \beta/\alpha\theta$, $k = -1/\alpha\theta$, and $h = -\alpha$ the cumulative distribution function (2) becomes equivalent to the cumulative distribution function of the three-parameter kappa distribution with shape parameter. Thus, the four-parameter kappa distribution may be regarded as a reparameterization of the three-parameter kappa distribution with an additional location parameter (Hosking 1994). Winchester (2000) estimated the unknown parameters of the four-parameter kappa distribution using maximum likelihood, moment and L-moment estimation methods, and maked a simulation study.

In Section 2 the maximum likelihood estimators for the unknown parameters and the corresponding asymptotic variance covariance matrix are obtained under type II censored sample. An illustrative example is carried out in Section 3.

2. Maximum likelihood estimators for Type II Censored Sample:

In a typical life test, *n* specimens are placed under observation and as each failure occurs the time is noted. When a predetermined total number of failures *r* is reached, the test is terminated. In this case the data

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collected consist of observations $x_{(1)} < x_{(2)} < ... < x_{(r)}$ plus the information that (*n-r*) items survived beyond the time of termination $x_{(r)}$, this censoring scheme is known as type II censoring and also the collected data is said to be censored type II data. Cohen (1965) gave the likelihood function for type II censoring:

$$L = C\prod_{i=1}^{r} f(x_{(i)};\theta)[1-F(x_{(r)};\theta)]^{n-r},$$

where C is a constant, r is the number of uncensored sample, $x_{(r)}$ is the observed value of the i^{th} order statistic, $f(x_{(i)};\theta)$ and $F(x_{(r)};\theta)$ are the density function and the cumulative function of the underlying distribution, respectively.

For the four-parameter kappa distribution (1), the likelihood function will be

$$L = C\alpha^{-r}\prod_{i=1}^{r}\left[\left(1-\frac{k(x_{(i)}-\xi)}{\alpha}\right)^{\frac{1}{k}-1}\left\{1-h\left(1-\frac{k(x_{(i)}-\xi)}{\alpha}\right)^{\frac{1}{k}}\right\}^{\frac{1}{h}-1}\right]\left[1-\left\{1-h\left(1-\frac{k(x_{(r)}-\xi)}{\alpha}\right)^{\frac{1}{k}}\right\}^{\frac{1}{h}}\right]^{n-r} \quad (3)$$

Taking the logarithm, (3) becomes

$$\ln L \propto -r\ln\alpha + \left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\ln\left[1-\frac{k(x_{(i)}-\xi)}{\alpha}\right] + \left(\frac{1}{h}-1\right)\sum_{i=1}^{r}\ln\left\{1-h\left(1-\frac{k(x_{(i)}-\xi)}{\alpha}\right)^{\frac{1}{k}}\right\}$$

$$+(n-r)\ln\left[1-\left\{1-h\left(1-\frac{k\ (x_{(r)}-\xi)}{\alpha}\right)^{\frac{1}{k}}\right\}^{\frac{1}{h}}\right]. \quad (4)$$

$$\ln L \propto -r\ln\alpha + \left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\ln(a_i) + \left(\frac{1}{h}-1\right)\sum_{i=1}^{r}\ln\left(d_i\right) + (n-r)\ln\left(1-\left(d_r\right)^{\frac{1}{h}}\right),$$

Where $a_i = 1-kb_i,\ b_i = \dfrac{x_{(i)}-\xi}{\alpha},\ d_i = 1-h\left(a_i\right)^{\frac{1}{k}}$.

Differentiate (4) with respect to the unknown parameters and equating to zero:

$$\frac{\partial \ln L}{\partial\alpha} = \frac{-r}{\alpha} + \left(\frac{1}{k}-1\right)\sum_{1=1}^{r}\left(\frac{k\ b_i}{\alpha\ a_i}\right) + \left(\frac{1}{h}\ -1\right)\sum_{i=1}^{r}\left[\frac{\left(-hb_i\left(a_i\right)^{\frac{1}{k}-1}\right)}{\alpha\ d_i}\right] + (n-r)\left[\frac{b_r\left(a_r\right)^{\frac{1}{k}-1}\left(d_r\right)^{\frac{1}{h}-1}}{\alpha\ \left(1-\ \left(d_r\right)^{\frac{1}{h}}\right)}\right] = 0,$$

$$\frac{\partial \ln L}{\partial\xi} = \left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\left(\frac{k}{\alpha\ \ a_i}\right) - \left(\frac{1}{h}-1\right)\sum_{i=1}^{r}\left[\frac{h\left(a_i\right)^{\frac{1}{k}-1}}{\alpha d_i}\right] + (n-r)\left[\frac{\left(a_r\right)^{\frac{1}{k}-1}\left(d_r\right)^{\frac{1}{h}-1}}{\alpha\left(1-\ \ \left(d_r\right)^{\frac{1}{h}}\right)}\right] = 0,$$

$$\frac{\partial \ln L}{\partial h}=\frac{-1}{h^2}\sum_{i=1}^{r}\ln d_i-\left(\frac{1}{h}-1\right)\sum_{i=1}^{r}\left[\frac{(a_i)^{\frac{1}{k}}}{d_i}\right]-(n-r)\left[\frac{(d_r)^{\frac{1}{h}}(f_r)}{\left(1-(d_r)^{\frac{1}{h}}\right)}\right]=0,\text{and}$$

$$\frac{\partial \ln L}{\partial k}=\frac{-1}{k^2}\sum_{i=1}^{r}\ln a_i-\left(\frac{1}{k}-1\right)\sum_{i=1}^{r}\left(\frac{b_i}{a_i}\right)-\left(\frac{1}{h}-1\right)\sum_{i=1}^{r}\left[\frac{h(a_i)^{\frac{1}{k}}(s_i)}{d_i}\right]+(n-r)\left[\frac{(a_r)^{\frac{1}{k}}(d_r)^{\frac{1}{h}-1}(s_r)}{\left(1-(d_r)^{\frac{1}{h}}\right)}\right]=0,$$

Where $a_i=(1-kb_i)$, $b_i=\left(\frac{x_{(i)}-\xi}{\alpha}\right)$, $d_i=\left(1-h(a_i)^{\frac{1}{k}}\right)$, $f_r=\left[\frac{-1}{h^2}\ln d_r-\left(\frac{(a_r)^{\frac{1}{k}}}{h\ \ d_r}\right)\right]$,

and $s_i=\left[\frac{-1}{k^2}\ln a_i-\left(\frac{b_i}{k\ \ a_i}\right)\right]$ Then, the maximum likelihood estimates of the parameters α,ξ,h and k can be obtained by solving the above system of equations. No explicit form for these parameters, we use a numerical technique using Mathcad2001 Package to obtain $\hat{\alpha}$, $\hat{\xi}$, $\hat{h}$ and $\hat{k}$.

The asymptotic variance-covariance matrix for the estimators $\hat{\alpha}$, $\hat{\xi}$, $\hat{h}$ and $\hat{k}$ can be obtained by inverting the information matrix with the elements that are negative of the expected values of the second order derivatives of logarithm of the likelihood function. The elements of the asymptotic variance covariance matrix can be approximated by Cohen's (1965) result:

$$-E\left(\frac{\partial^2 \ln L(\underline{\theta})}{\partial\theta_i\partial\theta_j}\right)\approx-\left.\frac{\partial^2 \ln L(\underline{\theta})}{\partial\theta_i\partial\theta_j}\right|_{\underline{\theta}=\hat{\underline{\theta}}}.$$

Let $A=\left(1-\hat{d}_r^{\frac{1}{\hat{h}}}\right)^2$, $B=\hat{d}_r^{\frac{1}{\hat{h}}-1}\ \hat{a}_r^{\frac{1}{\hat{k}}-1}$, $C=\hat{d}_r^{\frac{1}{\hat{h}}-1}\ \hat{a}_r^{\frac{1}{\hat{k}}}$, $D=\hat{d}_r^{\frac{1}{\hat{h}}-2}\ \hat{a}_r^{\frac{2}{\hat{k}}-2}$ and $E=\frac{n-r}{A}$

$$\left.\frac{-\partial^2 \ln L}{\partial\alpha^2}\right|_{\substack{\alpha=\hat{\alpha},\\ \xi=\hat{\xi},\\ h=\hat{h},\\ k=\hat{k}}}=\left(\hat{k}-1\right)\sum_{i=1}^{r}\left[\frac{-2\hat{b}_i\hat{a}_i-\hat{k}\left(\hat{b}_i\right)^2}{\left(\hat{\alpha}\ \hat{a}_i\right)^2}\right]-$$

$$\left(1-\hat{h}\right)\sum_{i=1}^{r}\left[\frac{\hat{d}_i\left(2\hat{b}_i(\hat{a}_i)^{\frac{1}{\hat{k}}-1}-\left(1-\hat{k}\right)\hat{b}_i^2\left(\hat{a}_i\right)^{\frac{1}{\hat{k}}-2}\right)-\hat{h}\left(\hat{b}_i\left(\hat{a}_i\right)^{\frac{1}{\hat{k}}-1}\right)^2}{\left(\hat{\alpha}\hat{d}_i\right)^2}\right]$$

$$-\frac{r}{\hat{\alpha}^2}-\left(E\hat{\alpha}^{-2}\right)\left[\sqrt{A}\ \left\{\hat{d}_r^{\frac{1}{\hat{h}}-1}\left(-2\hat{b}_r\hat{a}_r^{\frac{1}{\hat{k}}-1}+\left(1-\hat{k}\right)\hat{b}_r^2\hat{a}_r^{\frac{1}{\hat{k}}-2}\right)-\left(1-\hat{h}\right)\hat{b}_r^2 D\right\}-\left(\hat{b}_r B\right)^2\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\alpha\ \partial h}\right|_{\substack{\alpha=\hat\alpha,\\ \xi=\hat\xi,\\ h=\hat h,\\ k=\hat k}} = \left(\frac{-1}{\hat h^{2}}\right)\sum_{i=1}^{r}\left[\frac{\hat h\ \hat b_i\ \hat a_i^{\frac{1}{\hat k}-1}}{\left(\hat\alpha\ \hat d_i\right)}\right]-\left(\frac{1}{\hat h}-1\right)\sum_{i=1}^{r}\left[\frac{\left[-\hat\alpha\ \hat d_i\ \hat b_i\ \hat a_i^{\frac{1}{\hat k}-1}-\hat\alpha\ \hat h\ \hat b_i\ \hat a_i^{\frac{2}{\hat k}-1}\right]}{(\hat\alpha\ \hat d_i)^2}\right]\times$$

$$-E\left[\sqrt{A}\left\{\left(\frac{-\hat b_r}{\hat\alpha\ \hat h^{2}}\right)B\ln\hat d_r-\left(\frac{1}{\hat h}-1\right)\left(\hat a_r^{\frac{2}{\hat k}-1}\right)\hat\alpha^{-1}\ \hat d_r^{\frac{1}{\hat h}-2}\ \hat b_r\right\}+\left(\frac{\hat b_r}{\hat\alpha}\right)\hat a_r^{\frac{1}{\hat k}-1}\ \hat d_r^{\frac{2}{\hat h}-1}\ \hat f_r\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\xi\ \partial h}\right|_{\substack{\alpha=\hat\alpha,\\ \xi=\hat\xi,\\ h=\hat h,\\ k=\hat k}} = \left(\frac{-1}{\hat h}\right)\sum_{i=1}^{r}\left[\frac{\hat a_i^{\frac{1}{\hat k}-1}}{\left(\hat\alpha\ \hat d_i\right)}\right]-\left(\frac{1}{\hat h}-1\right)\sum_{i=1}^{r}\left[\frac{\left[-\hat\alpha\ \hat d_i\ \hat a_i^{\frac{1}{\hat k}-1}-\hat\alpha\ \hat h\ \hat a_i^{\frac{2}{\hat k}-1}\right]}{(\hat\alpha\ \hat d_i)^2}\right]\times$$

$$-E\left[\sqrt{A}\left\{\left(\frac{-1}{\hat\alpha\ \hat h^{2}}\right)B\ln\hat d_r-\left(\frac{1}{\hat h}-1\right)\left(\hat a_r^{\frac{2}{\hat k}-1}\right)\hat\alpha^{-1}\ \hat d_r^{\frac{1}{\hat h}-2}\right\}+\left(\frac{1}{\hat\alpha}\right)\hat a_r^{\frac{1}{\hat k}-1}\ \hat d_r^{\frac{2}{\hat h}-1}\ \hat f_r\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\xi^2}\right|_{\substack{\alpha=\hat\alpha,\\ \xi=\hat\xi,\\ h=\hat h,\\ k=\hat k}} = \left(\frac{1}{\hat k}-1\right)\sum_{i=1}^{r}\left[\frac{\hat k^{2}}{\left(\hat\alpha\ \hat a_i\right)^2}\right]-\left(\frac{1}{\hat h}-1\right)\sum_{i=1}^{r}\left[\frac{\left(\hat h\left(\hat k-1\right)\hat d_i\left(\hat a_i\right)^{\frac{1}{\hat k}-2}\right)-\left(\hat h\left(\hat a_i\right)^{\frac{1}{\hat k}-1}\right)^2}{\left(\hat\alpha\ \hat d_i\right)^2}\right]$$

$$-\left(E\,\hat\alpha^{-2}\right)\left[\sqrt{A}\left\{\hat d_r^{\frac{1}{\hat h}-1}\left(-\left(1-\hat h\right)D+\left(1-\hat k\right)\hat d_r^{\frac{1}{\hat h}-1}\ \hat a_r^{\frac{1}{\hat k}-2}\right)\right\}-B^2\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial h^2}\right|_{\substack{\alpha=\hat\alpha,\\ \xi=\hat\xi,\\ h=\hat h,\\ k=\hat k}} = -\left(\frac{2}{\hat h^{3}}\right)\sum_{i=1}^{r}\ln\ \hat d_i-\left(\frac{2}{\hat h^{2}}\right)\sum_{i=1}^{r}\left[\frac{\hat a_i^{\frac{1}{\hat k}}}{\hat d_i}\right]+\left(\frac{1}{\hat h}-1\right)\sum_{i=1}^{r}\left[\frac{\left(\left(\hat a_i\right)^{\frac{1}{\hat k}}\right)^2}{\left(\hat d_i\right)^2}\right]$$

$$+\ E.\left[\sqrt{A}\left\{\hat d_r^{\frac{1}{\hat h}}\left(\hat f_r\right)^2+\left(\frac{2}{\hat h^{2}}\right)C+\left(\frac{2}{\hat h^{3}}\right)\ln\hat d_r\left(\hat d_r^{\frac{1}{\hat h}}\right)-\hat h^{-1}\ \hat d_r^{\frac{1}{\hat h}-2}\ \hat a_r^{\frac{2}{\hat k}}\right\}+\left(\hat f_r\,\hat d_r^{\frac{1}{\hat h}}\right)^2\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\alpha\ \partial\xi}\right|_{\substack{\alpha=\hat{\alpha},\\ \xi=\hat{\xi},\\ h=\hat{h},\\ k=\hat{k}}} = \left(1-\hat{k}\right)\sum_{i=1}^{r}\left[\frac{1}{\left(\hat{\alpha}^2\ \hat{a}_i\right)}+\frac{\hat{k}\ \hat{b}_i}{\left(\hat{\alpha}\ \hat{a}_i\right)^2}\right]-$$

$$\left(\hat{h}-1\right)\sum_{i=1}^{r}\left[\frac{\left[\left(1-\hat{k}\right)\hat{d}_i\,\hat{b}_i\,\hat{a}_i^{\frac{1}{\hat{k}}-2}-\hat{d}_i\,\hat{a}_i^{\frac{1}{\hat{k}}-1}+\hat{h}\ \hat{b}_i\left(\hat{a}_i^{\frac{1}{\hat{k}}-1}\right)^2\right]}{(\hat{\alpha}\ \hat{d}_i)^2}\right]\times$$

$$-\left(E\,\hat{\alpha}^{-2}\right)\left[\sqrt{A}\ \left\{\left(1-\hat{k}\right)\hat{d}_r^{\frac{1}{\hat{h}}-1}\ \hat{a}_r^{\frac{1}{\hat{k}}-2}\ \hat{b}_r-\left(1-\hat{h}\right)D\,\hat{b}_r-B\right\}-\hat{b}_r\,B^2\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial k^2}\right|_{\substack{\alpha=\hat{\alpha},\\ \xi=\hat{\xi},\\ h=\hat{h},\\ k=\hat{k}}} = \left(\frac{-2}{\hat{k}^3}\right)\sum_{i=1}^{r}\ln\hat{a}_i-\left(\frac{2}{\hat{k}^2}\right)\sum_{i=1}^{r}\left[\frac{\hat{b}_i}{\hat{a}_i}\right]+\left(\frac{1}{\hat{k}}-1\right)\sum_{i=1}^{r}\left[\frac{\hat{b}_i}{\left(\hat{a}_i\right)}\right]^2-\left(\frac{1}{\hat{h}}-1\right).$$

$$\sum_{i=1}^{r}\left[\frac{\left[\left(-\hat{d}_i\,\hat{h}\ \hat{a}_i^{\frac{1}{\hat{k}}}\right)\left((\hat{s}_i)^2+\left(\frac{2}{\hat{k}^3}\right)\ln\hat{a}_i+\left(\frac{2\ \hat{b}_i}{\hat{k}^2\ \hat{a}_i}\right)-\left(\frac{(\hat{b}_i)^2}{\hat{k}\,(\hat{a}_i)^2}\right)\right)\right]-\left(\hat{s}_i\,\hat{a}_i^{\frac{1}{\hat{k}}}\,\hat{h}\right)^2}{(\hat{d}_i)^2}\right]$$

$$-E.\left[\sqrt{A}\ \left\{C\left(\hat{s}_r\right)^2+\left(\hat{h}-1\right)\ \hat{a}_r^{\frac{2}{\hat{k}}}\,\hat{d}_r^{\frac{1}{\hat{h}}-2}\,(\hat{s}_r)^2+C\left[\left(\frac{2}{\hat{k}^3}\right)\ln\hat{a}_r+\left(\frac{2\ \hat{b}_r}{\hat{k}^2\ \hat{a}_r}\right)-\left(\frac{\hat{b}_r^2}{\hat{k}\ \hat{a}_r^2}\right)\right]\right\}-\left(\hat{s}_r\,C\right)^2\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial\alpha\ \partial h}\right|_{\substack{\alpha=\hat{\alpha},\\ \xi=\hat{\xi},\\ h=\hat{h},\\ k=\hat{k}}} = \left(\frac{1}{\hat{k}}\right)\sum_{i=1}^{r}\left(\frac{\hat{b}_i}{\hat{\alpha}\ \hat{a}_i}\right)-\left(\frac{1}{\hat{k}}-1\right)\sum_{i=1}^{r}\left[\frac{\left(\hat{\alpha}\,\hat{a}_i\,\hat{b}_i+\hat{k}\,\hat{\alpha}\,\hat{b}_i^2\right)}{\left(\hat{\alpha}\ \hat{a}_i\right)^2}\right]+\left(1-\hat{h}\right)\left(\frac{1}{\hat{\alpha}}\right).$$

$$\sum_{i=1}^{r}\left[\frac{\left[\left(\frac{\hat{b}_i\ \ln\ \hat{a}_i}{\hat{k}^2}\right)\left(-\hat{d}_i\,\hat{a}_i^{\frac{1}{\hat{k}}-1}-\hat{h}\,\hat{a}_i^{\frac{2}{\hat{k}}-1}\right)\right]-\left(\frac{1}{\hat{k}}-1\right)\hat{b}_i^2\,\hat{a}_i^{\frac{1}{\hat{k}}-2}\,\hat{d}_i-\hat{b}_i^2\,\hat{a}_i^{\frac{2}{\hat{k}}-2}\left(\frac{\hat{h}}{\hat{k}}\right)}{(\hat{d}_i)^2}\right]-E\quad.$$

$$\left[\begin{matrix}\left\{\left(\frac{-\hat{b}_r\ \ln\hat{a}_r}{\hat{\alpha}\ \hat{k}^2}\right)B-\left(\frac{1}{\hat{k}}-1\right)\left(\frac{\hat{b}_r^2\,\hat{a}_r^{\frac{1}{\hat{k}}-2}\,\hat{d}_r^{\frac{1}{\hat{h}}-1}}{\hat{\alpha}}\right)+\left(1-\hat{h}\right)\left(\frac{\hat{b}_r\ln\hat{a}_r}{\hat{\alpha}\ \hat{k}^2}\right)\hat{a}_r^{\frac{2}{\hat{k}}-1}\ \hat{d}_r^{\frac{1}{\hat{h}}-2}+\left(1-\hat{h}\right)\left(\frac{\hat{b}_r^2}{\hat{\alpha}\ \hat{k}}\right)D\right\}.\\ \sqrt{A}+\left(\frac{1}{\hat{\alpha}\ \hat{k}}\right)\left(\hat{b}_r\,B\right)^2+\left(\frac{\hat{b}_r}{\hat{\alpha}\ \hat{k}^2}\right)\ \hat{a}_r^{\frac{2}{\hat{k}}-1}\ \hat{d}_r^{\frac{2}{\hat{h}}-2}\ \ln\hat{a}_r.\end{matrix}\right],$$

$$\left.\frac{-\partial^2 \ln L}{\partial k \partial h}\right|_{\substack{\alpha=\hat{\alpha},\\ \xi=\hat{\xi},\\ h=\hat{h},\\ k=\hat{k}}} = -\sum_{i=1}^{r}\left[\frac{\hat{s}_i\,\hat{a}_i^{\frac{1}{\hat{k}}}}{\left(\hat{d}_i\right)}\right] + \left(1-\hat{h}\right)\sum_{i=1}^{r}\left[\frac{\left[\hat{s}_i\ \hat{a}_i^{\frac{2}{\hat{k}}}\right]}{(\hat{d}_i)^2}\right] - E.$$

$$\left[\sqrt{A}\left\{\left(\hat{s}_r\ \hat{f}_r\right)C + \left(\hat{a}_r^{\frac{2}{\hat{k}}}\right)\hat{d}_r^{\frac{1}{\hat{h}}-2}\ \hat{s}_r\right\} + \left(\hat{s}_r\ \hat{f}_r\right)\hat{a}_r^{\frac{1}{\hat{k}}}\ \hat{d}_r^{\frac{2}{\hat{h}}-1}\right],$$

and

$$\left.\frac{-\partial^2 \ln L}{\partial \xi\ \partial k}\right|_{\substack{\alpha=\hat{\alpha},\\ \xi=\hat{\xi},\\ h=\hat{h},\\ k=\hat{k}}} = \left(\frac{1}{\hat{k}}\right)\sum_{i=1}^{r}\left(\frac{1}{\hat{\alpha}\ \hat{a}_i}\right) - \left(\frac{1}{\hat{k}}-1\right)\sum_{i=1}^{r}\left[\frac{\left(\hat{\alpha}\,\hat{a}_i + \hat{k}\ \hat{\alpha}\ \hat{b}_i\right)}{\left(\hat{\alpha}\ \hat{a}_i\right)^2}\right] - \left(\frac{1}{\hat{h}}-1\right).$$

$$\sum_{i=1}^{r}\left[\frac{\left[\left(\frac{\hat{h}\ \ln\ \hat{a}_i}{\hat{\alpha}\ \hat{k}^2}\right)\left(\hat{d}_i\ \hat{a}_i^{\frac{1}{\hat{k}}-1} + \hat{h}\ \hat{a}_i^{\frac{2}{\hat{k}}-1}\right)\right] + \left(\frac{1}{\hat{k}}-1\right)\left(\frac{\hat{h}}{\hat{\alpha}}\right)\hat{b}_i\,\hat{a}_i^{\frac{1}{\hat{k}}-2}\ \hat{d}_i + \hat{b}_i\,\hat{a}_i^{\frac{2}{\hat{k}}-2}\left(\frac{\hat{h}^2}{\hat{\alpha}\ \hat{k}}\right)}{(\hat{d}_i)^2}\right] - E.$$

$$\left[\begin{array}{l}\sqrt{A}\left\{\left(\frac{-\ \ln\hat{a}_r}{\hat{\alpha}\ \hat{k}^2}\right)B - \left(\frac{1}{\hat{k}}-1\right)\left(\frac{\hat{b}_r\,\hat{a}_r^{\frac{1}{\hat{k}}-2}\ \hat{d}_r^{\frac{1}{\hat{h}}-1}}{\hat{\alpha}}\right) + \left(1-\hat{h}\right)\left(\frac{\ln\hat{a}_r}{\hat{\alpha}\ \hat{k}^2}\right)\hat{a}_r^{\frac{2}{\hat{k}}-1}\ \hat{d}_r^{\frac{1}{\hat{h}}-2} + \left(1-\hat{h}\right)\left(\frac{\hat{b}_r}{\hat{\alpha}\ \hat{k}}\right)D\right\}\\ -\left(\frac{1}{\hat{\alpha}}\right)\hat{a}_r^{\frac{2}{\hat{k}}-1}\ \hat{d}_r^{\frac{2}{\hat{h}}-2}\ \hat{s}_r\end{array}\right].$$

Again, a numerical technique using Mathcad2001 Package and computer facilities are used to obtain the variance-covariance matrix.

3. A Numerical Illustration:

In this section, we present a numerical example to illustrate different maximum likelihood estimators, and their asymptotic variance covariance matrix. To generate random numbers from four-parameter kappa distribution (1) we have:

$$\mathrm{x} = \xi + \frac{\alpha}{k}\left(1-\ \left[\frac{1-u^h}{\mathrm{h}}\right]^k\right), \tag{5}$$

where u comes from the uniform distribution (0,1). Using Mathcad2001 package, we generate 50 numbers from (5) with $\alpha = 1$, $\xi = 1$, h = 0.2 and k =-3. For type II censored sample, suppose we have r = 20, 30, 40 and 50 respectively. When $r=n$ the results reduce to complete sample case [result of Winchester (2000) as a special case]. All results are displayed in the following table.

Table 1: The maximum likelihood estimates of the parameters of the four-parameter kappa distribution, the asymptotic variances and covariance's based on type II censored samples.

Sample Size"n"	r	$\hat{\alpha}$	$\hat{\xi}$	$\hat{h}$	$\hat{k}$	$Var(\hat{\alpha})$	$Var(\hat{\xi})$	$Var(\hat{h})$	$Var(\hat{k})$
50	20	0.916	0.838	0.531	-5.297	6.178E-3	5.483E-3	0.083	4.492
	30	0.652	0.828	0.534	-3.649	6.253E-3	7.703E-3	0.204	1.2
	40	0.61	0.827	0.555	-3.217	0.014	0.057	1.457	0.902
	50	0.642	0.838	0.516	-3.314	6.331E-3	0.011	0.346	0.434

According to these simulated data the variances increasing and then decreases with increasing sample size.

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