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Alpha Power Transformed Generalized Lomax Distribution with Application

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Abstract: The alpha power transformed generalized Lomax (APTGL) distribution, a novel four-parameter lifespan distribution, is developed throughout this work.The APTGL distribution seems to be more adaptable than the generalized Lomax (GL) distribution. Moments (MOs), MO generating function (MOGF), quantile function (QF), and order statistics (OS) are some of the statistical features we discovered.The maximum likelihood (MLLi) approach is used to estimate the model parameters. Eventually, a real-world data set is utilized to demonstrate the APTGL distribution's versatility.

Keywords: Alpha Power transformed generated family; generalized Lomax distribution; Order Statistics; maximum likelihood.

1. Introduction

The Lomax (L) distribution having numerous applications, including actuarial science, economics, biology, engineering, lifespan, and dependability.It is indeed a subset of the second-order generalized beta distribution (Kleiber and Kotz, 2003). This model is thought to be beneficial in engineering and survival analysis as an alternate distribution for survival issues and life-testing. (see Hassan and Al-Ghamdi, 2009).

Many researchers have investigated generalized and expanded variants of the L distribution; Examples include the following: beta L (BL) by Eugene et al (2004), Marshall Olkin extended L (Ghitany et al., 2007), exponentiated L (EL) (Abdel Moniem, 2012),McDonald L (MCL) (Lemonte and Cordeiro, 2013), gamma-L (GL) (Cordeiro et al., 2013); Weibull-L (WL) (Tahir et al., 2015), transmuted WL (Afify et al., 2015 b),Gumbel-L (Tahir et al., 2016), Power L (PL) (Rady et al., 2016), EL geometric (Hassan and Abd-Allah, 2017), PL Poisson (Hassan and Nassr, 2018), exponentiated WL (Hassan and Abd-Allah, 2018), inverse PL distribution (Hassan and Abd-Allah. 2018)

The GL distribution's density (pdf) and distribution function (cdf) which established Abdul-Moniem and Abdel-Hameed (2012) is as regards:

$$g(x;\lambda,\theta,\beta)=\frac{\theta\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\theta-1}\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta-1}\quad,\quad x,\lambda,\theta,\beta>0, \tag{1}$$

and

$$G(x;\lambda,\theta,\beta)=\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta}\quad,\quad x,\lambda,\theta,\beta>0. \tag{2}$$

The alpha power transformation (APT) approach was proposed by Mahdavi and Kundu (2017) to add an extra parameter to a family of distributions to enhance flexibility in that family.The APT- G family's cdf and pdf are available

$$F(x;\alpha)=\begin{cases}\frac{\alpha^{G(x)}-1}{\alpha-1} & if \;\; \alpha>0,\; \alpha\neq 1\\ G(x) & if \quad \alpha=1.\end{cases}, \tag{3}$$

and

$$f(x;\alpha)=\begin{cases}\frac{\ln\alpha}{\alpha-1}g(x)\alpha^{G(x)} & if \;\; \alpha>0, \alpha\neq 1\\ g(x) & if \quad \alpha=1.\end{cases}. \tag{4}$$

Many probability distributions in the literature have been generalized that use this strategy, for instance, APTWeibull (APTW) byDey et al. (2017a), APT G exponentialdistribution byDey et al. (2017b), APT Lindley by Dey et al. (2018), APT extended exponential byHassan et al. (2018), APT inverted exponential by Unal et al. (2018), APTinverse-W by Ramadan and Magdy (2018), APTinverse-Lindley byDey et al. (2018), APT power Lindley byHassan et al. (2019)and APT Pareto cited in Ihtisham et al. (2019).

The primary objective of this study is to present a different adaptable model known as the APT generalized Lomax (APTGL) distribution.Section 3 deduces several statistical characteristics of the APTGL distribution, as well as more appealing formulations for QF, median, MOs, and OS.In section 4, we use the MLLi technique of parameter estimation to estimate the parameters.In Section 5, a simulation study is conducted in order to determine the model parameters of the APTGL distribution.Section 6 examines one application to demonstrate the efficacy of the suggested approach. In section 7, there is a section titled "Final Remarks."

2. The APTGL model

The APTGL model for the random variable (RVr) X is indicated by APTGL(α,θ, λ, β)which having four-parameter, if X's pdf should be for$x \geq 0$ is

$$f(x)=\begin{cases}\frac{\theta\beta\log(\alpha)}{\lambda(\alpha-1)}\left(1+\frac{x}{\lambda}\right)^{-\theta-1}(1-\left(1+\frac{x}{\lambda}\right)^{-\theta})^{\beta-1}\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta}} & if \;\; \alpha\neq 1, \alpha,\theta,\lambda,\beta>0\\ \frac{\theta\beta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\theta-1}(1-\left(1+\frac{x}{\lambda}\right)^{-\theta})^{\beta-1} & if \;\alpha=1, \alpha,\theta,\lambda,\beta>0\end{cases} \tag{5}$$

and

$$F(x)=\begin{cases}\frac{\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta}}-1}{\alpha-1} & if \;\; \alpha\neq 1\\ \left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta} & if \;\; \alpha=1\end{cases} \tag{6}$$

The APTGL distribution's survival function (sf) and hazard rate function (hrf) take the appropriate configurations.

$$S(x)=\frac{\alpha-\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta}}}{\alpha-1},$$

and

$$h(x) = \frac{\frac{\theta\beta\log(\alpha)}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-\theta-1}(1-\left(1+\frac{x}{\lambda}\right)^{-\theta})^{\beta-1}\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta}}}{\alpha-\alpha^{\left(1-\left(1+\frac{x}{\lambda}\right)^{-\theta}\right)^{\beta}}} .$$

Figure 1 shows the pdf and hazard functions of the APTGL distribution for various values of α, θ, λ and β. Clearly, the pdf of APTGL distribution is decreasing, uni-modal and right skewed.The hrf of APTIL model can be decreasing and increasing.

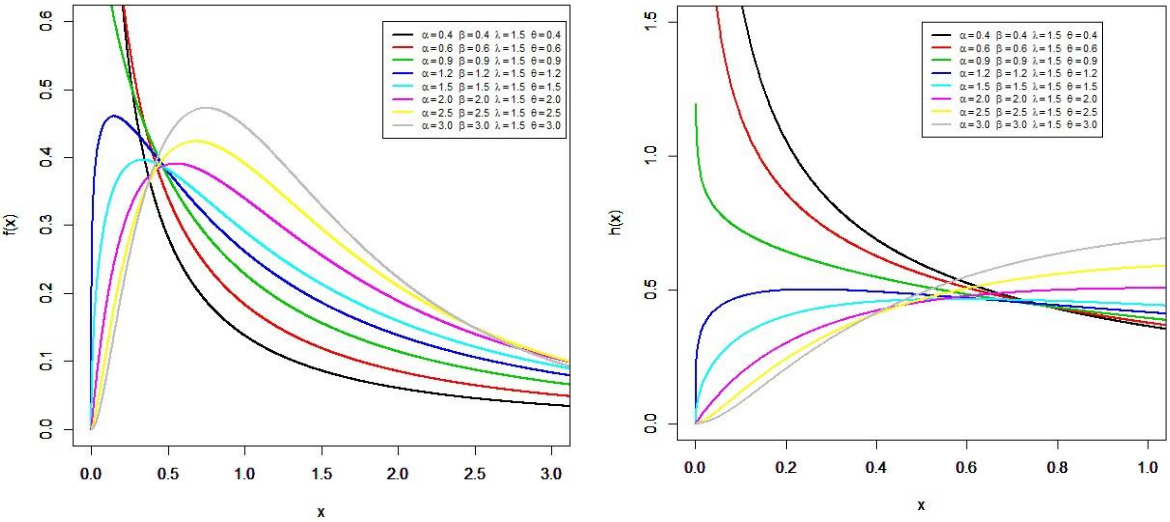

Figure 1. Plots of the pdf and hrf of the APTGL model

3. Basic Properties of APTGL Model

The statistical characteristics of the APTGL distribution are discussed in this section.

3.1. Important expansions

An explicit APTGL pdf formula is supplied here. Using the series structure shown below

$$\alpha^{\vartheta} = \sum_{i=0}^{\infty}\frac{(\log\alpha\,)^{i}}{i!}\vartheta^{i}. \tag{7}$$

We may rewrite the pdf by adding (7) into (5).

$$f(x) = \sum_{i=0}^{\infty} A_i\left(1+\frac{x}{\lambda}\right)^{-\theta-1}\left(1-(\,1+\frac{x}{\lambda})^{-\theta}\right)^{\beta(i+1)-1}, \tag{8}$$

where $A_i = \frac{\theta\beta\,(\log(\,\alpha\,))^{i+1}}{\lambda i!(\alpha-1)}$.

Using the appropriate binomial expansion

$$(1-z)^{\beta-1} = \sum_{j=0}^{\infty} (-1)^j \binom{\beta-1}{j} z^j .$$

Adding the preceding expansion to the equation (8)

$$f(x) = \sum_{i,j=0}^{\infty} w_{i,j} \left(1+\frac{x}{\lambda}\right)^{-\theta(j+1)-1}, \tag{9}$$

where $w_{i,j} = A_i (-1)^j \binom{\beta(i+1)-1}{j}$.

In addition, an expansion for $[F(x)]^h$ is obtained, where h is an integer, and the binomial expansion is performed out once again.

$$[F(x)]^h = \sum_{z=0}^{\infty} S_z \left(1+\frac{x}{\lambda}\right)^{-\theta z}, \tag{10}$$

where,

$$S_z = \left(\frac{1}{1-\alpha}\right)^h \sum_{k=0}^{h} \sum_{m=0}^{\infty} (-1)^{h+z} \binom{h}{k} \binom{\beta m}{z} \frac{(\log(\alpha))^m k^m}{m!}.$$

.

3.2. Quantile Function:

Through inverse equation (6), the generation from the APTGL distribution may be derived

$$x = \lambda \left[\left(1-\frac{\ln(\alpha u - u + 1)}{ln\alpha}\right)^{\frac{1}{\beta}}\right]^{\frac{-1}{\theta}} - \lambda . \tag{11}$$

If U ~ (0, 1) then X~ APTGL, the qth QF of APTGL is provided via

$$x_q = \lambda \left[\left(1-\frac{\ln(\alpha u - u + 1)}{ln\alpha}\right)^{\frac{1}{\beta}}\right]^{\frac{-1}{\theta}} - \lambda, \tag{12}$$

and the median may be computed as follows:

$$x_{0.5} = \lambda \left[\left(1-\frac{\ln(0.5(\alpha + 1))}{ln\alpha}\right)^{\frac{1}{\beta}}\right]^{\frac{-1}{\theta}} - \lambda, \tag{13}$$

3.3. Moments:

Theorem 1: Assume that X be a RVr from APTGL model then its r^{th}MO is

$$\acute{\mu}_r = \lambda^{r+1} \sum_{i,j=0}^{\infty} w_{i,j} B(r+1, \theta(j+1) - r). \tag{14}$$

Proof: Assume that X be a RVrhaving pdf (5). The r^{th}MOs of APTGL model are computed via

$$\acute{\mu}_r = \int_0^\infty x^r f(x;\ \alpha, \theta, \lambda, \beta) dx$$

Taking inspiration from (11)

$$\acute{\mu}_r = \int_0^\infty \sum_{i,j=0}^{\infty} w_{i,j} x^r \left(1 + \frac{x}{\lambda}\right)^{-\theta(j+1)-1} dx.$$

Let $y = \frac{x}{\lambda}$

$$\acute{\mu}_r = \lambda^{r+1} \int_0^\infty \sum_{i,j=0}^{\infty} w_{i,j} y^r (1+y)^{-\theta(j+1)-1} dy.$$

Then,

$$\acute{\mu}_r = \lambda^{r+1} \sum_{i,j=0}^{\infty} w_{i,j} B(r+1, \theta(j+1) - r).$$

$$\mu = \acute{\mu}_1 = E(X) = \lambda^2 \sum_{i,j=0}^{\infty} w_{i,j} B(2, \theta(j+1) - 1). \quad (15)$$

The variance of the APTGL distribution may be readily calculated as follows:

$$\sigma^2 = \acute{\mu}_2 - (\acute{\mu}_1)^2 = \lambda^3 \sum_{i,j=0}^{\infty} w_{i,j} B(3, \theta(j+1) - 2) - (\acute{\mu}_1)^2. \quad (16)$$

The MOGF of APTGL is investigated via

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \acute{\mu}_r = \lambda^{r+1} \sum_{r,i,j=0}^{\infty} \frac{t^r}{r!} w_{i,j} B(r+1, \theta(j+1) - r). \quad (17)$$

3.4. The probability weighted moments:

For a RVr X the probability-weighted moments (PWMs), indicated by $\tau_{r,s}$, may be computed using the following relationship

$$\tau_{r,s} = E(X^r F(x)^s) = \int_{-\infty}^{\infty} x^r f(x)\, F(x)^s dx \quad (18)$$

APTGL's PWMs are produced by inserting (9) and (10) into (18) and changing h by s, as shown below.

$$\tau_{r,s} = \sum_{i,j,z=0}^{\infty} \int_0^\infty w_{i,j} S_z x^r \left(1 + \frac{x}{\lambda}\right)^{-\theta(z+j+1)-1}.$$

Then,

$$\tau_{r,s} = \lambda^{r+1} \sum_{i,j,z=0}^{\infty} w_{i,j} S_z\, B(r+1, \theta(z+j+1) - r).$$

3.5. Order Statistics:

Assume that $X_1, X_2, \ldots, X_n$ be random sample from the APTGL model with OS $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. The pdf of RVr $X_{(r)}$, (r = 1, 2, ..., n) is computed via

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F^{r-1}(x) f(x) \left(1 - F(x)\right)^{n-r},$$

The pdf of $X_{(r)}$ can indeed be written simply

$$f_{X_{(r)}}(x) = \frac{n!\ \theta\beta \log(\alpha) \xi^{1-\frac{1}{\theta}} \alpha^{(1-\xi)^{\beta}}}{\lambda(r-1)!\,(n-r)!\,(\alpha-1)^{n}} (1-\xi)^{\beta-1} \left(\alpha^{(1-\xi)^{\beta}} - 1\right)^{r-1} \left(\alpha - \alpha^{(1-\xi)^{\beta}}\right)^{n-r}.$$

where $\left(1 + \frac{b}{x}\right)^{-a} = \xi$. The pdf of the first and nth OS, in contrast, may be calculated directly using the preceding equation as follows:

$$f_{X_{(1)}}(x) = \frac{n\,\theta\beta \log(\alpha) \xi^{1-\frac{1}{\theta}} \alpha^{(1-\xi)^{\beta}}}{\lambda(\alpha-1)^{n}} (1-\xi)^{\beta-1} \left(\alpha - \alpha^{(1-\xi)^{\beta}}\right)^{n-1}$$

and

$$f_{X_{(n)}}(x) = \frac{n\theta\beta \log(\alpha)\, \xi^{1-\frac{1}{\theta}} \alpha^{(1-\xi)^{\beta}}}{\lambda(\alpha-1)^{n}} (1-\xi)^{\beta-1} \left(\alpha^{(1-\xi)^{\beta}} - 1\right)^{n-1}.$$

4. MLLi Estimation

Suppose $X_1, \ldots, X_n$ features values derived from the APTGL distribution The (MLLiEs) of the postulated model parameters α, λ, θ and β and are determined through using log-likelihood function, marked by $\ln L$

$$\ln L = n \ln\theta + n \ln\beta + n \ln(log\alpha) - n \ln(\alpha-1) - n ln\,(\lambda) - (\theta+1) \sum_{i=1}^{n} \ln(1 + \frac{x}{\lambda}) + (\beta-1) \sum_{i=1}^{n} ln\left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right] + ln\alpha \sum_{i=1}^{n} \left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right]^{\beta}$$

The APTGL distribution's MLLi equations are provided via

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha \log \alpha} - \frac{n}{\alpha - 1} + \frac{1}{\alpha} \sum_{i=1}^{n} \left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right]^{\beta},$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln\left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right] + \ln\alpha \sum_{i=1}^{n} \left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right]^{\beta} \ln\left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right]$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n} ln\left(1 + \frac{x_i}{\lambda}\right) + (\beta-1) \sum_{i=1}^{n} \frac{\left(1 + \frac{x_i}{\lambda}\right)^{-\theta} \ln\left(1 + \frac{x_i}{\lambda}\right)}{1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}} + \beta ln\alpha \sum_{i=1}^{n} \left[1 - \left(1 + \frac{x_i}{\lambda}\right)^{-\theta}\right]^{\beta-1} \left(1 + \frac{x_i}{\lambda}\right)^{-\theta} ln\left(1 + \frac{x_i}{\lambda}\right)$$

and

$$\frac{\partial \ln L}{\partial \lambda}=\frac{-n}{\lambda}+(\theta+1)\sum_{i=1}^{n}\frac{\frac{x_i}{\lambda^2}}{1+\frac{x_i}{\lambda}}-(\beta-1)\sum_{i=1}^{n}\frac{\theta\left(1+\frac{x_i}{\lambda}\right)^{-\theta-1}\frac{x_i}{\lambda^2}\ln}{1-\left(1+\frac{x_i}{\lambda}\right)^{-\theta}}+\frac{\beta\theta ln\alpha}{\lambda^2}\sum_{i=1}^{n}x_i\left[1-\left(1+\frac{x_i}{\lambda}\right)^{-\theta}\right]^{\beta-1}\left(1+\frac{x_i}{\lambda}\right)^{-\theta-1}$$

Equating $\partial l/\partial\alpha$, $\partial l/\partial\lambda$, $\partial l/\partial\theta$ and $\partial l/\partial\beta$ to 0 and computing concurrently, we get the MLLiEs of α, λ, θ and β.

5. Numerical outcomes

We use a numerical analysis to compare the behavior of MLLiEs in this paper. From the APTGL distribution, we produce 1000 random samples of size n =50, 100, 200, and 500.The parameters are divided into six groups, as follows:set1=($\alpha=0.5$, $\lambda=0.5, \beta=0.5, \theta=0.5$), set2=($\alpha=0.5$, $\lambda=0.5, \beta=1.5, \theta=0.5$),set3=($\alpha=0.5$, $\lambda=1.5, \beta=0.8, \theta=1.5$), set4=($\alpha=0.5$, $\lambda=1.5, \beta=1.5, \theta=1.2$), set5=($\alpha=0.5$, $\lambda=0.5, \beta=2, \theta=0.5$) and set6=($\alpha=0.5$, $\lambda=0.5, \beta=2, \theta=1$). The MLLiEs of α, λ, β and $\theta$are investigated.
After that, the Es of MLLiapproach and their mean square errors (MSErs) are reveal in Tables 1 to 2.

Table 1:MLLiEs and MSErs of APTGLmodel for set1, set2 and se3.

n	Set1		Set2		Set3	
	MLLiEs	**MSErs**	**MLLiEs**	**MSErs**	**MLLiEs**	**MSErs**
50	0.428	0.105	0.317	0.147	0.762	0.47
	0.76	0.388	0.338	0.168	1.43	0.82
	0.54	0.023	2.421	2.041	0.916	0.112
	0.549	0.12	0.419	0.031	1.589	0.791
100	0.42	0.058	0.217	0.137	0.802	0.241
	0.685	0.206	0.312	0.089	1.562	0.431
	0.526	$9.928*10^{-3}$	2.1	0.61	0.789	0.023
	0.512	0.03	0.382	0.029	1.681	0.3
200	0.368	0.043	0.243	0.117	0.709	0.096
	0.636	0.076	0.356	0.08	1.273	0.183
	0.519	$3.67*10^{-3}$	1.959	0.466	0.808	$9.246*10^{-3}$
	0.49	0.014	0.391	0.025	1.482	0.078
500	0.343	0.042	0.201	0.117	0.696	0.065
	0.697	0.065	0.346	0.052	1.225	0.15
	0.502	$1.167*10^{-3}$	1.899	0.277	0.809	$5.13*10^{-3}$
	0.493	$4.146*10^{-3}$	0.38	0.024	1.45	0.048

Table 2:MLLiEs and MSErs of APTGL model for set4, set5 and se6.

n	Set4		Set5		Set6	
	MLLiEs	**MSErs**	**MLLiEs**	**MSErs**	**MLLiEs**	**MSErs**
50	0.729	0.519	0.649	0.395	0.648	0.454
	1.207	1.544	0.673	0.457	0.817	1.712
	2.406	2.966	2.439	1.346	2.598	2.385
	1.219	0.402	0.501	0.021	1.194	1.753
100	0.66	0.175	0.491	0.168	0.705	0.424
	1.124	0.448	0.576	0.096	0.591	0.219
	1.857	0.4	2.24	0.852	2.553	1.708
	1.136	0.172	0.481	0.011	1.02	0.074
200	0.676	0.118	0.446	0.06	0.558	0.212
	1.135	0.288	0.626	0.127	0.606	0.102
	1.731	0.2	2.046	0.377	2.084	0.336
	1.138	0.073	0.484	$6.113*10^{-3}$	0.994	0.038
500	0.643	0.065	0.433	0.03	0.491	0.095
	1.201	0.178	0.592	0.042	0.52	0.035
	1.619	0.084	1.951	0.187	2.088	0.169
	1.145	0.036	0.482	$2.787*10^{-3}$	0.963	0.015

6. Modelling

In this part, we demonstrate the efficacy of APTGL distribution by modeling two data sets. Several writers have utilized this data to demonstrate the applicability of competing models. We also give a performance assessments of the models' goodness of fit and draw comparisons to other distributions.The measures of goodness of fit metrics such; Anderson-Darling(A^*) and Cramér- von Mises (W^*) are produced toexamine the fitted models. Generally, the lower values of these statistics is the better fit to the data.

Aircraft Windshield Data

Murthy et al. (2004) investigated the data set, which depicts the failure times for a certain windshield device. For this data, we will compare the APTGL distribution's fits to the relevant distributions: (GL), (BL), (EL), (McL), and (L).Table 3 shows the estimated parameters of these distributions as well as the related standard error (SE) for windshield data. Table 3 also includes the statistics A and W. Figure 2 depicts the fitted models' estimated pdf, cdf, sf, and PP graphs for Aircraft Windshield data.

Table 3: Estimates, SE (in parentheses), A^* and W^* statistics for Aircraft Windshield data

distribution	Estimates and SE (in parentheses)					A^*	W^*
APTGL(α, β, λ, θ)	166.306 (243.18)	1.599 (0.59)	913.726 (3891)	931.133 (3955)		0.6649	0.0666
McL(a, b, c, α, β)	2.1875 (0.5211)	119.1751 (140.2970)	12.4171 (20.845)	19.9243 (38.96)	75.661 (147.242)	0.6672	0.0858
GL (a, α, β)	3.5876 (0.5133)	52001.5 (7955)	37029.658 (81.164)			1.3666	0.1618
BL(a, b, α, β)	3.6036 (0.619)	33.6387 (63.715)	4.831 (9.238)	118.84 (428.927)		1.4084	0.1680
EL(a, α, β)	3.6261 (0.624)	20074.51 (2041.826)	26257.681 (99.7417)			1.7435	0.2194
L (α, β)	51425.35 (5933.49)	131789.78 (2961198)	- -			1.3976	0.1665

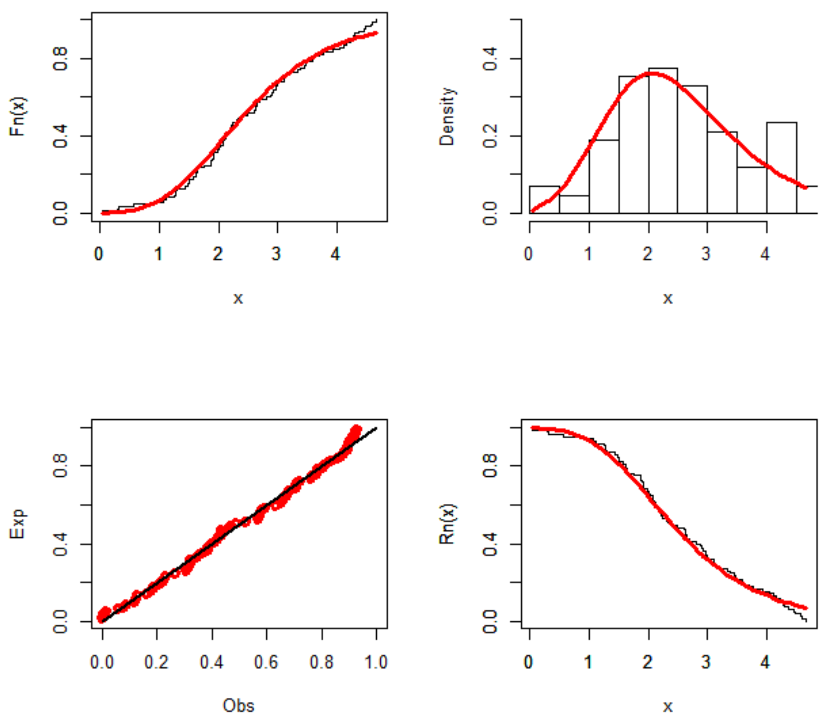

Figure 2. Plots of estimated pdf, cdf, sf and pp plots for Aircraft Windshield data

Table 3 and Figure 2 show that the APTGL gives us a better fit to the data and hence may be elected as the best model.

7. Summary and Conclusions

We suggested and investigated the APTGL distribution in this study. The APTGL distribution's structural characteristics are deduced.The MLLiEs technique of estimation is used to estimate the population parameters.A numerical outcomes is is conducted to assess the model parameters of the APTGL model. A real data collection is utilized in the application to demonstrate the versatility of the APTGL model.

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