A NEW GENERALIZED PARETO DISTRIBUTION

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Abstract

In this paper, we introduce a new generalized Pareto distribution and studied its properties. Some well known distributions can be derived as a special cases as well as some derived distributions.

1 – Introduction

The Pareto distribution was first proposed by Pareto (1897) as a model for the distribution of income. It can be used to represent various other forms of distributions (other than income data) that arise in human life. It has played a very important role in the investigation of city population sizes, occurrence of natural resources, insurance-risk, and business failures. Arnald (1983) gives an extensive historical survey of its use in the context of income distribution.

A random variable T is said to have a three – parameter Pareto distribution if its probability density function (p.d.f) is given by

$$f(t,\alpha,\beta,\lambda) = \frac{\alpha}{\beta}\left[1+\left(\frac{t-\lambda}{\beta}\right)\right]^{-(\alpha+1)} \qquad (1)$$

where $\lambda < x < \infty$, $\beta > 0$, $\alpha > 0$, where λ is the location parameter , β is the scale parameter and α is the shape parameter.

In this paper, we will generalized the three parameters Pareto distribution (1) by adding another shape parameter. The properties of the suggested distribution will be investigated as well as its relations with other well known distributions.

2 – Four Parameters Pareto Distribution.

Suppose X is distributed from Pareto distribution, the p.d.f of X will be

$$f(x;\alpha,\beta,\lambda,C) = \frac{C\alpha}{\beta}\left(\frac{x-\lambda}{\beta}\right)^{c-1}\left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]^{-(\alpha+1)} \qquad (2)$$

where $\lambda < x < \infty$, $\beta > 0$, $\alpha > 0$, $C > 0$, λ is the location parameter, β is the scale parameter and (α , C) are the shape parameters.

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The particular case c=1, the p.d.f. (2) reduces to the three parameter Pareto distribution (1) with parameters α, β and λ.

Figure (1) illustrates the shape of (2) for selected values of (α=1, β=1, λ=0, c=.5, 1, 5, 10). Another some possible shapes of (2) for (α=3, β=1, λ=0, c=.5, 1, 5, 10) are shown in Figure (2). The effect of the parameter c can be seen easily.

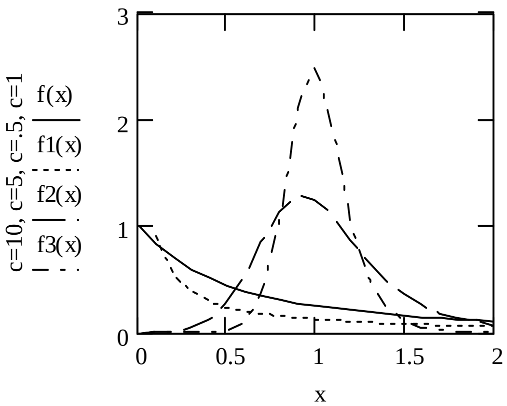

Figure (1)

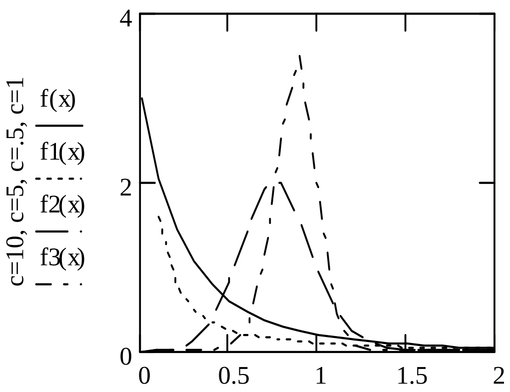

Figure (2)

The cumulative distribution function $F(x)$, survival $S(x)$ and hazard rate $H(x)$ functions of (2) is given by

$$F(x;\alpha,\beta,\lambda,C) = 1-\left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]^{-\alpha},$$

$$S(x) = \left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]^{-\alpha}$$

and

$$H(x) = \frac{C\alpha}{\beta}\left(\frac{x-\lambda}{\beta}\right)^{C-1}\left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]^{-1} \qquad (3)$$

The hazard rate function is an important quantity for characterizing life phenomena. Some possible shapes of (3) for (α=1, β=1, λ=0, c=.1, .5, .9, 2) are shown in Figure (3).

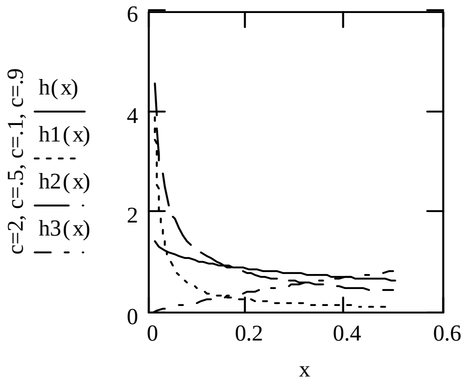

Figure (3)

The r[th] moment about mean of the generalized Pareto distribution(2) is given by:

$$\mu_r = \mathrm{E}(x-\mu)^r = \int_{\lambda}^{\infty} (x-\mu)^r f(x;\alpha,\beta,\lambda,C)dx$$

$$= \alpha\beta^r \sum_{j=0}^{r} \binom{r}{j} (-1)^j \left[\frac{\Gamma\left(\alpha-\frac{1}{C}\right)\Gamma\left(1+\frac{1}{C}\right)}{\Gamma(\alpha)}\right]^j \left[\frac{\Gamma\left(\alpha-\frac{r-j}{C}\right)\Gamma\left(1+\frac{r-j}{C}\right)}{\Gamma(\alpha+1)}\right]$$

where $\Gamma(.)$ is defined as the gamma function, and $r = 1,2,3,...$, Therefore, the mean, variance and coefficient of variation of X are given by

$$\mu = \beta \frac{\Gamma\left(\alpha-\frac{1}{C}\right)\Gamma\left(1+\frac{1}{C}\right)}{\Gamma(\alpha)} + \lambda ,$$

$$\sigma^2 = \beta^2 \left[\frac{\Gamma\left(1+\frac{2}{C}\right)\Gamma\left(\alpha-\frac{2}{C}\right)}{\Gamma(\alpha)} - \left(\frac{\Gamma\left(1+\frac{1}{C}\right)\Gamma\left(\alpha-\frac{1}{C}\right)}{\Gamma(\alpha)}\right)^2\right],$$

and

$$C.V(X) = \frac{\sigma}{\mu} \times 100$$

$$= \left[\frac{\left[\Gamma\left(1+\frac{2}{C}\right)\Gamma\left(\alpha-\frac{2}{C}\right)\Gamma(\alpha) - \Gamma^2\left(\alpha-\frac{1}{C}\right)\Gamma^2\left(\alpha-\frac{1}{C}\right)\right]^{\frac{1}{2}}}{\Gamma\left(1+\frac{1}{C}\right)\Gamma\left(\alpha-\frac{1}{C}\right) + \lambda\Gamma(\alpha)}\right]$$

3 – Some Special Cases

The 4-parametaer generalized Pareto distribution (2) can be specialized to different known distributions such as:

(i) For $C = 1$, (2) reduces to a three – parameter Pareto distribution (1).

(ii) When $\beta^C = A$, (2) reduces the 4-parameter compound Weibull – gamma distribution with p.d.f

$$f(x;\alpha,A,\lambda,C) = C\alpha \frac{(x-\lambda)^{C-1}}{A} \left[1+\frac{(x-\lambda)^C}{A}\right]^{-(\alpha+1)} \lambda < x < \infty \ , \ C, A, \alpha > 0$$

(iii) For $\beta^C = A \; and \; C = 2$, (2) reduces to the 3-parameter compound Rayleigh – gamma distribution with p.d.f

$$f(x;\alpha,A,\lambda) = 2\alpha \; \frac{(x-\lambda)}{A} \left[1+\left(\frac{x-\lambda}{A}\right)^2\right]^{-(\alpha+1)} \quad \lambda < x < \infty \; , \; \alpha, A > 0$$

(iv) When $\beta^C = A \; and \; \alpha = 1$, (2) reduces to the 3-parameter compound Weibull – exponential distribution with p.d.f

$$f(x;\alpha,A,\lambda,C) = C\left[1+\frac{(x-\lambda)^C}{A}\right]^{-2} \qquad \lambda < x < \infty \; , \quad C, A > 0$$

(v) For $\beta^C = A \; and \; C = \alpha = 1$, (2) reduces to the 2 – parameter compound exponential – exponential distribution defined as

$$f(x;A,\lambda) = \frac{1}{A}\left[1+\frac{x-\lambda}{A}\right]^{-2} \qquad \lambda < x < \infty \; , \quad A > 0$$

(vi) For $\beta = 1 and \; \lambda = 0$, (2) reduces the 2 – parameter Burr XII distribution.

$$f(x;\alpha,c) = c\alpha \; x^{C-1} \left(1+x^C\right)^{-(\alpha+1)}$$

where $x > 0$, $\alpha > 0$, $c > 0$ .Where c and α are the shape parameters.

(vii) For $C = 1 and \; \lambda = 0$, (2) reduces to the 2 – parameter Lomax distribution with p.d.f.

$$f(x;\alpha,\beta) = \frac{\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \qquad x > 0 \qquad \beta, \alpha > 0$$

(viii) For $(C = \beta = 1 and \; \lambda = 0)$, (2) reduces to the beta type II distribution with p.d.f

$$f(x;\alpha) = \alpha \; (1+x)^{-(\alpha+1)} \qquad x > 0 \; , \qquad \alpha > 0$$

4 – Some Transformed distributions.

The 4 – parameter generalized Pareto distribution (2) can be transformed to many different known distributions as follows:

(a) When $y=\left[\ln\left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]\right]^{\frac{1}{C}}+\lambda$, (2) can be transformed to 3-parameter Weibull distribution, i. e, the p.d.f of y will be

$$f(y;C,\alpha,\lambda) = C\alpha \ (y-\lambda)^{C-1} \ e^{-\alpha(y-\lambda)^{C}} \quad \lambda<y<\infty \quad C,\alpha>0$$

(b) When $y=\left[\ln\left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]\right]+\lambda$,(2) can be transformed to 2 –parameter exponential distribution with the following p.d.f.

$$f(y;\alpha,\lambda) = \alpha \ e^{-\alpha(y-\lambda)} \qquad \lambda<y<\infty \qquad \alpha>0$$

(c) For $y=\beta\left[\ln\left[1+\left(\frac{x-\lambda}{\beta}\right)^{C}\right]\right]+\lambda$, (2) can be transformed to 3 –parameter type I generalized logistic distribution with p.d.f. defined as

$$f(y;\alpha,\beta,\lambda) = \frac{\alpha}{\beta} \ e^{\left(\frac{y-\lambda}{\beta}\right)} \left[1+e^{\left(\frac{y-\lambda}{\beta}\right)}\right]^{-(\alpha+1)} \qquad -\infty<y<\infty$$

(d) When $y=\left(\frac{x-\lambda}{\beta}\right)^{C}$, (2)can be transformed to beta type II $\beta(1,\alpha)$ with p.d.f

$$f(y;\alpha) = \alpha \ [1+y]^{-(\alpha+1)} \qquad y>0 \ , \ \alpha>0$$

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