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ORIGINAL ARTICLES

On the Computation of Weighted Non-Central Chi-Square Distribution Function

[1]El-Sayed, A. Elsherpieny, [2]Sahar, A. Ibrahim, [3]Yassmen, Y. Abdelall

[1,2,3]Institute of Statistical Studies & Research, Cairo University, Egypt

ABSTRACT

Generally the cumulative distribution function is very important in calculating the power function of some statistical tests. The cumulative distribution function of weighted non-central chi-square distribution involves an integral which is difficult to obtain. In this paper we present two formulas for the cumulative distribution function of the weighted non-central chi-square distribution which are easy to handle in computation. The first one for any degrees of freedom, while the second is appropriate for only odd degrees of freedom. A numerical illustration is given. Ashour and Abdel-Samad (1990) results can be obtained as a special case of our results.

Key words: *Weighted non-central chi-square distribution, Gamma duplication.*

Introduction

Grau (2009) introduced the weighted non-central chi-square distribution with ν degrees of freedom. The probability density function of such distribution is expressed as an infinite sum of central chi-square distributions as follows:

$$f_\nu(z)=\sum_{k=0}^{\infty} a_k \left[\alpha_1^{-2} f_{\chi^2_{\nu+k}}\left(\alpha_1^{-2}z\right)+(-1)^k \alpha_2^{-2} f_{\chi^2_{\nu+k}}\left(\alpha_2^{-2}z\right)\right], z>0, \lambda\geq 0 \quad (1)$$

where

$$a_k=\frac{e^{\frac{-\lambda}{2}}}{2\sqrt{\pi}}\ \frac{(2\lambda)^{\frac{k}{2}}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k+1)}, \quad (2)$$

α_1 and α_2 are two positive numbers, λ is the non-centrality parameter, $\Gamma(.)$ is the gamma function, and $f_{\chi^2_{\nu+k}}(.)$ is the central chi-square distribution with $(\nu+k)$ degrees of freedom. This distribution denoted by $\chi^2_\nu(\lambda,\alpha_1,\alpha_2)$.

For the case where $\alpha_1=\alpha_2=1$, (1) reduced to the probability density function of the non-central chi-square distribution; $\chi^2_\nu(\lambda)$; with ν degrees of freedom and non-centrality parameter λ. When $\lambda=0$, this distribution is reduced to the probability density function of the central chi-square distribution; χ^2_ν; with ν degrees of freedom.

The cumulative distribution function of the weighted non-central chi-square distribution can be expressed as infinite sum of distribution functions of central chi-square distribution as follows:

$$F_\nu(z)=\int_0^z f_\nu(t)\,dt$$

$$=\sum_{k=0}^{\infty} a_k \left[\int_0^z \alpha_1^{-2} f_{\chi^2_{\nu+k}}\left(\alpha_1^{-2}t\right)dt+(-1)^k\int_0^z \alpha_2^{-2} f_{\chi^2_{\nu+k}}\left(\alpha_2^{-2}t\right)dt\right]$$

$$=\sum_{k=0}^{\infty} a_k \left[F_{\chi^2_{\nu+k}}(\alpha_1^{-2}z)+(-1)^k F_{\chi^2_{\nu+k}}(\alpha_2^{-2}z)\right] \quad (3)$$

Shea (1988) presented a formula to evaluate the central chi-square distribution integral. Patnaik (1949) gave two approximations for the non-central chi-square distribution; the first one is the chi-square

Corresponding Author: El-Sayed, A. Elsherpieny, Institute of Statistical Studies & Research, Cairo University, Egypt

approximation and the second is the normal approximation. Ashour and Abdel-Samad (1990) presented two formulas for computing the cumulative distribution function of the non-central chi-square distribution. The first one deals with the case of any degrees of freedom and the second deals with the case of odd degrees of freedom. They compared numerically between Patnaik's results (1949) and their results.

The main objective of the present paper is to obtain a formula for the weighted non-central chi-square distribution function with ν degrees of freedom which is easier in computation than that use the integral form.

Computational Formula for Any Degrees Of Freedom:

Shea (1988) presented formula to evaluate the central chi-square integral. He proved that:

$$F_{\chi_\nu^2}(y) = P^{\nu/2}\left(\frac{y}{2}\right)\sum_{s=0}^{\infty} C_s\left(\frac{y}{2},\frac{\nu}{2}\right), \tag{4}$$

where $P^{\nu/2}(y/2) = \dfrac{e^{-\frac{y}{2}}\left(\frac{y}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}+1\right)}$ is the pdf of the $(\nu/2)^{\text{th}}$ Poisson random variable with parameter $(y/2)$, and $C_s\left(\frac{y}{2},\frac{\nu}{2}\right)$ has a recurrence relation defined by:

$$C_s\left(\frac{y}{2},\frac{\nu}{2}\right) = \begin{cases} \dfrac{\left(\frac{y}{2}\right)}{\left(\frac{\nu}{2}+s\right)} C_{s-1}\left(\frac{y}{2},\frac{\nu}{2}\right) & \text{for } s = 1,2,3,... \\ 1 & \text{for } s = 0. \end{cases}$$

To obtain the cumulative distribution function of the weighted non-central chi-square distribution we use Equation (3) and (4) we get

$$F_\nu(z) = \sum_{k=0}^{\infty} a_k \left[\begin{array}{l} P^{(\nu+k)/2}\left(\frac{\alpha_1^{-2}z}{2}\right)\sum_{s=0}^{\infty} C_s\left(\frac{\alpha_1^{-2}z}{2},\frac{\nu+k}{2}\right) \\ +(-1)^k P^{(\nu+k)/2}\left(\frac{\alpha_2^{-2}z}{2}\right)\sum_{s=0}^{\infty} C_s\left(\frac{\alpha_2^{-2}z}{2},\frac{\nu+k}{2}\right) \end{array}\right], \tag{5}$$

where $C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+k}{2}\right)$ has a recurrence relation defined by:

$$C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+k}{2}\right) = \begin{cases} \dfrac{\left(\frac{\alpha_i^{-2}z}{2}\right)}{\left(\frac{\nu+k}{2}+s\right)} C_{s-1}\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+k}{2}\right) & \text{for } s = 1,2,3,... \\ 1 & \text{for } s = 0. \end{cases}$$

Now for k even and odd, Substitute $k = 2j$ and $k = 2j+1$; $j = 0,1,2,...$ in (5) respectively. We get

$$\begin{aligned} F_\nu(z) = &\sum_{j=0}^{\infty} a_{2j} \sum_{i=1}^{2} P^{(\nu+2j)/2}\left(\frac{\alpha_i^{-2}z}{2}\right) \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j}{2}\right) \\ &+ \sum_{j=0}^{\infty} a_{2j+1} \sum_{i=1}^{2} (-1)^{i+1} P^{(\nu+2j+1)/2}\left(\frac{\alpha_i^{-2}z}{2}\right) \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j+1}{2}\right), \end{aligned} \tag{6}$$

Note that the gamma duplication function is defined as follows:(see Andrews (1992))

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) \quad , \quad n \geq 0$$

So we can write

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\Gamma(2n)\sqrt{\pi}}{2^{2n-1}\Gamma(n)} \quad , \quad n \geq 0 \tag{7}$$

Using this formula together with Equation (2) in Equation (6), we get

$$F_\nu(z) = \sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{2\,\Gamma(j+1)} \sum_{i=1}^{2} P^{(\nu+2j)/2}\left(\frac{\alpha_i^{-2}z}{2}\right) \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2}, \frac{\nu+2j}{2}\right)$$

$$+\sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)} \sum_{i=1}^{2} (-1)^{i+1} P^{(\nu+2j+1)/2}\left(\frac{\alpha_i^{-2}z}{2}\right) \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2}, \frac{\nu+2j+1}{2}\right)$$

$$= \sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{2\,\Gamma(j+1)} \sum_{i=1}^{2} \frac{e^{\frac{-\alpha_i^{-2}z}{2}}\left(\frac{\alpha_i^{-2}z}{2}\right)^{\frac{\nu+2j}{2}}}{\Gamma\left(\frac{\nu+2j+2}{2}\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2}, \frac{\nu+2j}{2}\right)$$

$$+\sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)} \sum_{i=1}^{2} (-1)^{i+1} \frac{e^{\frac{-\alpha_i^{-2}z}{2}}\left(\frac{\alpha_i^{-2}z}{2}\right)^{\frac{\nu+2j+1}{2}}}{\Gamma\left(\frac{\nu+2j+3}{2}\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2}, \frac{\nu+2j+1}{2}\right). \tag{8}$$

The first term of Equation (8) can be abbreviated as follows:

$$\sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{2\,\Gamma(j+1)} \sum_{i=1}^{2} \frac{e^{\frac{-\alpha_i^{-2}z}{2}}\left(\frac{\alpha_i^{-2}z}{2}\right)^{\frac{\nu+2j}{2}}}{\Gamma\left(\frac{\nu+2j+2}{2}\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2}, \frac{\nu+2j}{2}\right)$$

$$= \frac{e^{-\lambda/2}}{2} \sum_{i=1}^{2} \frac{e^{\frac{-\alpha_i^{-2}z}{2}}\left(\frac{\alpha_i^{-2}z}{2}\right)^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}+1\right)} \left[\begin{array}{l} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}w'}{2}, \frac{\nu}{2}\right) + \frac{\left(\frac{\lambda\alpha_i^{-2}w'}{4}\right)}{\Gamma(2)\left(\frac{\nu}{2}+1\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}w'}{2}, \frac{\nu}{2}+1\right) \\ + \frac{\left(\frac{\lambda\alpha_i^{-2}w'}{4}\right)^2}{\Gamma(3)\left(\frac{\nu}{2}+2\right)\left(\frac{\nu}{2}+1\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}w'}{2}, \frac{\nu}{2}+2\right) + \ldots \\ + \frac{\left(\frac{\lambda\alpha_i^{-2}w'}{4}\right)^r}{\Gamma(r+1)\prod_{p=1}^{r}\left(\frac{\nu}{2}+p\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}w'}{2}, \frac{\nu}{2}+r\right) + \ldots \end{array}\right]$$

$$\sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{2\,\Gamma(j+1)} \sum_{i=1}^{2} \frac{e^{\frac{-\alpha_i^{-2}z}{2}}\left(\frac{\alpha_i^{-2}z}{2}\right)^{\frac{\nu+2j}{2}}}{\Gamma\left(\frac{\nu+2j+2}{2}\right)} \sum_{s=0}^{\infty} C_s\left(\frac{\alpha_i^{-2}z}{2}, \frac{\nu+2j}{2}\right)$$

$$= \frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2} P^{\nu/2}\left(\frac{\alpha_i^{-2}z}{2}\right)\sum_{j=0}^{\infty}\frac{1}{\Gamma(j+1)}C_j\left(\frac{\lambda\alpha_i^{-2}z}{4},\frac{\nu}{2}\right)\sum_{s=0}^{\infty}C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j}{2}\right), \tag{9}$$

where

$$C_j\left(\frac{\lambda\alpha_i^{-2}z}{4},\frac{\nu}{2}\right)=\begin{cases}\dfrac{\left(\dfrac{\lambda\alpha_i^{-2}z}{4}\right)^j}{\prod_{r=1}^{j}\left(\dfrac{\nu}{2}+r\right)} & \text{for } i=1,2,\ j=1,2,3,\ldots\ . \\ 1 & \text{for } i=1,2,\ j=0.\end{cases}$$

Following the same way of abbreviation, the second term of Equation (8) can be written as follows:

$$\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}\frac{e^{\frac{-\alpha_i^{-2}z}{2}}\left(\frac{\alpha_i^{-2}z}{2}\right)^{\frac{\nu+2j+1}{2}}}{\Gamma\left(\frac{\nu+2j+3}{2}\right)}\ \sum_{s=0}^{\infty}C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j+1}{2}\right)$$

$$=\frac{e^{-\lambda/2}}{2}\sqrt{\frac{\lambda}{2}}\sum_{i=1}^{2}(-1)^{i+1}P^{(\nu+1)/2}\left(\frac{\alpha_i^{-2}z}{2}\right)\sum_{j=0}^{\infty}\frac{1}{\Gamma\left(j+\frac{3}{2}\right)}C_j\left(\frac{\lambda\alpha_i^{-2}z}{4},\frac{\nu+1}{2}\right)\sum_{s=0}^{\infty}C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j+1}{2}\right) \tag{10}$$

Substituting from Equations (9), (10) in (8), the cumulative distribution function of the weighted non-central chi-square distribution with ν degrees of freedom takes the form

$$F_\nu(z)=\frac{e^{-\lambda/2}}{2}\sum_{i=1}^{2}P^{\nu/2}\left(\frac{\alpha_i^{-2}z}{2}\right)\sum_{j=0}^{\infty}\frac{1}{\Gamma(j+1)}C_j\left(\frac{\lambda\alpha_i^{-2}z}{4},\frac{\nu}{2}\right)\sum_{s=0}^{\infty}C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j}{2}\right)$$
$$+\frac{e^{-\lambda/2}}{2}\sqrt{\frac{\lambda}{2}}\sum_{i=1}^{2}(-1)^{i+1}P^{(\nu+1)/2}\left(\frac{\alpha_i^{-2}z}{2}\right)\sum_{j=0}^{\infty}\frac{1}{\Gamma\left(j+\frac{3}{2}\right)}C_j\left(\frac{\lambda\alpha_i^{-2}z}{4},\frac{\nu+1}{2}\right)$$
$$\times\sum_{s=0}^{\infty}C_s\left(\frac{\alpha_i^{-2}z}{2},\frac{\nu+2j+1}{2}\right), \tag{11}$$

It is worth mentioning here that when $\alpha_1=\alpha_2=1$, Equation (11) reduced to the cumulative distribution function of the non-central chi-square distribution with ν degrees of freedom (see Ashour and Abdel-Samad (1990)).

3. Computational Formula When Degrees of:

Freedom Are Odd:

Abramowitz and Stegun (1964) evaluated the cumulative distribution function of the central chi-square distribution with odd degrees of freedom. Following the same steps as that mentioned in Abramowitz and Stegun (1964) we can get another formula for the cumulative distribution function of the weighted non-central chi-square distribution with odd degrees of freedom, this formula is obtained as follows:

Equation (3) can be written as

$$\overline{F}_\nu(z)=\sum_{k=0}^{\infty}a_k\left[\overline{F}_{\chi^2_{\nu+k}}\left(\alpha_1^{-2}z\right)+(-1)^k\,\overline{F}_{\chi^2_{\nu+k}}\left(\alpha_2^{-2}z\right)\right],\quad z>0,\ \lambda\ge 0$$

where $\overline{F}_{\chi^2_\nu}(.)=1-F_{\chi^2_\nu}(.)$ is the complement of the cumulative distribution function of the central chi-square distribution. For k is even and odd substitute $k=2j$ and $k=2j+1$, respectively, we get:

$$\overline{F}_{\nu}(z)=\sum_{j=0}^{\infty}a_{2j}\sum_{i=1}^{2}\overline{F}_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}z\right)+\sum_{j=0}^{\infty}a_{2j+1}\sum_{i=1}^{2}(-1)^{i+1}\overline{F}_{\chi^2_{\nu+2j+1}}\left(\alpha_i^{-2}z\right).$$

Using gamma duplication formula (7) in Equation (2) when $k=2j$ and $k=2j+1$, we get

$$\overline{F}_{\nu}(z)=\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{2\,\Gamma(j+1)}\sum_{i=1}^{2}\overline{F}_{\chi^2_{\nu+2j}}\left(\alpha_i^{-2}z\right)+\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}\overline{F}_{\chi^2_{\nu+2j+1}}\left(\alpha_i^{-2}z\right).$$

When the degrees of freedom ν is odd, we have

$$\overline{F}_{2m+1}(w')=\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{2\,\Gamma(j+1)}\sum_{i=1}^{2}\overline{F}_{\chi^2_{2m+2j+1}}\left(\alpha_i^{-2}z\right)+\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}\overline{F}_{\chi^2_{2m+2j+2}}\left(\alpha_i^{-2}z\right)$$

$$=A+B\,. \qquad (12)$$

Abramowitz and Stegun (1964) presented formula to evaluate the central chi-square integral as follows

$$\overline{F}_{\chi^2_{\nu}}(y)=2\left(1-\Phi\left(\sqrt{y}\right)\right)+\sqrt{\frac{2}{\pi}}\,e^{\frac{-y}{2}}\sum_{r=1}^{\frac{\nu-1}{2}}\frac{y^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\ ,\ \nu\ \text{odd} \qquad (13)$$

where $\Phi(.)$ is the cumulative standard normal distribution. . Then from Equation (12), we have

$$A=\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{2\,\Gamma(j+1)}\sum_{i=1}^{2}\left[2\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right)+\sqrt{\frac{2}{\pi}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{r=1}^{m+j}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\right]$$

$$=\sum_{i=1}^{2}\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right)+\sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{\Gamma(j+1)}\sum_{r=1}^{m+j}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}$$

$$=\sum_{i=1}^{2}\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right)+\sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{\{(j,r):\ j\geq 0,\ r\geq 1,\ r-j\leq m\}}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{\Gamma(j+1)}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}$$

$$=\sum_{i=1}^{2}\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right)+\sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sum_{\{(j,r):\ j\geq 0,\ 1\leq r\leq m\}}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{\Gamma(j+1)}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\right.$$

$$\left.+\sum_{\{(j,r):\ r\geq m+1,\ r-j\leq m\}}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{\Gamma(j+1)}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\right)$$

$$=\sum_{i=1}^{2}\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right)+\sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{\Gamma(j+1)}\sum_{r=1}^{m}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\right.$$

$$\left.+\sum_{r=m+1}^{\infty}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\sum_{j=r-m}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j}}{\Gamma(j+1)}\right)$$

$$=\sum_{i=1}^{2}\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right)+\sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\,e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sum_{r=1}^{m}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\right.$$

$$\left.+\sum_{r=m+1}^{\infty}\frac{\left(\alpha_i^{-2}z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)}\sum_{j=r-m}^{\infty}P^{j}(\lambda/2)\right), \qquad (14)$$

where

$$\sum_{j=0}^{\infty} P^j(\lambda/2) = \sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{\Gamma(j+1)} = e^{-\lambda/2}\sum_{j=0}^{\infty}\frac{(\lambda/2)^j}{j!} = e^{-\lambda/2}e^{\lambda/2} = 1.$$

Using for Equation (14), the following relation between the chi-square and the Poisson distribution (see Johnson et al. (1970))

$$F_{\chi^2_{2\nu}}(y) = \sum_{s=\nu}^{\infty} P^s\left(\frac{y}{2}\right) \ , \qquad \nu = 1,2,\dots \text{ and } y > 0.$$

We have

$$A = \sum_{i=1}^{2}\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right) + \sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\, e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sum_{r=1}^{m}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} + \sum_{r=m+1}^{\infty}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} F_{\chi^2_{2r-2m}}(\lambda)\right)$$

$$= \sum_{i=1}^{2}\left\{\left(1-\Phi\left(\sqrt{\alpha_i^{-2}z}\right)\right) + \sqrt{\frac{1}{2\pi}}\, e^{\frac{-\alpha_i^{-2}z}{2}}\left[\sum_{r=1}^{m}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} + \sum_{r=m+1}^{\infty}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} F_{\chi^2_{2r-2m}}(\lambda)\right]\right\}.$$

Using Equation (13), we have

$$A = \sum_{i=1}^{2}\frac{1}{2}\overline{F}_{\chi^2_{2m+1}}(\alpha_i^{-2}z) + \sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\, e^{\frac{-\alpha_i^{-2}z}{2}}\left(\sum_{r=m+1}^{\infty}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} F_{\chi^2_{2r-2m}}(\lambda)\right). \qquad (15)$$

Note that the series between bracket on the right hand side of Equation (15) vanishes when $\lambda = 0$ and positive for $\lambda > 0$.

Notice that an equivalent version of Equation (15) can be obtained as follows:

$$\left\{1-\sum_{i=1}^{2}\frac{1}{2}\overline{F}_{\chi^2_{2m+1}}(\alpha_i^{-2}z)\right\} - \sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\, e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{r=m+1}^{\infty}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} F_{\chi^2_{2r-2m}}(\lambda)$$

$$= \sum_{i=1}^{2}\frac{1}{2}F_{\chi^2_{2m+1}}(\alpha_i^{-2}z) - \sum_{i=1}^{2}\sqrt{\frac{1}{2\pi}}\, e^{\frac{-\alpha_i^{-2}z}{2}}\sum_{r=m+1}^{\infty}\frac{(\alpha_i^{-2}z)^{r-\frac{1}{2}}}{1.3.5\dots(2r-1)} F_{\chi^2_{2r-2m}}(\lambda).$$

Also from Equation (12), we have

$$B = \sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}\overline{F}_{\chi^2_{2m+2j+2}}(\alpha_i^{-2}z). \qquad (16)$$

From Abramowitz and Stegun (1964) we have

$$\overline{F}_{\chi^2_\nu}(y) = e^{\frac{-y}{2}}\left[1+\sum_{r=1}^{\frac{\nu-2}{2}}\frac{y^r}{2.4.6\dots(2r)}\right], \ \nu \text{ even} \qquad (17)$$

Taking $y = \alpha_i^{-2}z$ and $\nu = 2m+2j+2$, $i = 1,2$ in (17), we get

$$\overline{F}_{\chi^2_{2m+2j+2}}(\alpha_i^{-2}z) = e^{\frac{-\alpha_i^{-2}z}{2}}\left[1+\sum_{r=1}^{m+j}\frac{(\alpha_i^{-2}z)^r}{2.4.6\dots(2r)}\right], \ i = 1,2 .$$

So Equation (16) can be written as follows

$$B = \sum_{j=0}^{\infty}\frac{e^{-\lambda/2}(\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)}\sum_{i=1}^{2}(-1)^{i+1}e^{\frac{-\alpha_i^{-2}z}{2}}\left[1+\sum_{r=1}^{m+j}\frac{(\alpha_i^{-2}z)^r}{2.4.6\dots(2r)}\right]. \qquad (18)$$

Using Equations (15) and (18) the cumulative distribution function of the weighted non-central chi-square distribution with odd degrees of freedom is given as

$$\overline{F}_{2m+1}(z) = \sum_{i=1}^{2} \frac{1}{2} \overline{F}_{\chi^2_{2m+1}}\left(\alpha_i^{-2} z\right) + \sum_{i=1}^{2} \sqrt{\frac{1}{2\pi}}\, e^{\frac{-\alpha_i^{-2} z}{2}} \left(\sum_{r=m+1}^{\infty} \frac{\left(\alpha_i^{-2} z\right)^{r-\frac{1}{2}}}{1.3.5\ldots(2r-1)} F_{\chi^2_{2r-2m}}(\lambda) \right)$$

$$+ \sum_{j=0}^{\infty} \frac{e^{-\lambda/2} (\lambda/2)^{j+\frac{1}{2}}}{2\,\Gamma\left(j+\frac{3}{2}\right)} \sum_{i=1}^{2} (-1)^{i+1} e^{\frac{-\alpha_i^{-2} z}{2}} \left[1 + \sum_{r=1}^{m+j} \frac{\left(\alpha_i^{-2} z\right)^r}{2.4.6\ldots(2r)} \right] \quad (19)$$

It is clear that for $\alpha_1 = \alpha_2 = 1$ in Equation (19), gives the cumulative distribution function of the non-central chi-square distribution with odd degrees of freedom (see Ashour and Abdel-Samad (1990)).

A Numerical Illustration:

In this subsection, we present a numerical illustration for computing the cumulative distribution function of the weighted non-central chi-square distribution presented in Equations (11) and (19). The infinite sum is appropriately truncated. Mathcad package is used. We use sample size n = 50, $z = 1.5, 1.765, 2.5, 3, 3.5, 4, 5, \ldots, 17, 24$ and parameters $\nu = 3, 4, \ldots, 14$; $\lambda = 1, 2, 3, 4, 6$; and $\alpha_1(\alpha_2) = 0.8(0.4), 1(2), 1(1), 1(3), 0.5(1.5), 1(4), 0.6(3), 0.7(1.75), 0.8(1.33)$. (for the details you can contact with the authors)

Ashour and Abdel-Samad (1990) obtained the probability integral of the non-central chi-square distribution for the same results when $\alpha_1 = \alpha_2 = 1$. All results are listed in the table.

Comments:

From Table : we observe that

1. For $\alpha_1 = \alpha_2 = 1$, the approximation values of the cdf of the weighted non-central chi-square distribution is coincide with that approximation values of the cdf of the non-central chi-square distribution.

2. For $\alpha_1 = \alpha_2 = 1$, $\nu = 3, 4, 5, 6, 9, 10, 12, 14$ and $\nu = 3, 5, 9.$ using Equation (11) and (19) respectively, the approximation values of the cdf of the weighted non-central chi-square are coincide with that approximation values of the cdf of the non-central chi-square distribution of Ashour and Abdel-Samad (1990) .

3. For $\alpha_1 = \alpha_2 = 1$ and $\lambda = 0$ the values of the cdf of the weighted non-central chi-square is equivalent to the values of the cdf of the central chi-square.

Table 1: The approximation values of the cdf of the weighted non-central chi-square distribution using Equation (11) for any degrees of freedom, Equation (19) for odd degrees of freedom respectively

ν	λ	z	α_1	α_2	c.d.f using Eq(11)	c.d.f using Eq(19) for odd
3	1	4	0.8	0.4	0.814	0.814
			1	2	0.517	0.517
			1	1	0.602*	0.603*
	2	5	1	3	0.535	0.535
			1	1	0.593*	0.594*
	3	6	0.5	1.5	0.985	0.985
			1	4	0.551	0.551
			1	1	0.585*	0.586*
	4	1.5	0.6	3	0.299	0.299
			0.7	1.75	0.194	0.194
			0.8	1.33	0.133	0.133
			1	1	0.072	0.072
4	1	5	0.8	0.4	0.830	-
			1	2	0.495	-
			1	1	0.590*	-
	2	6	1	3	0.522	-
			1	1	0.584*	-
		7	0.5	1.5	0.981	-

	3		1	4	0.542	-
			1	1	0.579*	-
	4	1.765	0.6	3	0.283	-
			0.7	1.75	0.169	-
			0.8	1.33	0.106	-
			1	1	0.050*	-
5	1	6	0.8	0.4	0.845	0.845
			1	2	0.481	0.481
			1	1	0.582*	0.582*
	2	7	1	3	0.514	0.514
			1	1	0.578*	0.578*
	3	8	0.5	1.5	0.978	0.978

*means values of the cdf of the non-central chi-square distribution.

Table (Cont.)

ν	λ	z	α_1	α_2	c.d.f using Eq(11)	c.d.f using Eq(19) for odd
5	3	8	1	4	0.537	0.537
			1	1	0.573*	0.573*
	4	1.5	0.6	3	0.150	0.150
			0.7	1.75	0.078	0.078
			0.8	1.33	0.043	0.043
			1	1	0.017	0.017
6	1	7	0.8	0.4	0.858	-
			1	2	0.471	-
			1	1	0.575*	-
	2	8	1	3	0.509	-
			1	1	0.572*	-
	3	9	0.5	1.5	0.975	-
			1	4	0.532	-
			1	1	0.569*	-
	4	1.5	0.6	3	0.100	-
			0.7	1.75	0.047	-
			0.8	1.33	0.023	-
			1	1	0.007	-
7	1	8	0.8	0.4	0.870	0.845
			1	2	0.464	0.464
			1	1	0.570	0.570
	2	9	1	3	0.505	0.505
			1	1	0.567	0.567
	3	10	0.5	1.5	0.973	0.973
			1	4	0.529	0.529
			1	1	0.565	0.565
	4	1.5	0.6	3	0.064	0.064
			0.7	1.75	0.026	0.026
			0.8	1.33	0.011	0.011
			1	1	0.003	0.003
8	1	9	0.8	0.4	0.880	-
			1	2	0.459	-
			1	1	0.566	-

*means values of the cdf of the non-central chi-square distribution.

Table (Cont.)

ν	λ	z	α_1	α_2	c.d.f using Eq(11)	c.d.f using Eq(19) for odd
8	2	10	1	3	0.502	-
			1	1	0.564	-
	3	11	0.5	1.5	0.971	-
			1	4	0.526	-
			1	1	0.561	-
	4	1.5	0.6	3	0.039	-
			0.7	1.75	0.014	-
			0.8	1.33	0.005	-
			1	1	0.001	-
9	1	10	0.8	0.4	0.890	0.890
			1	2	0.456	0.456
			1	1	0.562*	0.562*
	2	11	1	3	0.500	0.500
			1	1	0.560*	0.561*
	3	12	0.5	1.5	0.969	0.969
			1	4	0.524	0.524

			1	1	0.559*	0.559*
	4	1.5	0.6	3	0.023	0.023
			0.7	1.75	0.007	0.007
			0.8	1.33	0.002	0.002
			1	1	5.016E-4	5.016E-4
10	1	11	0.8	0.4	0.898	-
			1	2	0.453	-
			1	1	0.559*	-
	2	12	1	3	0.498	-
			1	1	0.558*	-
	3	13	0.5	1.5	0.967	-
			1	4	0.522	-
			1	1	0.556*	-
	4	2.5	0.6	3	0.089	-
			0.7	1.75	0.030	-
			0.8	1.33	0.010	-
			1	1	0.001	-

*means values of the cdf of the non-central chi-square distribution.

Table (Cont.)

ν	λ	z	α_1	α_2	c.d.f using Eq(11)	c.d.f using Eq(19) for odd
11	1	12	0.8	0.4	0.906	0.906
			1	2	0.451	0.451
			1	1	0.556	0.556
	2	13	1	3	0.496	0.496
			1	1	0.555	0.555
	3	14	0.5	1.5	0.966	0.966
			1	4	0.520	0.520
			1	1	0.554	0.554
	4	2.5	0.6	3	0.061	0.061
			0.7	1.75	0.018	0.018
			0.8	1.33	0.005	0.005
			1	1	8.143E-4	8.143E-4
12	1	13	0.8	0.4	0.912	-
			1	2	0.449	-
			1	1	0.554	-
	2	14	1	3	0.495	-
			1	1	0.553	-
	3	15	0.5	1.5	0.965	-
			1	4	0.519	-
			1	1	0.552	-
	6	24	0.6	3	0.007	-
			0.7	1.75	0.171	-
			0.8	1.33	0.567	-
			1	1	0.817*	-
13	1	14	0.8	0.4	0.916	0.920
			1	2	0.447	0.447
			1	1	0.552	0.552
	2	15	1	3	0.493	0.493
			1	1	0.551	0.551
	3	16	0.5	1.5	0.964	0.964
			1	4	0.517	0.517

*means values of the cdf of the non-central chi-square distribution.

Table: (Cont.)

ν	λ	z	α_1	α_2	c.d.f using Eq(11)	c.d.f using Eq(19) for odd
13	3	16	1	1	0.550	0.550
	4	3.5	0.6	3	0.107	0.107
			0.7	1.75	0.030	0.030
			0.8	1.33	0.008	0.008
			1	1	9.591E-4	9.591E-4
14	1	15	0.8	0.4	0.915	-
			1	2	0.446	-
			1	1	0.550*	-
	2	16	1	3	0.492	-
			1	1	0.549*	-

	3	17	0.5	1.5	0.963	-
			1	4	0.516	-
			1	1	0.549*	-
	4	6	0.6	3	0.448	-
			0.7	1.75	0.186	-
			0.8	1.33	0.067	-
			1	1	0.009	-

*means values of the cdf of the non-central chi-square distribution.

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