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Statistical inference of the Burr Type III distribution under joint progressively Type-II censoring

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ABSTRACT

The joint censoring technique is essential when the study's objective is to assess the relative benefits of products in relation to their service times. To reduce the cost and duration of the experiment, progressive censoring has gained a lot of attention in recent years. This article examines the statistical inference for the Burr Type III distribution using a joint progressive Type II censoring method on two samples. For model parameters, both the maximum likelihood and Bayesian methods are considered. Next, approximate confidence intervals are obtained based on the observed information matrix. Confidence intervals are also obtained using the procedures of Bootstrap-P and Bootstrap-T. Bayesian estimators are provided for symmetric and asymmetric loss functions. The Bayesian estimators cannot be produced in closed forms; hence, we compute the Bayesian estimators and the related credible intervals using the Markov chain Monte Carlo method. To evaluate the performance of the estimators, we conduct comprehensive simulation experiments. Finally, for purposes of illustration, we analyze two real data sets.

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Introduction

Burr Type III distribution

Burr [1] established twelve distinct classifications of cumulative distribution functions that provide a range of density arrangements. The main goal of selecting one of these distribution types is to make the mathematical analysis it undergoes as simple as possible while still producing a fair approximation. Among these distributions, the Burr Type III distribution (BIID) can accommodate different hazard lifetime data, so it has received considerable attention in the recent past. This distribution has been widely used in numerous fields of science with different parameterizations under other names. In studies of income, wage, and wealth distributions, it is referred to as the Dagum distribution [2]. It is referred to as the inverse Burr distribution in actuarial literature [3] and the Kappa distribution in meteorological literature [4]. The BIIID has several applications in statistical modeling fields. It has also been employed in reliability theory [5]. Gove et al. [6] fitted BIIID to data related to forestry. Lindsay et al. [7] employed BIIID as an alternative to the Weibull distribution to simulate

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the diameter distributions of forest stands. BIIID was used by Shao [8] to model the NOEC toxicity data. Given that the BIIID can approximate a number of conventional lifetime models, including Weibull, gamma, and lognormal (see Zoraghi et al. [9]), it makes sense to model failure data with this distribution. Therefore, the BIIID is crucial to lifetime analysis.

The lifetime *X* of the product has BIIID if the cumulative distribution function, probability distribution function, and survival function for x > 0, have the following specifications:

$$F(x;\alpha,\beta) = \left[1 + x^{-\alpha}\right]^{-\beta},\tag{1}$$

$$f(\mathbf{x};\alpha,\beta) = \alpha\beta \mathbf{x}^{-(\alpha+1)} \left[1 + \mathbf{x}^{-\alpha}\right]^{-(\beta+1)},\tag{2}$$

and,

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$$(x; \alpha, \beta) = 1 - [1 + x^{-\alpha}]^{-\beta}, \tag{3}$$

where $\alpha > 0$, and $\beta > 0$ are shape parameters. Some researchers offered some of the most significant studies about the BIID. Altindag et al. [10] presented the estimation and prediction problems for the BIID with Type II censored samples. The statistical inference of a BIID was established by Panahi [11] using the unified hybrid censored sample. Gamchi et al. [12] conducted research on the estimation and prediction problems for the BIID under progressive Type II hybrid censored data. A maximum likelihood (ML) estimation of the lifetime performance index for BIID, based on progressive censoring, was taken into consideration by Hassan et al. [13]. Based on unified progressive hybrid censoring, Dutta and Kayal [14] presented estimation and prediction for the BIID.

The joint progressive Type-II censoring scheme

The most common censorship techniques employed are Types I and II. A Type I censoring approach is one in which the observations are terminated at a predetermined time and the failure timings are recorded. When a sufficient and predetermined number of units fail, the observations are terminated, which is the definition of a Type II censoring method. Although the lifetimes of the tested units are relatively long, neither of these two censoring strategies performs well in this situation. Later, further censorship plans were put forth; the most well-liked and attractive of these are progressive censoring (PC) plans, which eliminate test units each time they fail rather than simply the last time. Information on the PC plans was provided by Balakrishnan and Aggarwala [15]. The two types of PC are progressive Type-I and progressive Type II. The PC Type I and Type II for some lifetime distributions have been researched by many academics [16,17,18,19,20,21].

The aforementioned censoring techniques are all applied to one-sample issues. In real life, we encounter and must take into account two or more samples from various assembly lines. The joint progressive Type II censoring (JP–IIC) scheme has drawn a lot of attention recently and is particularly helpful in comparing the lives of goods from various manufacturing lines. To analyze two populations from various exponential distributions, the JP–IIC scheme was first introduced by Balakrishnan and Rasouli [22]. For the JP–IIC method, Parsi et al. [23] explored the conditional ML and the interval estimators of the Weibull distribution. The Bayes estimator was produced by Doostparast et al. [24], when data were sampled using the JP–IIC scheme from a general class of distributions. Shafay et al. [25] addressed Bayesian inference based on a JP–IIC sample from two exponential populations. Balakrishnan et al. [26] considered a JP–IIC sample arising from k independent exponential populations. The procedure of estimating lifetime using multiple exponential and Weibull distributions was investigated by Mondal and Kundu [27,28]. Abo-Kasem [29] discussed inferences for two Rayleigh populations based on JP–IIC data. Krishna and Goel [30] discussed inferences for two Lindley populations, based on JP–IIC data. Chen and Gui [31] provided statistical inference of the generalized inverted exponential distribution via JP–IIC. Fan and Gui [32] studied the inference of an inverted exponential Rayleigh distribution under JP–IIC.

Motivated by the various applications of BIIID in many fields, in this study, we use a JP–IIC strategy to construct statistical inferences and evaluate two independent samples from BIIID. Point and interval estimators are obtained by using Bayesian and ML estimation procedures. On the basis of the observed information matrix, asymptotic confidence intervals (Asy-CIs) are then calculated. The CIs are computed using the Bootstrap-P and Bootstrap-T techniques. For both shape parameters, a gamma prior is assumed. Using the Metropolis-Hastings (MH) method, it is possible to obtain the Bayes estimates and credible intervals for the informative prior, under the squared error loss function (SELF), linear exponential loss function (LiLF), minimum expected loss function (MELF), and Degroot loss function (DeLF). To assess the effectiveness of various approaches, Monte Carlo simulation and actual data analysis are used.

The remainder of the paper is set up as follows. The model is described in Section 2 along with the ML estimators and the Asy-CIs of the BIIID parameters. Bootstrap-P, Bootstrap-T CIs and highest posterior density (HPD) credible intervals are provided in Section 3. Section 4 examines Bayesian estimation with gamma priors under some loss functions. Section 5 uses a simulation study and actual data sets as examples. Bootstrap-P, as well as Bootstrap-T CIs are provided in Section 3.

Estimating the maximum likelihood using the model

Assume that the lifetime of *m* units of product A, W_1 , W_2 ,..., W_m are identically independent distributed (iid) random variables (RVs) possessing BIIID (α , β_1). The same is true for the lifetime of *n* units of product B, Z_1 , Z_2 ,..., Z_n , which are iid RVs possessing BIIID (α , β_2).

The first step is to organize N = m + n RVs in combined order, represented by $W_1 \le W_2 \le ... \le W_k$, then the JP–IIC implementation for the next two samples. N units are initially subjected to a lifetime experiment, and when the first failure occurs (either at W or Z), R_1 live units are subtracted from the remaining N - 1 live units. When the second component (W or Z) fails, R_2 live units are subtracted from the remaining N, R_1 , and R_2 live units, and so on. The remaining live $R_k = N - r - R_1 - R_2 - ... - R_{k-1}$ units are pulled from the test upon the occurrence of the *k*-th failure, which may come from W or Z.

In this case, the total number of failures k and the censoring strategy $R=(R_1,R_2,...,R_k)$ are pre-fixed before the experiment is run, where $R_i = S_i + T_i$ and S_i and T_i are unknown RVs denoting the number of units removed from the A and B populations, respectively, at the time of the *i*th failure. As a result, the collected data are represented as (*W*,*Z*,*S*), where $W=(W_1,W_2,...,W_k)$ with $1 \le k \le N$ and $Z=(Z_1,Z_2,...,Z_k)$ are defined as:

$$Z_i = \begin{cases} 1; & if W_i \in A \text{ population} \\ 0; & 0.w. \end{cases}$$

In other words, Z_i only accepts one of two values, 1 or 0, depending on whether W_i is a W failure or a Z failure. We further divide the censoring scheme $\underline{R}=(R_1, R_2,...,R_k)$ into $S + T=(S_1, S_2,...,S_k) + (T_1,T_2,...,T_k)$. For $R_1=R_2=...=R_{k-1}=0$, $R_k=N-k$, $S_k=m-k_1$ and $T_k=n-k_2$ the results are very consistent with joint Type II.

Here $k_1 = \sum_{i=1}^{k} z_i$, this is case refers to the number of failures from line A.

Similarly, $k_2 = \sum_{i=1}^{k} (1 - z_i) = k - k_1$ represents the quantity of failures from line B.

The likelihood function (LF), based on (1), (2), (3) for Z and W, in case of JP–IIC scheme can be written as

$$L(\phi, data) = C \prod_{i=1}^{k} \left[\{ \bar{F}(w_i) \}^{s_i} \{ \bar{G}(w_i) \}^{t_i} \{ f(w_i) \}^{z_i} \{ g(w_i) \}^{(1-z_i)} \right],$$
(4)

where $0 \le w_1 \le w_2 \le ... \le w_k, \phi = (\beta_1, \beta_2, \alpha), \bar{F}(.) = 1 - F(.), \bar{G}(.) = 1 - G(.), \text{and}$

$$C = \prod_{j=1}^{k} \left[Z_j \left(m - \sum_{i=1}^{j-1} s_i - \sum_{i=1}^{j-1} z_i \right) + \left(1 - z_j \right) \left(n - \sum_{i=1}^{j-1} (R_i - s_i) - \sum_{i=1}^{j-1} (1 - z_i) \right) \right] \\ \times \prod_{j=1}^{k-1} \left(m - \sum_{i=1}^{j-1} s_i - \sum_{i=1}^{j-1} z_i \right) \left(n - \sum_{i=1}^{j-1} (R_i - s_i) - \sum_{i=1}^{j-1} (1 - z_i) \right) \left(m - j - n - \sum_{i=1}^{j-1} R_i \right)^{-1} .$$

The LF (4) can be written as:

$$L(\phi, data) = C\alpha^{k}\beta_{1}^{k_{1}}\beta_{2}^{k_{2}}\prod_{i=1}^{k}w_{i}^{-(\alpha+1)}D_{i}(\alpha)^{-[\beta_{1}z_{i}+\beta_{2}(1-z_{i})+1]}\left[1-(D_{i}(\alpha))^{-\beta_{1}}\right]^{s_{i}}\left[1-(D_{i}(\alpha))^{-\beta_{2}}\right]^{t_{i}},$$
(5)

where $D_i(\alpha) = (1 + w_i^{-\alpha})$. The log-LF of (5), denoted by l^* , is written as

$$l^{*} = k \ln \alpha + k_{1} \ln \beta_{1} + k_{2} \ln \beta_{2} - (\alpha + 1) \sum_{i=1}^{k} \ln w_{i} - \sum_{i=1}^{k} [\beta_{1} z_{i} + \beta_{2} (1 - z_{i}) + 1] \ln D_{i}(\alpha) + \sum_{i=1}^{k} \left[s_{i} \ln (1 - D_{i}(\alpha)^{-\beta_{1}}) + t_{i} \ln (1 - D_{i}(\alpha)^{-\beta_{2}}) \right].$$

The first partial derivative of β_1 , β_2 and α are obtained, respectively, as follows:

$$\frac{\partial l^*}{\partial \beta_1} = \frac{k_1}{\beta_1} - \sum_{i=1}^k z_i ln(D_i(\alpha)) + \sum_{i=1}^k \frac{s_i ln(D_i(\alpha))}{(D_i(\alpha))^{\beta_1} - 1},\tag{6}$$

$$\frac{\partial l^*}{\partial \beta_2} = \frac{k_2}{\beta_2} - \sum_{i=1}^k (1 - z_i) \ln(D_i(\alpha)) + \sum_{i=1}^k \frac{t_i \ln(D_i(\alpha))}{(D_i(\alpha))^{\beta_2} - 1},\tag{7}$$

$$\frac{\partial l^{*}}{\partial \alpha} = \frac{k}{\alpha} - \sum_{i=1}^{k} ln w_{i} - \sum_{i=1}^{k} \frac{[\beta_{1}z_{i} + \beta_{2}(1-z_{i}) + 1]ln w_{i}}{1 + w_{i}^{\alpha}} - \sum_{i=1}^{k} \frac{s_{i}\beta_{1}(D_{i}(\alpha))^{-1} w_{i}^{-\alpha} ln w_{i}}{[(D_{i}(\alpha))^{\beta_{1}} - 1]} - \sum_{i=1}^{k} \frac{t_{i}\beta_{2}(D_{i}(\alpha))^{-1} w_{i}^{-\alpha} ln w_{i}}{[(D_{i}(\alpha))^{\beta_{2}} - 1]}.$$
(8)

The ML estimators of β_1 , β_2 and α are obtained by settling (6)–(8) with zero and solving numerically via R-statistical programming language.

Furthermore, for evaluating estimated variance-covariance matrix (VCM) and related asymptotic CIs of ML estimators, with JP–IIC data, the observed Fisher information matrix (FIM) is defined as:

$$\hat{I}(\phi) = \begin{bmatrix} \frac{-\partial^{2}l^{*}}{\partial \beta_{1}} & \frac{\partial^{2}l^{*}}{\partial \beta_{1}\beta_{2}} & \frac{-\partial^{2}l^{*}}{\partial \beta_{1}\beta_{2}} \\ \frac{\partial^{2}l^{*}}{\partial \beta_{2}\beta_{1}} & \frac{\partial^{2}l^{*}}{\partial \beta_{2}\beta_{2}} & \frac{-\partial^{2}l^{*}}{\partial \beta_{2}\beta_{2}} \\ \frac{-\partial^{2}l^{*}}{\partial \alpha \partial \beta_{1}} & \frac{-\partial^{2}l^{*}}{\partial \alpha \partial \beta_{2}} & \frac{-\partial^{2}l^{*}}{\partial \alpha^{2}} \end{bmatrix}_{\substack{\beta_{1}=\hat{\beta}_{1}\\\beta_{2}=\hat{\beta}_{2}\\\alpha=\hat{\alpha}}} = \begin{bmatrix} I_{11} & I_{12} & I_{13}\\ I_{21} & I_{22} & I_{23}\\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{\substack{\beta_{1}=\hat{\beta}_{1}\\\beta_{2}=\hat{\beta}_{2}\\\alpha=\hat{\alpha}}} = \begin{bmatrix} \hat{I}_{11} & \hat{I}_{12} & \hat{I}_{13}\\ \hat{I}_{21} & \hat{I}_{22} & \hat{I}_{23}\\ \hat{I}_{31} & \hat{I}_{32} & \hat{I}_{33} \end{bmatrix}$$

Note that the expressions for second-order partial derivatives are provided in Appendix. The $(1 - \varepsilon)$ 100% Asy-Cls for $\phi = (\beta_1, \beta_2, \alpha)$ by using the approximated standard normal distribution are given by

$$\hat{\hat{\phi}} \pm Z^*_{\varepsilon/2} \sqrt{\widehat{\operatorname{var}}(\hat{\hat{\phi}})},$$

where $Z^*_{\epsilon/2}$ denoted the upper $\epsilon/2$ percent point of standard normal distribution and $\widehat{\operatorname{var}}(\hat{\phi})$ is the estimated variance.

Bootstrap methods

The construction of CIs is done in this part using the Bootstrap methodology. Bootstrap-P (Boot-P) and Bootstrap-T (Boot-T) approaches' algorithms are provided in Algorithms 1 and 2, respectively.

$$\left[\hat{\hat{\phi}} - \sqrt{\operatorname{Var}(\hat{\phi})} T_{\phi(hb)}, \hat{\hat{\phi}} - \sqrt{\operatorname{Var}(\hat{\phi})} T_{\phi(lb)}\right]$$

where $Ib = \left[\frac{\varepsilon}{2}N\right]$, $hb = \left[\frac{1-\varepsilon}{2}N\right]$. Here [x] means the largest integer not exceeding *x*.

Bayesian estimation

Bayes estimate takes into account the previous knowledge of life factors, in contrast to classical statistics. As a result, Bayesian estimation considers both the available data and the prior probability to infer the relevant parameters.

According to different gamma distributions, that β_1, β_2 and α are independent, where

$$\begin{aligned} \pi_i(\beta_i) &= \frac{b_i^{a_i}}{\Gamma(a_i)} \beta_i^{a_i-1} e^{-b_i \beta_i}, \qquad a_i, \ b_i, \ \beta_i > 0, \ i = 1, 2, \\ \pi_3(\alpha) &= \frac{d^c}{\Gamma(c)} \alpha^{c-1} e^{-d\alpha}, \qquad \alpha > 0, c, d > 0, \end{aligned}$$

where $a_i, b_i, c, d, i = 1$, 2 are the hyper-parameters that contain the prior information. The joint prior distribution can be written as.

$$\pi_0(\phi) \propto \beta_1^{a_1-1} \beta_2^{a_2-1} \alpha^{c-1} e^{-(b_1\beta_1+b_2\beta_2+d\alpha)}$$

The joint posterior probability distribution is

$$\pi(\phi|data) = \frac{\pi_0(\phi)L(\phi, data)}{\int_0^\infty \int_0^\infty \int_0^\infty \pi_0(\phi)L(\phi, data)d\beta_1 d\beta_2 d\alpha}.$$
(9)

The denominator of (9) is a function of the observed data. Thus, $L(\phi, data)$ and $\pi(\phi|data)$ have a coefficient-proportional relationship. Therefore, the joint posterior probability distribution is

$$\pi (\phi | data) \propto \beta_1^{k_1 + a_1 - 1} \beta_2^{k_2 + a_2 - 1} \alpha^{k + c - 1} e^{-(b_1 \beta_1 + b_2 \beta_2 + d\alpha)} \\ \times \exp \left(\beta_1 \sum_{i=1}^k \psi_{1i}(\alpha, \beta_1) + \beta_2 \sum_{i=1}^k \psi_{2i}(\alpha, \beta_2) + \alpha \sum_{i=1}^k \psi_{3i}(\alpha) \right),$$
(10)

where

$$\begin{split} \psi_{1i}(\alpha, \beta_1) &= z_i \ln D_i(\alpha) - s_i \ln(1 - (D_i(\alpha))^{-\beta_1}), \\ \psi_{2i}(\alpha, \beta_2) &= (1 - z_i) \ln D_i(\alpha) - t_i \ln(1 - (D_i(\alpha))^{-\beta_2}), \\ \psi_{3i}(\alpha) &= \ln w_i + \ln D_i(\alpha). \end{split}$$

Hence, the marginal posterior distributions of β_1 , β_2 and α take the following forms:

$$\begin{aligned} \pi^{*}(\beta_{1}|\beta_{2},\alpha,data) &= \int_{0}^{\infty} \int_{0}^{\infty} \pi(\phi|data)d\beta_{2}d\alpha \\ &= N\beta_{1}^{k_{1}+a_{1}-1} \int_{0}^{\infty} \int_{0}^{\beta} \beta_{2}^{k_{2}+a_{2}-1} \alpha^{k+c-1} e^{-\left(\beta_{1}\left(\sum\limits_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum\limits_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum\limits_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{2}d\alpha, \\ \pi^{**}(\beta_{2}|\beta_{1},\alpha,data) &= \int_{0}^{\infty} \int_{0}^{\infty} \pi(\phi|data)d\beta_{1}d\alpha \\ &= N\beta_{2}^{k_{2}+a_{2}-1} \int_{0}^{\infty} \int_{0}^{\beta} \beta_{1}^{k_{1}+a_{1}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum\limits_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum\limits_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum\limits_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\alpha, \\ \pi^{**}(\beta_{2}|\beta_{1},\alpha,data) &= \int_{0}^{\infty} \int_{0}^{\infty} \pi(\phi|data)d\beta_{1}d\alpha \\ &= N\beta_{2}^{k_{2}+a_{2}-1} \int_{0}^{\infty} \int_{0}^{\beta} \beta_{1}^{k_{1}+a_{1}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum\limits_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum\limits_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum\limits_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\alpha, \end{aligned}$$

where, $N^{-1} = \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1} \beta_{2}^{k_{2}+a_{2}-1} \alpha^{k+c-1} e^{-(\beta_{1}(\sum_{i=1}^{k} \psi_{1i}(\alpha,\beta_{1})+b_{1})+\beta_{2}(\sum_{i=1}^{k} \psi_{2i}(\alpha,\beta_{2})+b_{2})+\alpha(\sum_{i=1}^{k} \psi_{3i}(\alpha)+d))} d\beta_{1} d\beta_{2} d\alpha.$

In the following, the Bayesian estimators are provided under different loss functions.

Loss functions

Here the Bayesian estimators of $\phi = (\beta_1, \beta_2, \alpha)$ are all obtained under symmetric and asymmetric loss functions.

(1) Squared Error loss function

One of the helpful symmetric loss functions seen in nature, a quadratic or SELF, prioritizes both over and under estimation equally. A definition of the SELF is

$$L_{SELF}(\hat{\phi},\phi) = (\hat{\phi}-\phi)^2.$$

Therefore, the Bayesian estimators of β_1 , β_2 , and α under SELF, say $\hat{\beta}_{1(SE)}$, $\hat{\beta}_{2(SE)}$ and $\hat{\alpha}_{(SE)}$ are obtained as a posterior mean as follows

$$\hat{\boldsymbol{\beta}}_{1(SE)} = E(\boldsymbol{\beta}_{1}|\underline{w}) = N \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}} \boldsymbol{\beta}_{2}^{k_{2}+a_{2}-1} \boldsymbol{\alpha}^{k+c-1} e^{-\left(\beta_{1}\left(\sum_{i=1}^{k} \psi_{1i}(\boldsymbol{\alpha},\boldsymbol{\beta}_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k} \psi_{2i}(\boldsymbol{\alpha},\boldsymbol{\beta}_{2})+b_{2}\right)\right)} \times e^{-\boldsymbol{\alpha}\left(\sum_{i=1}^{k} \psi_{3i}(\boldsymbol{\alpha})+d\right)} d\boldsymbol{\beta}_{1} d\boldsymbol{\beta}_{2} d\boldsymbol{\alpha},$$

$$(11)$$

$$\hat{\hat{\beta}}_{2(SE)} = E(\beta_{2}|\underline{w}) = N \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{2}^{k_{2}+a_{2}} \beta_{1}^{k_{1}+a_{1}-1} \alpha^{k+c-1} e^{-\left(\beta_{1}\left(\sum_{i=1}^{k} \psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k} \psi_{2i}(\alpha,\beta_{2})+b_{2}\right)\right)} \times e^{-\alpha\left(\sum_{i=1}^{k} \psi_{3i}(\alpha)+d\right)} d\beta_{1} d\beta_{2} d\alpha,$$
(12)

$$\hat{\hat{\alpha}}_{(SE)} = E(\alpha|\underline{w}) = N \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \alpha^{k+c} \beta_{1}^{k_{1}+a_{1}-1} \beta_{2}^{k_{2}+a_{2}-1} e^{-\left(\beta_{1}\left(\sum_{i=1}^{k} \psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k} \psi_{2i}(\alpha,\beta_{2})+b_{2}\right)\right)} \times e^{-\alpha\left(\sum_{i=1}^{k} \psi_{3i}(\alpha)+d\right)} d\beta_{1} d\beta_{2} d\alpha.$$
(13)

(1) Linear - Exponential loss function

The LiLF is the most universally used asymmetric loss function. The asymmetric loss function is considered more comprehensive in many respects. The LiLF is given below:

$$l_{\text{Lilf}}(\phi,\hat{\phi}) = e^{h(\hat{\phi}-\phi)} - h(\hat{\phi}-\phi) - 1, h \neq 0,$$

where $\hat{\phi}$ is an estimate of ϕ and h is the real number and stands for the sign, which presents the asymmetry. The corresponding Bayesian estimate $\hat{\phi}_{ijjF}$ of ϕ can be derived from

$$\hat{\hat{\phi}}_{_{LiLF}} = -\frac{1}{h} ln[E(e^{-h\phi}|\underline{w})].$$

Hence, the Bayesian estimators of β_1 , β_2 , and α under LiLF, say $\hat{\hat{\beta}}_{1(LiLF)}$, $\hat{\hat{\beta}}_{2(LiLF)}$ and $\hat{\hat{\alpha}}_{(LiLF)}$ are obtained as follows:

$$\hat{\hat{\beta}}_{1(LiLF)} = \frac{-1}{h} ln \left[N \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1} \alpha^{k+c-1} \beta_{2}^{k_{2}+a_{2}-1} \right] \times e^{-\left(\beta_{1}\left(\sum_{i=1}^{k} \psi_{1i}(\alpha,\beta_{1})+b_{1}+h\right)+\beta_{2}\left(\sum_{i=1}^{k} \psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k} \psi_{3i}(\alpha)+d\right)\right)} d\beta_{1} d\beta_{2} d\alpha \right],$$
(14)

$$\hat{\beta}_{2(LiLF)} = \frac{-1}{h} ln \left[N \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \alpha^{k+c-1} \beta_{1}^{k_{1}+a_{1}-1} \beta_{2}^{k_{2}+a_{2}-1} \right] \\ \times e^{-\left(\beta_{1} \left(\sum_{i=1}^{k} \psi_{1i}(\alpha,\beta_{1})+b_{1}\right) + \beta_{2} \left(\sum_{i=1}^{k} \psi_{2i}(\alpha,\beta_{2})+b_{2}+h\right) + \alpha \left(\sum_{i=1}^{k} \psi_{3i}(\alpha)+d\right) \right)} d\beta_{1} d\beta_{2} d\alpha \right],$$
(15)

$$\hat{\hat{\alpha}}_{(LiLF)} = \frac{-1}{h} ln \left[N \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{2}^{k_{2}+a_{2}-1} \alpha^{k+c-1} \beta_{1}^{k_{1}+a_{1}-1} \right] \\ \times e^{-\left(\beta_{1} \left(\sum_{i=1}^{k} \psi_{1i}(\alpha,\beta_{1})+b_{1} \right) + \beta_{2} \left(\sum_{i=1}^{k} \psi_{2i}(\alpha,\beta_{2})+b_{2} \right) + \alpha \left(\sum_{i=1}^{k} \psi_{3i}(\alpha)+d+h \right) \right)} d\beta_{1} d\beta_{2} d\alpha \right].$$
(16)

(1) Minimum expected loss function

The MELF was developed in Tummala and Sathe [33], which is defined by:

$$L_{MELF}(\hat{\hat{\phi}},\phi) = \frac{(\hat{\hat{\phi}}-\phi)^2}{\phi^2},$$

where, $\hat{\phi}_{MELF}$ is an estimator of ϕ . Hence, the Bayesian estimators of β_1 , β_2 and α under MELF, $\hat{\beta}_{1(MELF)}$, $\hat{\beta}_{2(MELF)}$ and $\hat{\alpha}_{(MELF)}$, are derived as follows:

$$\hat{\hat{\beta}}_{1(MELF)} = \frac{E(\beta_{1}^{-1}|\underline{w})}{E(\beta_{1}^{-2}|\underline{w})} = \begin{bmatrix} \sum_{0}^{\infty} \beta_{1}^{-1}\pi^{*}(\beta_{1}|\beta_{2},\alpha,data)d\beta_{1} \\ \sum_{0}^{\infty} \beta_{1}^{-2}\pi^{*}(\beta_{1}|\beta_{2},\alpha,data)d\beta_{1} \\ \sum_{0}^{\infty} \beta_{1}^{-2}\pi^{*}(\beta_{1}|\beta_{2},\alpha,data)d\beta_{1} \end{bmatrix},$$

$$\hat{\hat{\beta}}_{1(MELF)} = \begin{bmatrix} \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-2}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-3}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\hat{\beta}}_{2(MELF)} = \begin{bmatrix} \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-3}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\hat{\alpha}}_{(MELF)} = \begin{bmatrix} \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-3}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\hat{\alpha}}_{(MELF)} = \begin{bmatrix} \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-2}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\hat{\alpha}}_{(MELF)} = \begin{bmatrix} \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-2}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\hat{\beta}}_{0} = 0 \begin{bmatrix} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-2}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-2}e^{-\left(\beta_{1}\left(\sum_{l=1}^{k}\psi_{1l}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{l=1}^{k}\psi_{2l}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{l=1}^{k}\psi_{3l}(\alpha)+d\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \sum_{0}^{\infty} \sum_{0}^{\infty} \beta_{1}^{$$

(1) Degroot loss function

The DeLF was provided in Degroot [34], as follows:

$$L_{DeLF}(\hat{\phi},\phi) = \frac{(\hat{\phi}-\phi)^2}{\phi^2},$$

where, $\hat{\phi}_{DeLF}$ is an estimator of ϕ . Hence, the Bayesian estimators of β_1 , β_2 and α under DeLF, $\hat{\beta}_{1(DeLF)}$, $\hat{\beta}_{2(DeLF)}$, and $\hat{\alpha}_{(DeLF)}$ are derived as follows:

$$\hat{\beta}_{1(DeLF)} = \frac{E(\beta_{1}^{2}|\underline{w})}{E(\beta_{1}|\underline{w})} = \begin{bmatrix} \int_{0}^{\infty} \beta_{1}^{2}\pi^{*}(\beta_{1}|\beta_{2},\alpha,data)d\beta_{1} \\ \int_{0}^{\infty} \beta_{1}\pi^{*}(\beta_{1}|\beta_{2},\alpha,data)d\beta_{1} \\ \int_{0}^{\infty} \beta_{1}\pi^{*}(\beta_{1}|\beta_{2},\alpha,data)d\beta_{1} \end{bmatrix},$$

$$\hat{\beta}_{1(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}+1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c-1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c+1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c+1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int_{0}^{\infty} \beta_{1}^{k_{1}+a_{1}-1}\beta_{2}^{k_{2}+a_{2}-1}\alpha^{k+c+1}e^{-\left(\beta_{1}\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_{1})+b_{1}\right)+\beta_{2}\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_{2})+b_{2}\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)}d\beta_{1}d\beta_{2}d\alpha} \\ \hat{\beta}_{2(DeLF)} = \begin{bmatrix} \int_{0}^{\infty} \int$$

Integrals (11) to (22) are difficult to obtain; therefore, the MH algorithm is employed to generate MCMC samples from posterior density functions (10). After acquiring Markov chain Monte Carlo (MCMC) samples from the posterior distribution, we can get the Bayes estimate of β_1 , β_2 and α .

Highest Posterior Density Credible Interval (HPD)

An approach provided by Chen and Shao [35] is utilized to create the HPD credible intervals for ϕ . The MCMC samples $\phi_1, \phi_2, ..., \phi_M$ are taken into consideration for generating the $(1 - \varepsilon)$ 100% symmetric HPD credible intervals for ϕ . These samples are arranged in ascending order $\phi_{(1)}, \phi_{(2)}, ..., \phi_{(M)}$, and has an HPD credible interval of $[\phi_{[(M(\varepsilon/2))]}, \phi_{[(M(1-\varepsilon/2))]}]$.

Simulation study and data analysis

Analyzing the effectiveness of the different estimating methods described in the parts above is the goal of this section. A simulation study is used to assess the statistical performances of the different estimates given a JP–IIC scheme for BIIID. A real dataset is used for illustration purposes in order to investigate the behavior of the recommended approaches. The R statistical programming language has been used to do the calculations. With the help of the bbmle and HPDInterval packages, MLEs and HPD intervals are calculated in R.

Simulation study

A Monte Carlo simulation analysis is used in this section, using the JP–IIC scheme for BIIID, to assess the effectiveness of the ML and Bayesian estimation methods. Based on the subsequent hypotheses, 1000 observations for the MLEs are obtained from BIIID:

- 1 Assume the following selected values of parameters, $\alpha = 0.5$, $\beta_1 = 1.5$, $\beta_2 = 1.75$
- 2 The sum of sample sizes (N = n + m) of the two samples are given as: N = 80, 160, and 240, where n = m.
- 3 Removed items R_i are assumed at different sample sizes N and number of stages k.
- 4 Five different censorning schemes, namely S_1 , S_2 , S_3 , S_4 , and S_5 are selected as shown in Table 1.

The MLEs and related 95% Asy-CI are produced based on the generated data. When deriving MLEs, be aware that the initial assumed values are regarded as true parameter values. Also, Boot- P and Boot-T are computed.

We compute Bayesian estimates (BEs) using gamma (informative prior (IP)) for the Bayesian estimation method. As historical samples, we construct 500 completed samples of size 60 each from the BIIID, then determine the values of the hyper-parameter given below as:

$$a_1 = 91.08, b_1 = 179.09, a_2 = 50.84, b_2 = 33.44, c = 59.06, d = 33.16$$

| Table 1 | | | | | | | | | |
|----------|-----|----------|-------|----|------|----|-----------|-------|---------|
| Patterns | for | removing | items | in | test | at | different | stage | numbers |

| | | | | Censoring Schemes | | |
|-----------|-----|----------------------------|--------------------------------------|--|----------------------------|-----------------------|
| (n, m) | k | <i>S</i> 1 | \$2 | \$3 | S4 | \$5 |
| (40,40) | 20 | $(60, 0 \times {}^{19})$ | $(30, 0 \times {}^{18}, 30)$ | $(0 \times {}^{9},30,30, 0 \times {}^{9})$ | $(0 \times {}^{19}, 60)$ | $(3 \times {}^{20})$ |
| | 40 | $(40, 0 \times {}^{39})$ | $(20, 0 \times {}^{38}, 20)$ | $(0 \times {}^{19},20,20, 0 \times {}^{19})$ | $(0 \times {}^{39}, 40)$ | $(1 \times {}^{40})$ |
| (80,80) | 40 | $(120, 0 \times {}^{39})$ | $(60, 0 \times {}^{38}, 60)$ | $(0 \times {}^{19},\!60,\!60, 0 \times {}^{19})$ | $(0 \times {}^{39}, 120)$ | $(3 \times {}^{40})$ |
| | 80 | $(80, 0 \times {}^{79})$ | $(40, 0 \times {}^{78}, 40)$ | $(0 \times {}^{39},40,40, 0 \times {}^{39})$ | $(0 \times {}^{79}, 80)$ | $(1 \times {}^{80})$ |
| (120,120) | 80 | $(160, 0 \times {}^{79})$ | $(80, 0 \times {}^{78}, 80)$ | $(0 \times {}^{39},\!80,\!80, 0 \times {}^{39})$ | $(0 \times {}^{79}, 160)$ | $(2 \times {}^{80})$ |
| | 120 | $(120, 0 \times {}^{119})$ | (60, 0 \times ¹¹⁸ , 60) | (0 \times ⁵⁹ ,60,60, 0 \times ⁵⁹) | $(0 \times {}^{119}, 120)$ | $(1 \times {}^{120})$ |

Here, $(5 \times {}^{3},0)$, for example, means that the censoring scheme employed is (5,5,5,0).

To analyze the necessary estimates, such IP values are fed in. The MLEs are used as starting guess values when the MH algorithm is implemented, along with the associated VCM S_{ϕ} of $(\ln(\hat{\alpha}), \ln(\hat{\beta}_1), \ln(\hat{\beta}_2))$. In the end, 2000 burn-in samples were deleted from the total of 10,000 generated samples by the posterior density, and produced BEs under different loss functions, namely: SELF, LiLF at h = 0.5, MELF, and DeLF. Also, HPD interval estimates have been computed according to the technique of Chen and Shao [35].

All the average estimates for methods are reported in Table 2(a,b,c) for total sample sizes N = 80, 160, and 240, respectively. Further, the first column denotes the average estimates (AVEs), and, in the second column, related mean squared errors (MSEs). For CIs, we have Asy-CI for MLEs and HPD for BEs based on MCMC, which is reported in Table 3(a,b,c) for given parameter values and N. Further, the first column represents average interval lengths (AILs), and, in the second column, related coverage probabilities (CPs).

From the tabulated results, one can conclude that:

- As *n* and *m* increase, the accuracy of the estimated values increases.
- Scheme S_3 usually gives the highest MSE values over the S_2 and S_4 schemes, which sometimes give peaks in MSE values over the average estimated parameters.
- The ML method gives larger MSE values over the estimated parameters compared to the Bayesian method.
- In approximately most situations, the MSEs of all parameter estimates based on Scheme S_1 usually give the smallest values compared to the other schemes.
- **E**stimates of β_2 usually provide the largest MSE values, and the range of AVE values is [1.64 2.86].
- From Table 2(a, b, c), we can find that the MLEs perform better in terms of the MSE as the value of k increases. The MSEs of α estimates are always much smaller than those of β_1 and β_2 estimates which means α estimates are better. This is reasonable because α is the same in the two populations.
- The BEs using MCMC for different schemes under DeLF provide better estimates in terms of MSE than the others under MELF, LiLF, and SELF, for total sample sizes N = 80, 160, and 240, respectively (see Table 2(a, b, c)).
- The BEs have good bias and MSE under SELF, LiLF, DeLF, and MELF compared to MLEs. Nonetheless, the BEs using MCMC under DeLF frequently perform better than those under SELF, LiLF, and MELF in terms of bias.
- The BEs of all parameters under DeLF are preferable to the others in almost all situations.
- The quality of Bayesian and ML methods based on MSE is listed in the following order:

 $DeLF \rightarrow MLE \rightarrow MELF \rightarrow LiLF \rightarrow SELF$ in most situations.

- The coverage probability for the parameters in the case of interval estimation is 95% of their nominal values.
- Table 3 (a, b, c) display the AIL and CP of confidence intervals for all methods. The contrast between Boot-P and Boot-T indicates that the AILs of Boot-T are wider than those of Boot-P. Therefore, the Boot-P method is more appropriate to get the confidence intervals than the Boot-T.
- The AIL of the Asy-CI is larger than the HPD credible intervals.
- The quality of CI methods, based on the average shortest interval, is listed in the following order:

The HPD credible interval (Good) \rightarrow Asy-Cl \rightarrow Boot-P \rightarrow Boot-T (worst), in approximately most situations.

The convergence of MCMC estimation in the case of scheme S_1 of JP-IIC can be shown for α , β_1 and and is represented in Fig. 1. This figure showed a scatter plot, histogram, and cumulative mean of posterior samples for each estimated parameter, which showed the normality of the generated posterior samples.

Data analysis

The strength values provided by Badar and Priest [36] are analysed in this sub-section. For single carbon fibres and impregnated 1000-carbon fiber tows, the strength measurements are represented using GPA units. Tension tests on single fibres were conducted at gage lengths of 1, 10, 20, and 50 mm. At gage lengths of 20, 50, 150, and 300 mm, 1000-fiber impregnated tows were tested.

Table 2

(a): AVE values and MSEs of both estimation methods under JP–IIC for BIIID at m = 40 and n = 40.

| | Para- | MLE | | MCMC-SELF | | MCMC-LiLF | | MCMC-DeLF | | MCMC-MELF | |
|--|--|---|---|--|--|--|---|--|---|---|--|
| Sch | motor | ΔVE | MSE | AVE | MSE | ΔVE | MSE | ΔVE | MSE | AVE | MSE |
| Juli, | meter | AVL. | WIJL | AV L | WIJL | TWL . | WIJL | AVL | IVISE | I V L | IVISE |
| 1 - 2 | 0 | | | | | | | | | | |
| $\kappa = 2$ | 0 | | | | | | | | | | |
| S1 | α | 0.5400 | 0.0100 | 0.5107 | 0.0008 | 0.5102 | 0.0008 | 0.5028 | 0.0007 | 0.5146 | 0.0009 |
| | B1 | 1.7166 | 0.5471 | 1.5213 | 0.0158 | 1.5141 | 0.0154 | 1.4838 | 0.0154 | 1.5402 | 0.0170 |
| | β. | 2.0562 | 0.4010 | 1 7977 | 0.0002 | 1 7770 | 0.0094 | 1 7205 | 0.0078 | 1 9120 | 0.0119 |
| | p_2 | 2.0303 | 0.4315 | 0.5120 | 0.0035 | 0.5150 | 0.0004 | 1.7555 | 0.0076 | 0.5164 | 0.0110 |
| S2 | α | 0.6997 | 0.4295 | 0.5120 | 0.0012 | 0.5153 | 0.0028 | 0.5044 | 0.0006 | 0.5164 | 0.0014 |
| | β_1 | 1.0724 | 0.2796 | 1.3866 | 0.0251 | 1.3803 | 0.0267 | 1.3497 | 0.0386 | 1.4054 | 0.0199 |
| | Ba | 1 9536 | 0 3725 | 1 8484 | 0.0257 | 1 8369 | 0.0240 | 1 7987 | 0.0347 | 1 8756 | 0 0484 |
| | P_2 | 1.5550 | 0.5725 | 1.0404 | 0.0257 | 1.0505 | 0.0240 | 1.7507 | 0.0347 | 1.0750 | 0.0404 |
| S3 | α | 0.3241 | 0.0795 | 0.3752 | 0.0284 | 0.3749 | 0.0285 | 0.3687 | 0.0299 | 0.3785 | 0.0278 |
| | β_1 | 2.0727 | 0.8031 | 1.6136 | 0.0395 | 1.6064 | 0.0376 | 1.5774 | 0.0321 | 1.6318 | 0.0442 |
| | Ba | 2 6880 | 1 7897 | 1 9889 | 0 1198 | 1 9781 | 0 1 1 3 7 | 1 9452 | 0 0993 | 2 0107 | 0 1314 |
| | P2 | 2.0000 | 0.2000 | 0.5100 | 0.0000 | 0.5104 | 0.0000 | 0.5017 | 0.0000 | 2.0107 | 0.1511 |
| S4 | α | 0.6949 | 0.2886 | 0.5109 | 0.0006 | 0.5104 | 0.0006 | 0.5017 | 0.0004 | 0.5151 | 0.0007 |
| | β_1 | 1.1633 | 0.1995 | 1.4133 | 0.0174 | 1.4443 | 0.6053 | 1.4257 | 0.9752 | 1.4298 | 0.0146 |
| | Ba | 1 7463 | 0 2019 | 1 8095 | 0.0225 | 1 8008 | 0.0213 | 1 7675 | 0.0259 | 1 8292 | 0 0242 |
| | P2 | 0.5775 | 0.02010 | 0.5001 | 0.0007 | 0.5070 | 0.0007 | 0.4000 | 0.0000 | 0.5100 | 0.0000 |
| S5 | α | 0.5775 | 0.0295 | 0.5081 | 0.0007 | 0.5076 | 0.0007 | 0.4996 | 0.0006 | 0.5123 | 0.0008 |
| | β_1 | 1.2864 | 0.1317 | 1.4327 | 0.0104 | 1.4266 | 0.0112 | 1.3985 | 0.0160 | 1.4499 | 0.0085 |
| | Ba | 1 9355 | 0 1993 | 1 8276 | 0.0166 | 1 8185 | 0.0151 | 1 7877 | 0.0116 | 1 8476 | 0.0204 |
| 1. 4 | 0 ^{P2} | 110500 | 011000 | 110270 | 0.0100 | 110100 | 010101 | | 0.0110 | 110 17 0 | 010201 |
| K = 4 | 0 | | | | | | | | | | |
| S1 | α | 0.5210 | 0.0047 | 0.5100 | 0.0009 | 0.5096 | 0.0008 | 0.5037 | 0.0008 | 0.5132 | 0.0009 |
| | B1 | 1.5964 | 0.1537 | 1.5185 | 0.0135 | 1.5110 | 0.0131 | 1.4790 | 0.0134 | 1.5384 | 0.0148 |
| | PI | 1 9701 | 0.2024 | 1 7661 | 0.0100 | 1 7569 | 0.0006 | 1 7220 | 0.0100 | 1 7072 | 0.0114 |
| | ρ_2 | 1.0701 | 0.2024 | 1.7001 | 0.0100 | 1.7508 | 0.0090 | 1.7256 | 0.0100 | 1./0/5 | 0.0114 |
| S2 | α | 0.5645 | 0.0120 | 0.5180 | 0.0011 | 0.5176 | 0.0011 | 0.5113 | 0.0009 | 0.5213 | 0.0013 |
| | β_1 | 1.1287 | 0.1967 | 1.3594 | 0.0292 | 1.3532 | 0.0309 | 1.3230 | 0.0405 | 1.3777 | 0.0247 |
| | R- | 2 2 2 7 2 | 0.4579 | 1 0056 | 0.0372 | 1 8065 | 0.03/1 | 1 8672 | 0.0261 | 1 02/9 | 0.0420 |
| | P_2 | 2.2313 | 0.4070 | 1.3030 | 0.0372 | 1.0503 | 0.0341 | 1.0075 | 0.0201 | 1.3240 | 0.0430 |
| \$3 | α | 0.3500 | 0.0525 | 0.3943 | 0.0201 | 0.3940 | 0.0201 | 0.3896 | 0.0208 | 0.3966 | 0.0197 |
| | β_1 | 1.6468 | 0.1808 | 1.5276 | 0.0187 | 1.5205 | 0.0181 | 1.4901 | 0.0175 | 1.5465 | 0.0203 |
| | ß | 2 20/2 | 0 7201 | 1 02/19 | 0.0700 | 1 0154 | 0.0761 | 1 9960 | 0.0660 | 1 0//2 | 0.0976 |
| | ρ_2 | 2.3045 | 0.7391 | 1.5240 | 0.0799 | 1.9134 | 0.0701 | 1.0000 | 0.0009 | 1.9442 | 0.0870 |
| S4 | α | 0.5954 | 0.0202 | 0.5204 | 0.0012 | 0.5199 | 0.0012 | 0.5132 | 0.0009 | 0.5240 | 0.0013 |
| | β_1 | 0.8691 | 0.4253 | 1.2547 | 0.0663 | 1.2492 | 0.0690 | 1.2193 | 0.0846 | 1.2724 | 0.0581 |
| | β. | 2 2215 | 0 3705 | 1 96/0 | 0.0508 | 1 0556 | 0.0560 | 1 0206 | 0.0457 | 1 0812 | 0.0677 |
| | p_2 | 2.2313 | 0.3703 | 1.5040 | 0.0330 | 1.5550 | 0.0500 | 1.5250 | 0.0437 | 1.5012 | 0.0077 |
| S5 | α | 0.5367 | 0.0066 | 0.5125 | 0.0008 | 0.5121 | 0.0008 | 0.5058 | 0.0006 | 0.5159 | 0.0009 |
| | β_1 | 1.2751 | 0.1103 | 1.4071 | 0.0175 | 1.4009 | 0.0185 | 1.3716 | 0.0250 | 1.4249 | 0.0147 |
| | Ba | 1 9731 | 0 1713 | 1 8324 | 0.0207 | 1 8241 | 0.0192 | 1 7961 | 0.0155 | 1 8506 | 0 0243 |
| (1.). A | P2 | and MCEs of | hoth optimoti | 1.0521 | nder ID IIC fe | DUID of m | 0.0152 | 1.7501 | 0.0155 | 1.0500 | 0.02 15 |
| | | | | | | | Ull and n | | | | |
| (D): A | VE values | | Dotti estimati | on memous u | nder jP-nc id | or BIIID at $m =$ | = 80 and $n = 3$ | 50 | | | |
| (b): A Sch. | Para- | MLE | Dotti estilliati | MCMC-SELF | nder jP-nc id | MCMC-LiLF | = 80 and $n = 3$ | MCMC-DeLF | | MCMC-MELF | |
| (b): A Sch. | Para- meter | MLE AVE | MSE | MCMC-SELF AVE | MSE | MCMC-LiLF AVE | = 80 and $n = 3$ MSE | MCMC-DeLF AVE | MSE | MCMC-MELF AVE | MSE |
| (D): A Sch. | Para- meter | MLE AVE | MSE | MCMC-SELF AVE | MSE | MCMC-LiLF AVE | = 80 and $n = 3$ MSE | MCMC-DeLF AVE | MSE | MCMC-MELF AVE | MSE |
| (D): A Sch. k = 4 | Para- meter | MLE AVE | MSE | MCMC-SELF AVE | MSE | MCMC-LiLF AVE | = 80 and <i>n</i> = 3 | MCMC-DeLF AVE | MSE | MCMC-MELF AVE | MSE |
| (D): A Sch. k = 4 | Para- meter | MLE AVE | MSE | MCMC-SELF AVE | MSE | $m = \frac{1}{MCMC-LiLF}$ $AVE = \frac{1}{MCMC-LiLF}$ | = 80 and $n = 3$ MSE | MCMC-DeLF AVE | MSE | MCMC-MELF AVE | MSE |
| (D): A Sch. k = 4 | Para- meter 0 α | MLE AVE 0.5306 | 0.0061 | 0.5152 | 0.0011 | MCMC-LiLF AVE 0.5148 | 80 and n = 3 MSE 0.0011 | MCMC-DeLF AVE 0.5091 | MSE | MCMC-MELF AVE | MSE |
| (D): A Sch. k = 4 | Para- meter 0 α β_1 | 0.5306 1.6106 | 0.0061 0.2116 | 0.5152 1.5374 | 0.0011 0.0237 | 0.5148 1.5304 $m = m = m = m = m = m = m = m = m = m =$ | 80 and n = 3 MSE 0.0011 0.0230 | MCMC-DeLF AVE 0.5091 1.5010 | MSE 0.0009 0.0222 | MCMC-MELF AVE 0.5182 1.5556 | MSE 0.0012 0.0255 |
| (b): A Sch. k = 4 | Para- meter 0 α β_1 β_2 | 0.5306 1.6106 1.8909 | 0.0061 0.2116 0.1637 | 0.5152 0.5374 1.7831 | 0.0011 0.0237 0.0107 | 0.5148 0.5304 1.7741 | 0.0011 0.0230 0.0100 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 | MSE 0.0009 0.0222 0.0094 | MCMC-MELF AVE 0.5182 1.5556 1.8034 | MSE 0.0012 0.0255 0.0126 |
| (b): A Sch. k = 4 | Para- meter 0 α β_1 β_2 α | 0.5306 1.6106 1.8909 0.528 | 0.0061 0.2116 0.1637 0.0214 | 0.5152 0.5152 1.5374 1.7831 0.5197 | 0.0011 0.0237 0.0107 0.010 | 0.5148 0.5148 1.5304 1.7741 0.5193 | 0.0011 0.0230 0.0100 0.010 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 | MSE 0.0009 0.0222 0.0094 0.0007 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 | MSE 0.0012 0.0255 0.0126 0.0011 |
| (D): A Sch. k = 4 S1 | Para- meter 0 α β_1 β_2 α | 0.5306 1.6106 1.8909 0.5928 | 0.0061 0.2116 0.1637 0.0214 | 0.5152 0.5152 1.5374 1.7831 0.5197 | 0.0011 0.0237 0.0107 0.0010 | 0.5148 0.5148 1.5304 1.7741 0.5193 | 0.0011 0.0230 0.0100 0.0010 | 0.5091 1.5010 1.7428 0.5126 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 | MSE 0.0012 0.0255 0.0126 0.0011 |
| (D): A Sch. k = 4 S1 | Para- meter 0 α β_1 β_2 α β_1 | 0.5306 1.6106 1.8909 0.5928 1.0886 | 0.0061 0.2116 0.1637 0.0214 0.2141 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 | 0.0011 0.0237 0.0107 0.0010 0.0406 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 | 0.0011 0.0230 0.0100 0.0010 0.0423 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 |
| (b): A Sch. k = 4 S1 | Para- meter 0 α β_1 β_2 α β_1 β_2 α β_1 β_2 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2141 0.2000 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 | 0.0011 0.0230 0.0100 0.0100 0.0423 0.0399 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 |
| (b): A Sch. $k = 4$ s_1 s_2 | Para- meter 0 α β_1 β_2 α β_1 β_2 α β_1 β_2 α | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 | 0.0011 0.0230 0.0100 0.0100 0.0423 0.0399 0.0486 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 |
| (b): A Sch. k = 4 S1 S2 S3 | Para- meter 0 α β_1 β_2 α β_1 β_2 α β_1 β_2 α | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1025 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.0582 | MCMC-SELF AVE 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7961 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.720 | 0.0011 0.0230 0.0100 0.0100 0.0423 0.0399 0.0486 0.1590 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1452 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.01724 |
| (b): A Sch. k = 4 S1 S2 S3 | Para- meter 0 α β_1 β_2 α β_1 β_2 α β_1 β_2 α β_1 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 | 0.0011 0.0230 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.1734 |
| (b): A Sch. k = 4 S1 S2 S3 | Para- meter 0 α β_1 β_2 α β_1 β_2 α β_1 β_2 α β_1 β_2 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.1635 0.3639 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.1734 0.3806 |
| (b): A Sch. k = 4 S1 S2 S3 S4 | Para- meter $0 \qquad \alpha \qquad \beta_1 \qquad \beta_2 \alpha \qquad \beta_1 \beta_2 \alpha \beta_1 \beta_2 \beta_2 \beta_2 \beta_3 \beta_3 \beta_4 \beta_5 \beta_5 $ | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0585 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3223 0.0012 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.1734 0.3806 0.0010 |
| (b): A Sch. k = 4 s_1 s_2 s_3 s_4 | Para- meter 0 α β_1 β_2 β_2 β_3 β_1 β_2 β_3 | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0585 0.0280 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.323 0.0012 0.0386 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.1734 0.3806 0.0010 |
| (b): A Sch. k = 4 S1 S2 S3 S3 | Para- meter 0 α β_1 β_2 β_2 α β_1 β_2 β_2 β_3 β_3 β_4 β_3 β_3 β_4 β_3 β_4 β_4 β_3 β_4 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.0208 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1135 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.972 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.9205 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0585 0.0280 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.955 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0322 0.0498 0.1453 0.323 0.0012 0.0386 0.0222 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.9552 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.0010 0.0201 0.0201 |
| (b): A Sch. k = 4 s_1 s_2 s_3 s_4 | Para- meter β_1 β_2 α β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_3 β | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 | MCMC-SELF AVE 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 | .00011 0.0230 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0280 0.0259 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0480 0.1734 0.3806 0.0010 0.0201 0.02217 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 | Para- meter 0 α β_1 β_2 α β_3 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0585 0.0280 0.0259 0.0008 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.0480 0.0480 0.0480 0.0480 0.0480 0.0211 0.0201 0.0209 |
| (b): A Sch. k = 4 s_1 s_2 s_3 s_4 s_5 | Para- meter 0 α β_1 β_2 α β_2 α β_1 β_2 α β_2 β_3 $\beta_$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 | MCMC-LiLF AVE 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0585 0.0280 0.0259 0.0008 0.0233 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0478 0.1734 0.3806 0.0010 0.0201 0.0201 0.0207 0.0009 0.0192 |
| (b): A Sch. k = 4 s_1 s_2 s_3 s_4 s_5 | Para- meter β_1 β_2 α β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_3 β_1 β_3 β_1 β_3 β_3 β_1 β_3 | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.2001 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1005 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.9457 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0008 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.9298 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0585 0.0280 0.0259 0.0008 0.0233 0.0233 0.0233 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.9150 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.9000 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0480 0.1734 0.3806 0.0010 0.0201 0.0201 0.0227 0.0009 0.0192 0.0055 |
| (b): A Sch. k = 4 s_1 s_2 s_3 s_4 s_5 | Para- meter 0 α β_1 β_2 β_2 β_2 β_2 β_3 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0222 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.1586 0.0585 0.0280 0.0259 0.0008 0.0233 0.0208 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.3538 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.00170 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.010 0.0201 0.0201 0.0027 0.0009 0.0192 0.0255 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 | Para- meter α β_1 β_2 α β_2 β_2 α β_1 β_2 β_2 β_2 β_3 $\beta_$ | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0222 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0585 0.0280 0.0259 0.0008 0.0233 0.0208 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0478 0.1734 0.3806 0.0010 0.0201 0.0201 0.0207 0.0099 0.0192 0.0255 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 | Para- meter β_1 β_2 α β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_3 β_3 β_4 β_3 β_4 β_3 β_4 | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0008 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 | .00011 0.0230 0.0100 0.0423 0.0399 0.0486 0.3536 0.0259 0.0008 0.0233 0.0208 0.0008 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0480 0.1734 0.3806 0.0010 0.0201 0.0201 0.0201 0.0225 0.0009 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 | Para- meter 0 α β_1 β_2 β_2 β_3 $\beta_$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5580 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0211 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5102 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 0.0222 0.00222 0.0008 0.0156 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5122 | 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0585 0.0280 0.0259 0.0008 0.0259 0.0008 0.0233 0.0208 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5246 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.010 0.0201 0.0201 0.0201 0.0201 0.0255 0.0009 0.0156 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 | Para- meter β_1 β_2 α β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_3 β_1 β_3 | MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0222 0.0008 0.0156 | 0.5148 1.5304 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 | .00011 0.0230 0.0100 0.0100 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.0208 0.0008 0.0152 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0478 0.0478 0.0478 0.0470 0.0201 0.0201 0.0201 0.0277 0.0009 0.0192 0.0255 0.0009 0.0166 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 | $\begin{array}{c} \text{Para-} \\ \text{meter} \\ \hline \\ 0 \\ \alpha \\ \beta_1 \\ \beta_2 \\ \beta_1 \\ \beta_2 \\ \alpha \\ \beta_1 \\ \beta_2 \\ \beta_1 \\ \beta_1 \\ \beta_1 \\ \beta_2 \\ \beta_1 \\ \beta$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.00222 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 | .00011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0259 0.0008 0.0233 0.0233 0.02159 0.0008 0.0152 0.0152 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.00170 0.0008 0.0149 0.0120 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.0480 0.0480 0.0480 0.0201 0.0201 0.0201 0.0009 0.0192 0.0009 0.0166 0.0137 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S2 S3 S4 S5 S1 S2 S2 S2 S3 S3 S4 S5 S1 S2 S2 S3 S3 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 | Para- meter α β_1 β_2 α β_2 α β_1 β_2 α β_2 α β_1 β_2 α β_2 α β_1 β_2 β_2 α β_1 β_2 β_2 β_3 β_3 β_3 β_3 β_3 β_3 β_4 β_3 β_3 β_4 β_3 β_4 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 0.0262 0.0008 0.0222 0.0008 0.0222 0.0008 0.0156 0.0126 0.0010 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 | .00011 0.0230 0.0100 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0585 0.0280 0.0259 0.0008 0.0208 0.0008 0.0152 0.0121 0.0010 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.51187 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.010 0.0201 0.0201 0.0201 0.0205 0.0192 0.0192 0.0166 0.0137 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S2 S2 S2 | Para- meter β_1 β_2 α β_1 β_2 β_3 | MLE MLE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 | 0.51152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2027 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0008 0.0222 0.0008 0.0156 0.0126 0.0126 0.0126 | 0.5148 1.5304 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 | .00011 0.0230 0.0100 0.0100 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.0233 0.0208 0.0152 0.0121 0.0010 0.052% | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2650 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0009 0.0009 0.0009 0.0009 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0478 0.0478 0.0478 0.0201 0.0201 0.0201 0.0205 0.0009 0.0166 0.0137 0.0011 0.025 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S2 | Para- meter 0 α β_1 β_2 β_1 β_2 β_1 β_2 β_3 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 1.1192 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.1698 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.2927 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.00222 0.0008 0.0156 0.0126 0.0126 0.0010 0.0511 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.5145 1.2884 | 0.0011 0.0230 0.0100 0.0010 0.0423 0.0399 0.0486 0.1586 0.1586 0.0585 0.0280 0.0259 0.0008 0.0233 0.0208 0.0008 0.0152 0.0121 0.0010 0.0528 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0077 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.009 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.010 0.0201 0.0207 0.0009 0.0192 0.0255 0.0009 0.0166 0.01137 0.0011 0.0213 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S2 | Para- meter α β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_3 $\beta_$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 0.0222 0.0008 0.0222 0.0008 0.0156 0.0126 0.0010 0.05511 0.0593 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 | 80 and n = 3 MSE 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.0259 0.0008 0.0259 0.0008 0.0233 0.0208 0.0008 0.0152 0.0121 0.0010 0.0528 0.0563 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0009 0.0027 0.0479 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.3062 1.9727 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.1734 0.3806 0.0010 0.0201 0.0277 0.0099 0.01255 0.0009 0.0166 0.0137 0.00137 0.00458 0.0458 0.0455 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 | Para- meter 0 α β_1 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_3 β_3 β_1 β_2 β_3 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.1226 0.0222 0.0008 0.0222 0.0008 0.0156 0.0126 0.0126 0.0126 0.0126 0.0010 0.0511 0.0593 0.0369 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 | and n = 3 MSE 0.0011 0.0230 0.0100 0.0100 0.0399 0.0486 0.1586 0.0259 0.0008 0.0233 0.0208 0.0008 0.0152 0.0121 0.0012 0.0528 0.0563 0.0369 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0478 0.0478 0.0010 0.0201 0.0201 0.0277 0.0009 0.0192 0.0255 0.0009 0.0166 0.0137 0.0011 0.0458 0.0366 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S1 S2 S3 S2 S3 | Para- meter 0 α β_1 β_2 β_3 β_3 β_3 β_4 β_3 β_4 β_3 β_4 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.0789 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0022 0.0008 0.0222 0.0008 0.0222 0.0008 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0126 0.0010 0.0511 0.0593 0.0369 0.0415 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.568 | .00011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0585 0.0280 0.0259 0.0008 0.0259 0.0008 0.0152 0.0152 0.0152 0.0152 0.0152 0.0563 0.0563 0.0404 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0120 0.0009 0.0627 0.0479 0.0375 0.0375 0.0375 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.3866 1.3866 1.3866 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.0010 0.0201 0.0227 0.0009 0.0192 0.0255 0.0009 0.0137 0.0011 0.0213 0.0011 0.0255 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S5 S2 S3 S2 S3 | Para- meter β_1 β_2 α β_1 β_2 β_1 β_2 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_3 β_1 β_3 β_1 β_3 β_1 β_3 β_3 β_3 β_3 β_1 β_3 β_1 β_3 β_1 β_3 | MLE MLE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.255 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.9586 0.3472 1.5723 0.5723 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0008 0.0222 0.0008 0.0126 0.0126 0.0126 0.0010 0.00126 0.0010 0.00126 0.0010 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 | | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0377 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.3806 0.0010 0.0201 0.0207 0.0009 0.0125 0.0009 0.0166 0.0137 0.0011 0.0458 0.0657 0.0366 0.0440 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S3 | Para- meter 0 α β_1 β_2 α β_3 $\beta_$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.00222 0.0008 0.0156 0.0126 0.0126 0.0016 0.0126 0.0126 0.0015 0.0126 0.0015 0.0126 0.0015 0.0126 0.0015 0.0011 0.0593 0.0369 0.0415 0.2011 | G SiliD at m = MCMC-LiLF AVE 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 | and n = 3 MSE 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.0259 0.0008 0.0233 0.0208 0.0008 0.0152 0.0121 0.0010 0.0528 0.0563 0.0369 0.0404 0.1954 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0377 0.1825 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 2.0724 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.0201 0.0201 0.0207 0.0009 0.0162 0.0009 0.0166 0.0137 0.0011 0.0458 0.0657 0.0366 0.0440 0.0440 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S5 S2 S3 S3 S3 S4 | $\begin{array}{c} \text{Paraameter} \\ \text{meter} \\ \hline \\ 0 \\ \alpha \\ \beta_1 \\ \beta_2 \\ \alpha \\ $ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.5784 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 0.0104 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5150 1.3823 1.8457 0.5086 1.5192 1.7687 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 0.1226 0.0022 0.0008 0.0222 0.0008 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0015 0.0010 0.0511 0.0593 0.0369 0.0415 0.2011 0.0016 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 | 80 and n = 3 MSE 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0088 0.0259 0.0008 0.0152 0.0110 0.0528 0.0563 0.0369 0.0404 0.1954 0.0015 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0377 0.1825 0.0013 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.3966 1.8606 0.5109 1.5346 1.7856 0.51187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.0010 0.0201 0.0201 0.0201 0.0205 0.0009 0.0166 0.0137 0.0011 0.0458 0.0657 0.0366 0.0440 0.2110 0.0017 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S1 S2 S3 S4 S1 S2 S3 S3 S4 S2 S3 S3 S3 S4 S3 S3 S4 S5 S2 S3 S3 S3 S3 S5 S4 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 S5 | Para- meter α β_1 β_2 α β_3 | MLE MLE AVE 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.5784 0.8685 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 0.0104 0.4109 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 1.1487 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0222 0.0008 0.0126 0.0126 0.0126 0.0010 0.0511 0.0593 0.0369 0.0415 0.2011 0.0016 0.016 0.018 | 0.5148 0.5148 1.5304 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 1.450 | and n = 3 MSE 0.0011 0.0230 0.0100 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.02233 0.0008 0.0152 0.0121 0.0010 0.0528 0.0369 0.0404 0.150 0.015 0.1309 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 1.1231 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0377 0.1825 0.0013 0.1468 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.3966 1.38606 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 1.1615 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.3806 0.0010 0.0201 0.0207 0.0009 0.0192 0.0025 0.0009 0.0166 0.0117 0.0011 0.0458 0.0657 0.0366 0.0440 0.0117 0.0017 0.0117 0.0017 0.0117 0.0017 0.017 0.0012 0.0255 0.0009 0.0166 0.0440 0.0117 0.0012 0.0012 0.0012 0.0255 0.0126 0.0217 0.0012 0.0255 0.0126 0.0255 0.0255 0.0126 0.0255 0.0255 0.009 0.0166 0.0211 0.0257 0.0011 0.0255 0.0011 0.0255 0.0012 0.0255 0.0012 0.0012 0.0255 0.0010 0.0255 0.0010 0.0255 0.0010 0.0255 0.0010 0.0255 0.0010 0.0255 0.0010 0.0255 0.0010 0.0255 0.0011 0.0255 0.0011 0.0255 0.0012 0.0057 0.00366 0.0137 0.0012 0.0255 0.0012 0.0057 0.00366 0.0147 0.0017 0.0010 0.0157 0.0010 0.0157 0.0010 0.0157 0.0010 0.0157 0.0010 0.0157 0.0010 0.0157 0.0010 0.0157 0.0010 0.0157 0.0110 0.0177 0.0110 0.0177 0.0110 0.0177 0.0110 0.0177 0.0107 0.0107 0.0110 0.0177 0.0107 0.0110 0.0177 0.0117 0.0177 0.0177 0.0107 0.0177 0 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S3 S3 S4 S2 S3 S3 S3 S3 S3 | Para- meter 0 α β_1 β_2 β_1 β_2 β_1 β_3 β_3 β_4 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.5784 0.5784 0.8685 2.3192 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 0.0104 0.4109 0.222 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 1.1487 2.2041 | 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.00222 0.0008 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0126 0.0127 0.0107 0.0222 | 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 1.1450 2.9002 | 0.0011 0.0230 0.0100 0.0100 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.0233 0.0208 0.0008 0.0152 0.0121 0.0010 0.0528 0.0369 0.0404 0.1954 0.0015 0.1309 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 1.1231 2.2020 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0375 0.0375 0.0377 0.1825 0.0013 0.1468 0.0003 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 1.1615 2.0752 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.010 0.0201 0.0275 0.0009 0.0192 0.0255 0.0009 0.0166 0.0137 0.0011 0.0458 0.057 0.0366 0.0440 0.0440 0.04410 0.04458 0.057 0.0366 0.0440 0.04410 0.04458 0.057 0.0366 0.0440 0.04410 0.04458 0.057 0.0366 0.04410 0.04458 0.0457 0.0366 0.04410 0.04458 0.0457 0.0366 0.0440 0.04458 0.0457 0.0457 0.0457 0.0458 0.0457 0.0457 0.0458 0.0440 0.04458 0.0440 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.04458 0.0457 0.0458 0.0457 0.0458 0.0457 0.0458 0.0457 0.0458 0.0457 0.0458 0.0448 0.0457 0.0448 0.0457 0.0448 0.0457 0.0448 0.0457 0.0448 0.0457 0.0448 0.0488 0.0488 0.0448 0.0488 |
| (b): A Sch. $k = 4$ S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S1 S2 S3 S3 S4 | $\begin{array}{c} \text{Paraa-}\\ \text{meter} \\ \hline \\ 0 \\ \alpha \\ \beta_1 \\ \beta_2 \\ \beta_1 \\ \beta_2 \\ \beta_1 \\ \beta_2 \\ \beta_2 \\ \beta_2 \\ \beta_1 \\ \beta_2$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.8685 2.2189 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.0103 0.1005 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 0.0104 0.4109 0.2929 | 0.5152 1.5374 1.7831 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8457 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 1.1487 2.0441 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 0.1226 0.0008 0.0222 0.0008 0.0222 0.0008 0.0156 0.0156 0.0156 0.0156 0.0156 0.0156 0.0126 0.0011 0.0593 0.0369 0.0415 0.2011 0.0593 0.0369 0.0415 0.2011 0.0593 0.0369 0.0415 0.2011 0.0593 0.0369 0.0415 0.2011 0.0593 0.0369 0.0415 0.2011 0.0593 0.0016 0.1284 0.1046 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 1.1450 2.0380 | a 80 and n = 3 MSE 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.0223 0.0208 0.0008 0.0152 0.0121 0.0010 0.0528 0.0563 0.0369 0.4044 0.1954 0.0115 0.1309 0.1008 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 1.1231 2.0200 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0375 0.0377 0.1825 0.0013 0.1468 0.0906 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.3868 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.51187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 1.1615 2.0562 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.1734 0.3806 0.010 0.0201 0.0201 0.0201 0.0277 0.0009 0.0166 0.0137 0.0011 0.0458 0.0657 0.0366 0.0440 0.2110 0.0017 0.1197 0.1120 |
| (b): A Sch. k = 4 S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S2 S3 S4 S2 S3 S4 S2 S3 S4 S5 S4 S5 S5 | Para- meter 0 α β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 β_3 β_3 β_3 β_4 β_1 β_3 β_3 β_4 β_3 β_4 β_3 β_4 β | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.5784 0.5784 0.5784 0.5784 0.5784 0.5285 2.2189 0.5245 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 0.0104 0.4109 0.2929 0.0035 | 0.5152 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 1.1487 2.0441 0.5121 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0262 0.0008 0.0222 0.0222 0.0008 0.0126 0.0222 0.0008 0.0126 0.0126 0.0010 0.0511 0.0511 0.0593 0.0369 0.0415 0.2011 0.0016 0.1284 0.1046 0.0009 | 0.5148 1.5304 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 1.1450 2.0380 0.5118 | a 80 and n = 3 MSE 0.0011 0.0230 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0280 0.0259 0.0008 0.0233 0.0208 0.0208 0.0152 0.0121 0.0010 0.0528 0.0369 0.0404 0.1954 0.0015 0.1309 0.1008 0.0009 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 1.1231 2.0200 0.5073 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0120 0.0009 0.0627 0.0479 0.0375 0.0377 0.1825 0.0013 0.1468 0.0906 0.0008 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 1.1615 2.0562 0.5145 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0478 0.0478 0.3806 0.0010 0.0201 0.0201 0.0207 0.0009 0.0192 0.0012 0.0009 0.0166 0.0137 0.0011 0.0458 0.0657 0.0366 0.0440 0.2110 0.0017 0.1120 0.0010 |
| (b): A Sch. $k = 4$ S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S5 S4 S5 S4 S5 | Para- meter 0 α β_1 β_2 β_3 β_3 β_4 β_1 β_2 β_3 β_3 β_4 $\beta_$ | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.5784 0.8685 2.2189 0.5245 1.2528 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.1028 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.7823 0.0104 0.4109 0.2929 0.0035 0.0879 | 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 1.1487 2.0441 0.5121 1.3579 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0022 0.0022 0.0022 0.0022 0.0008 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.00126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0126 0.0010 0.0222 0.0008 0.0126 0.0010 0.02511 0.0016 0.02011 0.0016 0.0290 0.0016 0.0290 0.0016 0.0290 0.0016 0.0290 0.0016 0.0290 0.00290 0.0290 0.00290 0.0290 | of BillD at m = MCMC-LiLF AVE 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 1.1450 2.0380 0.5118 1.3535 | and n = 3 MSE 0.0011 0.0230 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0259 0.0008 0.0223 0.0208 0.0008 0.0152 0.0101 0.0528 0.0369 0.0404 0.152 0.1309 0.0404 0.152 0.1309 0.0015 0.1309 0.1008 0.0015 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 1.1231 2.0200 0.5073 1.3320 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0120 0.0009 0.0627 0.0479 0.0375 0.0375 0.0375 0.0377 0.1825 0.0013 0.1468 0.0906 0.0008 0.0008 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.8606 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 1.1615 2.0562 0.5145 1.3709 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0480 0.01734 0.3806 0.0010 0.0201 0.0277 0.0009 0.0137 0.0011 0.0137 0.0011 0.0458 0.0657 0.0366 0.0440 0.2110 0.0117 0.1197 0.1197 0.1197 0.025 |
| (b): A Sch. $k = 4$ S1 S2 S3 S4 S5 k = 8 S1 S2 S3 S4 S1 S2 S3 S4 S5 S4 S5 | Para- meter 0 α β_1 β_2 β_1 β_2 β_1 β_3 β_1 β_2 β_1 β_3 β_1 β_3 β_1 β_3 β_3 β_1 β_3 β_3 β_1 β_3 β_3 β_3 β_3 β_4 β_1 β_3 β_3 β_4 β_3 β_4 | 0.5306 1.6106 1.8909 0.5928 1.0886 2.0169 0.2796 2.1925 2.8350 0.6055 1.2113 1.8028 0.5468 1.2562 1.9001 0.5119 1.5589 1.8190 0.5372 1.1192 2.1577 0.3186 1.6737 2.3554 0.5245 1.2528 1.962 | MSE 0.0061 0.2116 0.1637 0.0214 0.2141 0.2000 0.0868 0.9583 2.1002 0.0451 0.1439 0.1125 0.0103 0.1005 0.0025 0.0811 0.0844 0.0043 0.1698 0.2460 0.0604 0.1789 0.2460 0.0604 0.1789 0.7823 0.0104 0.4109 0.2929 0.0035 0.0879 0.1104 | MCMC-SELF AVE 0.5152 1.5374 1.7831 0.5197 1.3232 1.9079 0.3199 1.7861 2.2034 0.5151 1.4272 1.8372 0.5120 1.3823 1.8457 0.5086 1.5192 1.7687 0.5163 1.2927 1.9586 0.3472 1.5723 2.0577 0.5290 1.1487 2.0441 0.5121 1.3579 1.9817 | MSE 0.0011 0.0237 0.0107 0.0010 0.0406 0.0424 0.0485 0.1635 0.3639 0.0008 0.1226 0.0008 0.1226 0.0008 0.0222 0.0008 0.0222 0.0008 0.0156 0.0156 0.0156 0.0156 0.0126 0.0010 0.0511 0.0593 0.0369 0.0415 0.2011 0.0016 0.1284 0.1046 0.0009 0.0290 0.0214 | 0.5148 0.5148 1.5304 1.7741 0.5193 1.3181 1.9005 0.3197 1.7789 2.1939 0.5269 1.3723 1.8296 0.5115 1.3774 1.8388 0.5084 1.5133 1.7612 0.5160 1.2884 1.9517 0.3470 1.5668 2.0501 0.5287 1.1450 2.0380 0.5118 1.3535 1.9754 | 80 and n = 3 MSE 0.0011 0.0230 0.0100 0.0100 0.0423 0.0399 0.0486 0.1586 0.3536 0.0585 0.0280 0.0259 0.0008 0.0152 0.0150 0.1309 0.1008 0.0009 0.0302 0.0302 | MCMC-DeLF AVE 0.5091 1.5010 1.7428 0.5126 1.2924 1.8769 0.3157 1.7540 2.1695 0.5062 1.3476 1.8055 0.5053 1.3538 1.8160 0.5042 1.4884 1.7349 0.5117 1.2659 1.9303 0.3443 1.5444 2.0284 0.5237 1.1231 2.0200 0.5073 1.3200 1.9572 | MSE 0.0009 0.0222 0.0094 0.0007 0.0521 0.0332 0.0498 0.1453 0.3323 0.0012 0.0386 0.0292 0.0007 0.0294 0.0170 0.0008 0.0149 0.0170 0.0009 0.0120 0.0009 0.0479 0.0375 0.0375 0.0377 0.1825 0.0013 0.1468 0.0906 0.0008 0.0368 0.0368 0.0368 0.0327 | MCMC-MELF AVE 0.5182 1.5556 1.8034 0.5233 1.3386 1.9235 0.3219 1.8021 2.2204 0.5188 1.3950 1.8568 0.5153 1.3966 1.3668 0.5109 1.5346 1.7856 0.5187 1.3062 1.9727 0.3486 1.5862 2.0724 0.5317 1.1615 2.0562 0.5145 1.3709 | MSE 0.0012 0.0255 0.0126 0.0011 0.0356 0.0478 0.0480 0.1734 0.3806 0.0010 0.0201 0.0277 0.0009 0.0166 0.0137 0.0016 0.0458 0.0657 0.0366 0.0440 0.2110 0.0458 0.0657 0.0366 0.0440 0.2110 0.01197 0.1120 0.0010 0.0256 0.0010 0.0256 0.0010 0.0256 0.0400 0.0125 0.0000 0.0125 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.00000 0.000000 0.000000 0.00000000 |

(continued on next page)

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Table 2 (continued)

| $ \begin{array}{ c c c c c c c c c c c c c c c c c c c$ | Sch. | Para- | MLE | | MCMC-SELF | | MCMC-LiLF | | MCMC-DeLF | | MCMC-MELF | |
|---|------------|-----------|-------------|-----------------|--------------|------------------|----------------|---------------|------------|--------|-----------|--------|
| | | meter | AVE | MSE | AVE | MSE | AVE | MSE | AVE | MSE | AVE | MSE |
| $ \begin{array}{c} \begin{tabular}{ c c c c c c c c c c c c c c c c c c c$ | (c) · A | VE values | and MSEs of | both estimation | on methods u | oder IP IIC fo | r BIIID at m _ | 120 and n - | 120 | | | |
| $ \begin{array}{c c c c c c c c c c c c c c c c c c c $ | Sch | Para_ | MI F | both cothilati | MCMC-SELE | lider ji –lie io | MCMC-LilF | 120 and $n =$ | MCMC_Del F | | MCMC-MFLF | |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | Sen. | meter | AVE | MSE | AVE | MSE | AVE | MSE | AVE | MSE | AVE | MSE |
| $ \begin{array}{c c c c c c c c c c c c c c c c c c c $ | | | | | | | | moe | | | | |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | k = 8 | 0 | | | | | | | | | | |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | S 1 | α | 0.5131 | 0.0023 | 0.5086 | 0.0009 | 0.5083 | 0.0009 | 0.5041 | 0.0008 | 0.5109 | 0.0010 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_1 | 1.5375 | 0.0678 | 1.5165 | 0.0150 | 1.5110 | 0.0147 | 1.4875 | 0.0146 | 1.5311 | 0.0158 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 1.8249 | 0.0826 | 1.7817 | 0.0160 | 1.7737 | 0.0152 | 1.7458 | 0.0143 | 1.7998 | 0.0178 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | S2 | α | 0.5729 | 0.0093 | 0.5262 | 0.0015 | 0.5258 | 0.0015 | 0.5208 | 0.0013 | 0.5289 | 0.0017 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_1 | 0.9688 | 0.2975 | 1.2011 | 0.0951 | 1.1976 | 0.0972 | 1.1777 | 0.1095 | 1.2128 | 0.0883 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 2.1801 | 0.2559 | 2.0447 | 0.1075 | 2.0383 | 0.1036 | 2.0197 | 0.0930 | 2.0573 | 0.1153 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | \$3 | α | 0.2740 | 0.0852 | 0.2925 | 0.0620 | 0.2924 | 0.0620 | 0.2900 | 0.0626 | 0.2937 | 0.0617 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_1 | 2.0391 | 0.6401 | 1.7934 | 0.1869 | 1.7879 | 0.1829 | 1.7689 | 0.1723 | 1.8056 | 0.1946 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 2.6805 | 1.5748 | 2.3152 | 0.5640 | 2.3069 | 0.5525 | 2.2868 | 0.5304 | 2.3294 | 0.5814 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | S4 | α | 0.5546 | 0.0089 | 0.5158 | 0.0011 | 0.5154 | 0.0011 | 0.5097 | 0.0009 | 0.5188 | 0.0012 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_1 | 1.1414 | 0.1468 | 1.2861 | 0.0517 | 1.2827 | 0.0531 | 1.2652 | 0.0609 | 1.2965 | 0.0474 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 1.9550 | 0.0832 | 1.9298 | 0.0463 | 1.9247 | 0.0443 | 1.9084 | 0.0389 | 1.9406 | 0.0504 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | S 5 | α | 0.5272 | 0.0038 | 0.5103 | 0.0008 | 0.5100 | 0.0008 | 0.5052 | 0.0007 | 0.5129 | 0.0009 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_1 | 1.2459 | 0.0873 | 1.3459 | 0.0315 | 1.3422 | 0.0326 | 1.3243 | 0.0384 | 1.3567 | 0.0284 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | | β_2 | 1.9570 | 0.0790 | 1.9036 | 0.0359 | 1.8980 | 0.0341 | 1.8798 | 0.0290 | 1.9155 | 0.0399 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | k = 1 | 20 | | | | | | | | | | |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | S1 | α | 0.5089 | 0.0015 | 0.5096 | 0.0031 | 0.5090 | 0.0029 | 0.5024 | 0.0025 | 0.5138 | 0.0040 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | | β_1 | 1.5242 | 0.0418 | 1.7048 | 6.5233 | 1.6436 | 2.6488 | 1.5129 | 1.1320 | 1.5571 | 0.4332 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 1.7906 | 0.0558 | 1.7846 | 1.1381 | 1.7818 | 1.1264 | 1.7497 | 2.1769 | 1.9310 | 5.1967 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | S2 | α | 0.5384 | 0.0035 | 0.5204 | 0.0012 | 0.5201 | 0.0012 | 0.5166 | 0.0010 | 0.5223 | 0.0013 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | | β_1 | 1.1034 | 0.1721 | 1.2572 | 0.0648 | 1.2541 | 0.0663 | 1.2373 | 0.0748 | 1.2672 | 0.0601 |
| $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 2.1590 | 0.2198 | 2.0054 | 0.0815 | 1.9996 | 0.0784 | 1.9824 | 0.0698 | 2.0170 | 0.0877 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | S3 | α | 0.2942 | 0.0671 | 0.3084 | 0.0548 | 0.3083 | 0.0548 | 0.3048 | 0.0561 | 0.3102 | 0.0541 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | | β_1 | 1.7362 | 0.1967 | 2.4507 | 2.3879 | 2.2881 | 1.7116 | 1.7015 | 9.0081 | 2.2377 | 1.4598 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 2.4258 | 0.8419 | 2.8923 | 2.8592 | 2.8627 | 1.7030 | 2.4717 | 1.9209 | 2.8544 | 2.2357 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | S 4 | α | 0.5624 | 0.0068 | 0.5272 | 0.0016 | 0.5269 | 0.0016 | 0.5229 | 0.0014 | 0.5293 | 0.0017 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | | β_1 | 0.8672 | 0.4080 | 1.1049 | 0.1598 | 1.1023 | 0.1618 | 1.0861 | 0.1749 | 1.1143 | 0.1525 |
| $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | | β_2 | 2.2228 | 0.2629 | 2.0952 | 0.1342 | 2.0902 | 0.1306 | 2.0763 | 0.1212 | 2.1046 | 0.1409 |
| $ \begin{array}{c ccccccccccccccccccccccccccccccccccc$ | S5 | α | 0.5167 | 0.0020 | 0.5099 | 0.0008 | 0.5096 | 0.0008 | 0.5061 | 0.0007 | 0.5118 | 0.0009 |
| β_2 1.9837 0.0928 1.9040 0.0384 1.8990 0.0367 1.8829 0.0321 1.9145 0.0419 | | β_1 | 1.2611 | 0.0739 | 1.3437 | 0.0308 | 1.3406 | 0.0318 | 1.3249 | 0.0370 | 1.3532 | 0.0280 |
| | | β_2 | 1.9837 | 0.0928 | 1.9040 | 0.0384 | 1.8990 | 0.0367 | 1.8829 | 0.0321 | 1.9145 | 0.0419 |

To illustrate the findings of the paper, we will be considering the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gage length, with sample sizes, m = 69 and n = 63, respectively, as seen in Table 4.

We fit BIID to each sample using the Kolmogorov- Smirnov (K-S) test statistics, where the K-S distance between the empirical and the fitted for the first population (W) is 0.1219 and its p-value is 0.2349 where $\hat{\alpha} = 4.2441$ and $\hat{\beta}_1 = 26.4297$. Also, for the second population (Z) the K-S distance is 0.1384 and its p-value is 0.1266 where $\hat{\alpha} = 5.6704$ and $\hat{\beta}_2 = 19.0345$ which indicate that this distribution can be considered an adequate model for the given two data sets (W and Z).

From the original data, one can generate, e.g., three JP–IIC schemes with different numbers of stages k = (20, 40, 60) and the removed items R_i are assumed as given in the following table (Table 5).

We compute the MLEs of α , β_1 , and β_2 together with the associated 95% Asy-CI estimates. The BE is computed using the MH algorithm, where $a_1 = b_1 = a_2 = b_2 = c = d = 0$. While generating samples from the posterior distribution utilizing the MH algorithm, initial values of $(\alpha, \beta_1, \beta_2)$ are considered as $(\alpha^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}) = (\hat{\alpha}, \hat{\beta}_1, \hat{\beta}_2)$, where $\hat{\alpha}, \hat{\beta}_1$ and $\hat{\beta}_2$ are the MLEs of the parameters α, β_1 and β_2 respectively. Thus, we considered the VCM S_{ϕ} of $(\ln(\hat{\alpha}), \ln(\hat{\beta}_1), \ln(\hat{\beta}_2))$. Eventually, 2000 burn-in samples are terminated from the entire 10,000 samples generated by the posterior density, and the technique is adopted to produce BEs: SELF, LiLF at h = 0.5, MELF, and DeLF. Also, HPD interval estimates have been computed according to the technique of Chen and Shao [35].

All values of MLEs and BEs utilizing MCMC under different proposed loss functions employing the MH algorithm, as well as the related standard error (St.Er) are illustrated in Table 6. Also, Asy-CI and associated HPD intervals are calculated based on the SELF in Table 7. Here, LB denotes the lower bound, and UB denotes the upper bound.

Conclusion

In this study, we used a JP-IIC approach to analyze two samples for the Burr IIID. When estimating model parameters, both the Bayesian and maximum likelihood techniques are taken into consideration. The asymptotic confidence intervals are then calculated using the observed information matrix. Confidence intervals can also be obtained using the Bootstrap-P and

Table 3

| (a): AILs and CP(%) values of | all methods under JP–IIC | for BIIID at $m = 40$ and $n = 40$. |
|-------------------------------|--------------------------|--------------------------------------|
|-------------------------------|--------------------------|--------------------------------------|

| | Para- | Asy-CI | | Boot-P | | Boot-T | | HPD | |
|--|--|---|---|--|---|---|---|--|---|
| Sch. | meter | AIL | СР | AIL | СР | AIL | СР | AIL | СР |
| 1, 20 | | | | | | | | | |
| $\kappa = 20$ | | 0 2726 | 07.60 | 0.2062 | 08.20 | 0.4120 | 02.40 | 0 1052 | 07.90 |
| S1 | β | 1 7059 | 97.00 | 0.5902 | 96.20 | 0.4129 | 92.40 | 0.1052 | 97.80 |
| | ρ_1 β_2 | 2 2839 | 96.40 | 2.7255 | 97.00 | 2.2100 | 88.40 | 0.3400 | 97.00 |
| 62 | ρ_2 | 0 7624 | 96.60 | 1 0662 | 97.60 | 0.5230 | 90.00 | 0.0843 | 97.60 |
| 52 | B1 | 1 1025 | 98.00 | 1 4191 | 96.40 | 1 2372 | 90.10 | 0.3569 | 96.40 |
| | β | 1.5273 | 93.30 | 2.5880 | 97.00 | 2.0066 | 91.10 | 0.4798 | 96.60 |
| 52 | α | 0.2518 | 73.60 | 0.7860 | 96.40 | 0.4352 | 71.40 | 0.3653 | 99.00 |
| 33 | β_1 | 1.5701 | 93.60 | 1.9486 | 79.80 | 2.2473 | 61.80 | 0.5871 | 97.80 |
| | β ₂ | 1.9309 | 93.20 | 2.8134 | 89.40 | 2,9992 | 88.00 | 0.8510 | 98.00 |
| 54 | α | 0.7849 | 96.60 | 1.3088 | 97.70 | 0.5430 | 91.50 | 0.0802 | 97.20 |
| 5. | β_1 | 1.0728 | 98.60 | 1.7103 | 97.50 | 1.2389 | 96.80 | 0.3147 | 97.50 |
| | β_2 | 1.3686 | 95.20 | 2.4790 | 95.20 | 1.8351 | 95.90 | 0.4504 | 97.30 |
| \$5 | α | 0.5213 | 95.40 | 0.5945 | 96.60 | 0.5210 | 91.70 | 0.0807 | 98.40 |
| | β_1 | 1.1851 | 99.20 | 1.2299 | 96.00 | 1.3022 | 89.80 | 0.3349 | 98.40 |
| | β_2 | 1.5178 | 97.60 | 1.7583 | 96.80 | 1.7084 | 91.20 | 0.4480 | 97.00 |
| k = 40 | | | | | | | | | |
| <i>S</i> 1 | α | 0.2569 | 97.60 | 0.2603 | 97.60 | 0.2747 | 93.40 | 0.1220 | 97.80 |
| | β_1 | 1.2791 | 95.60 | 1.4814 | 96.40 | 1.5445 | 90.40 | 0.4199 | 96.80 |
| | β_2 | 1.5441 | 95.20 | 1.5524 | 95.20 | 1.5746 | 88.00 | 0.3801 | 98.00 |
| S2 | α | 0.2859 | 94.80 | 0.3164 | 95.60 | 0.3380 | 90.20 | 0.1295 | 96.80 |
| | β_1 | 0.9286 | 96.60 | 0.8840 | 95.00 | 0.9716 | 88.40 | 0.3412 | 96.20 |
| | β_2 | 1.4205 | 94.20 | 1.5903 | 94.60 | 1.7277 | 87.60 | 0.4757 | 97.40 |
| S3 | α | 0.1809 | 72.00 | 0.6944 | 96.80 | 0.4681 | 72.80 | 0.3416 | 98.80 |
| | β_1 | 1.1757 | 98.00 | 1.3828 | 90.40 | 1.7846 | 76.60 | 0.4707 | 97.60 |
| | β_2 | 1.4486 | 95.00 | 2.3093 | 96.40 | 2.6511 | 76.60 | 0.7503 | 98.40 |
| S4 | α | 0.3346 | 95.40 | 0.3995 | 97.40 | 0.4141 | 89.80 | 0.1200 | 96.60 |
| | β_1 | 0.7875 | 99.00 | 0.6486 | 96.80 | 0.7354 | 89.60 | 0.2870 | 98.00 |
| | β_2 | 1.2110 | 95.00 | 1.4990 | 96.40 | 1.5820 | 92.20 | 0.4501 | 97.60 |
| S5 | α | 0.2853 | 96.80 | 0.2872 | 96.60 | 0.3126 | 92.00 | 0.1167 | 97.60 |
| | β_1 | 0.9752 | 97.60 | 0.9350 | 96.80 | 1.0195 | 90.60 | 0.3629 | 97.20 |
| | β_2 | 1.2429 | 96.60 | 1.3077 | 96.60 | 1.3596 | 91.00 | 0.4616 | 96.80 |
| (h) All s | \rightarrow nnd (D(%)) | values of all me | thode under ID 1 | IC for DIID at m | 00 and n 00 | | | | |
| (D). ML3 | | values of all file | thous under $J^p - I$ | IC IOI DIIID at III : | = 80 and n = 80 | | | | |
| Sch. | Para- | Asy-Cl | cn | Boot-P | = 00 and n = 00 | Boot-T | CD | HPD | CD |
| Sch. | Para- meter | Asy-CI AIL | CP | Boot-P AIL | = 80 and n = 80 | Boot-T AIL | СР | HPD AIL | СР |
| Sch. k = 40 | Para- meter | Asy-Cl AIL | CP | Boot-P AIL | CP | Boot-T AIL | СР | HPD AIL | СР |
| Sch. k = 40 | Para- meter | Asy-CI AIL 0.2571 | CP 95.60 | Boot-P AIL 0.2580 | CP 95.00 | Boot-T AIL 0.2809 | CP 88.80 | HPD AIL 0.1235 | CP 97.20 |
| $\frac{k}{s_1} = 40$ | Para- meter α β_1 | Asy-CI AIL 0.2571 1.1874 | CP 95.60 93.20 | 0.2580 1.5905 | CP 95.00 96.00 | Boot-T AIL 0.2809 1.6074 | CP 88.80 90.40 | HPD AIL 0.1235 0.5227 | CP 97.20 95.80 |
| $\frac{k}{s_1} = 40$ | Para- meter α β_1 β_2 | Asy-Cl AIL 0.2571 1.1874 1.4842 | CP 95.60 93.20 97.20 | 0.2580 1.5905 1.4902 | CP 95.00 96.00 96.80 | Boot-T AIL 0.2809 1.6074 1.5374 | CP 88.80 90.40 90.00 | HPD AIL 0.1235 0.5227 0.3864 | CP 97.20 95.80 96.60 |
| $\frac{k}{s_1} = 40$ | Para- meter α β_1 β_2 α | Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 | CP 95.60 93.20 97.20 96.40 | 0.2580 0.2580 1.5905 1.4902 0.4155 | CP 95.00 96.00 96.80 96.80 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 | CP 88.80 90.40 90.00 88.40 | HPD AIL 0.1235 0.5227 0.3864 0.1042 | CP 97.20 95.80 96.60 98.60 |
| $\frac{(0)}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ | Para- meter α β_1 β_2 α β_1 | Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 | CP 95.60 93.20 97.20 96.40 97.20 | 0.2580 0.2580 1.5905 1.4902 0.4155 0.7918 | CP 95.00 96.00 96.80 96.80 94.80 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 | CP 88.80 90.40 90.00 88.40 91.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 | CP 97.20 95.80 96.60 98.60 97.00 |
| $\frac{(0)}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ | $\begin{array}{c} \alpha \\ \beta_1 \\ \beta_2 \\ \alpha \\ \beta_1 \\ \beta_2 \\ \alpha \\ \beta_1 \\ \beta_2 \end{array}$ | Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 | CP 95.60 93.20 97.20 96.40 97.20 93.60 | 0.2580 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 | CP 95.00 96.00 96.80 96.80 94.80 96.00 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 | CP 88.80 90.40 90.00 88.40 91.00 90.60 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 | CP 97.20 95.80 96.60 98.60 97.00 95.80 |
| $\frac{(0)}{\text{Sch.}}$ $\frac{k}{\text{Sch.}}$ $\frac{k}{Sch$ | $\begin{array}{c} \alpha \\ Para-\\ meter \end{array}$ $\begin{array}{c} \alpha \\ \beta_1 \\ \beta_2 \\ \alpha \\ \beta_1 \\ \beta_2 \\ \alpha \\ \beta_2 \\ \alpha \end{array}$ | Alles of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 | 0.2580 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 | CP 95.00 96.00 96.80 96.80 94.80 96.00 95.80 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 | CP 97.20 95.80 96.60 98.60 97.00 95.80 97.80 |
| (b): All s_{sch} k = 40 s_{sch} s_{sch} | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{1} \end{array}$ | Aites of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 | 20 20 20 20 20 20 20 20 20 20 20 20 20 2 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 | CP 97.20 95.80 96.60 98.60 97.00 95.80 97.80 99.00 |
| (b): $h = 3$ Sch. k = 40 S1 S2 S3 | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \end{array}$ | Alter of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 | 200 and <i>n</i> = 80 CP 95.00 96.00 96.80 96.80 94.80 96.00 95.80 77.60 61.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 |
| (b): All s Sch. k = 40 s_1 s_2 s_3 s_4 | Para- meter α β_1 β_2 α β_1 β_2 α β_1 β_2 α β_1 β_2 α | Autes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 | CP 95.00 96.00 96.80 96.80 94.80 96.00 95.80 77.60 61.20 95.60 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 97.20 |
| (b): All s Sch. k = 40 s_1 s_2 s_3 s_4 | Para- meter α β_1 β_2 β_2 α β_3 β_1 β_2 β_3 | Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 | CP 95.00 96.00 96.80 96.80 94.80 96.00 95.80 77.60 61.20 95.60 93.30 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 97.20 97.60 |
| (b). And Sch. k = 40 S1 S2 S3 S4 | $\begin{array}{c} \alpha \\ Para-\\ meter \end{array}$ $\begin{array}{c} \alpha \\ \beta_1 \\ \beta_2 \\ \alpha \\ \beta_1 \\ \beta_2 \end{array}$ | Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 95.00 95.00 95.00 95.00 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 | CP 95.00 96.00 96.80 96.80 94.80 96.00 95.80 77.60 61.20 95.60 93.30 94.00 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 97.20 97.60 97.80 |
| (b): All s Sch. k = 40 s_1 s_2 s_3 s_4 s_5 | Para- meter α β_1 β_2 α | Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.40 86.20 95.00 97.20 96.60 96.60 96.60 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 | CP 95.00 96.00 96.80 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 97.20 97.60 97.80 97.80 98.80 |
| (b): h_{123} Sch. k = 40 S1 S2 S3 S4 S5 | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{$ | Autes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.40 86.20 95.00 97.20 95.00 97.20 96.60 96.60 98.40 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.80 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 | CP 97.20 95.80 96.60 97.00 97.00 97.80 99.00 98.00 97.20 97.60 97.80 97.80 97.80 |
| (b): All S Sch. k = 40 S1 S2 S3 S4 S5 | Para- meter α β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_3 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_3 β_3 β_3 β_4 β_3 β_3 β_4 β_3 β_4 β_3 β_4 β_3 β_4 β_4 β_4 β_4 β_4 β_4 β_4 β_4 β_4 β_4 | Autor of an ine Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.40 86.40 86.20 95.00 97.20 96.60 96.60 96.60 98.40 97.00 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 | 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.80 96.60 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 | CP 97.20 95.80 96.60 98.60 97.80 97.80 99.00 98.00 97.20 97.20 97.60 97.80 97.80 98.80 97.40 96.40 |
| (b): $h = 10^{-10}$ Sch. $k = 40^{-10}$ S1 S2 S3 S4 S5 $k = 80^{-10}$ | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \\ \beta_{3} \\ \beta_{4} \\ \beta_{5} \\ \beta_{$ | Autor of an ine Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 98.40 97.00 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 | CP 95.00 96.00 96.80 96.80 94.80 96.00 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.80 96.60 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.8383 1.1308 0.83655 1.1388 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 97.20 97.60 97.60 97.80 97.40 98.80 97.40 |
| (b): h_{123} Sch. k = 40 s_1 s_2 s_3 s_4 s_5 k = 80 s_1 | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{2} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \beta_{$ | Altes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.00 97.00 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.789 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.80 96.60 96.60 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.1868 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.20 97.60 97.80 97.80 97.40 96.40 |
| (b). And Sch. k = 40 s_1 s_2 s_3 s_4 s_5 k = 80 s_1 | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \beta_{2} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{2} \\ \beta_{1} \\ \beta_{2} \\ \beta_{2} \\ \beta_{2} \\ \beta_{2} \\ \beta_{2} \\ \beta_{2} \\ \beta_{$ | Altes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 0.0770 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 96.60 97.00 97.00 97.00 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.5722 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 97.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 96.60 95.00 93.80 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.1868 1.0584 1.0584 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 89.60 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.20 97.60 97.80 97.40 96.40 97.80 96.40 |
| (b). And Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 | $\begin{array}{c} \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\ \alpha \\ \beta_{1} \\ \beta_{2} \\ \alpha \\ \beta_{2} \\$ | Altes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1010 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 98.40 97.00 95.60 93.80 97.20 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 2.2560 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 96.60 95.00 93.80 95.20 27 20 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.1868 1.0584 1.0584 1.1423 0.2142 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 89.60 88.80 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 98.00 97.20 97.60 97.80 97.80 96.40 97.80 96.00 96.00 96.00 96.00 |
| $k = 40$ s_1 s_2 s_3 s_4 s_5 $k = 80$ s_1 s_2 s_2 | $\begin{array}{c} \alpha \\ Para-meter \\ \hline \\ meter \\ \hline \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ $ | Alter of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.577 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.00 95.60 93.80 97.20 95.40 95.40 95.40 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.0605 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 96.60 95.20 95.20 95.20 97.00 96.20 | Boot-T AIL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.3383 1.1308 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6276 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 1.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.2402 | CP 97.20 95.80 96.60 97.00 97.00 97.80 97.80 97.80 97.60 97.80 97.80 98.80 97.40 96.40 97.80 96.00 96.00 96.00 96.00 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 | $\begin{array}{c} \alpha \\ Para-\\meter \\ \hline \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ $ | Autor of an ine Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.6575 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 96.60 96.60 95.40 95.40 95.40 95.40 95.40 98.40 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0221 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 95.00 93.80 95.20 97.00 96.00 95.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.3868 1.0584 1.1423 0.2142 0.6576 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 89.60 88.80 92.00 91.80 89.60 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.20 97.20 97.60 97.80 96.40 97.80 96.00 96.00 97.60 96.00 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 | Para- meter α β_1 β_2 β_2 β_2 β_1 β_2 β_2 β_2 β_1 β_2 β_2 β_2 β_3 β_3 β_4 β_3 β_3 β_4 β_4 β_3 β_4 | Autor of an ine Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1155 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.20 95.60 93.80 97.20 95.40 93.80 97.20 95.40 93.80 97.20 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 95.00 93.80 95.20 95.20 95.20 95.20 95.20 95.20 95.20 95.20 96.40 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6576 1.1341 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.20 72.90 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.2706 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.20 97.20 97.60 97.80 96.40 97.80 96.00 96.00 97.60 96.80 96.80 96.80 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 | Para- meter α β_1 β_2 α β_2 α β_3 β_3 β_4 β_3 β_4 β_3 β_4 β | Altes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.9220 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.20 95.60 93.80 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.60 97.20 97.20 96.60 97.20 97.20 96.60 97.20 97.20 96.60 97.20 97.20 97.20 97.20 97.20 96.60 97.20 97.40 97.20 97.40 97.40 97.20 97.40 97. | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2600 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.80 96.60 95.20 97.00 95.20 97.00 95.20 97.00 95.20 96.40 84 92 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.8383 1.1308 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7002 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.20 73.80 71.40 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.621 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.80 97.20 97.60 97.80 97.40 96.40 97.80 96.00 96.00 96.00 97.60 96.80 96.80 96.80 97.40 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 | Para- meter α β_1 β_2 β_1 β_2 β_3 β_1 β_2 β_3 β_3 β_1 β_2 β_3 | Altes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0270 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.00 95.60 93.80 97.20 95.40 95.40 95.40 95.40 95.40 95.40 95.40 98.00 94.40 73.60 94.80 86 60 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 | CP 95.00 96.00 96.80 96.80 94.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 95.20 97.00 95.20 97.00 95.20 97.00 96.40 84.80 82.60 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.1868 1.0584 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6320 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.20 73.80 71.40 68.20 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.6091 0.9692 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.20 97.60 97.20 97.60 97.80 96.40 97.80 96.00 97.60 96.00 96.00 96.00 96.80 96.80 96.80 96.80 |
| (b): h_{113} Sch. k = 40 s_1 s_2 s_3 s_4 s_5 k = 80 s_1 s_2 s_3 s_4 s_5 s_5 s_4 s_5 $s_$ | Para- meter α β_1 β_2 β_1 β_2 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_3 β_3 β_1 β_2 β_3 β_3 β_3 β_4 β_3 β_3 β_4 β_3 β_4 β_3 β_4 | Altes of all life Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0370 0.2371 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 96.60 96.60 97.00 95.40 97.20 95.40 97.20 95.40 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.40 97.20 95.60 97.20 95.60 97.20 95.60 97.20 96.60 97.20 97.20 96.60 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.20 97.40 97.20 95.60 97.20 95.40 97.20 95.40 95. | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 0.3524 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 77.60 61.20 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 95.20 95.20 96.40 84.80 93.60 96.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.3785 2.2704 2.9084 5.0848 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6230 0.2727 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.80 91.20 73.80 71.40 68.20 91.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.6091 0.9682 0.1071 | CP 97.20 95.80 96.60 97.00 95.80 97.80 97.80 97.20 97.60 97.80 97.40 96.40 97.80 96.40 97.80 96.00 96.00 96.00 96.80 96.80 98.80 97.40 98.80 97.40 98.00 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 S4 S2 S3 S4 | $\begin{array}{c} \alpha \\ Para-meter \\ \hline \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ $ | Altes of all life Asy-CI Altes of all life Asy-CI Altes 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.8420 1.0654 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0370 0.2271 0.5527 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 96.60 97.00 95.60 93.80 97.20 95.40 95.40 95.40 95.40 95.40 95.40 95.40 95.60 93.80 97.20 95.40 95.60 95.60 95.60 95.60 95.60 94.40 73.60 94.40 73.60 94.60 95.60 94.60 95.60 90 90.60 90 90.00 90 90 90 90 90 90 90 90 90 90 90 90 9 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 0.2524 0.4428 | CP 95.00 96.00 96.80 96.80 94.80 96.80 95.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 96.60 95.20 97.00 96.00 95.20 96.40 84.80 93.60 96.20 97.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.3785 2.2704 2.9084 5.0848 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6230 0.2737 0.4820 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 89.60 88.80 92.00 91.80 89.60 88.80 92.00 91.80 81.40 91.20 73.80 71.40 68.20 91.00 91.00 91.00 91.20 91.00 91.20 91.00 91.20 91.20 91.00 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.6091 0.9682 0.1071 | CP 97.20 95.80 96.60 97.00 97.00 97.80 97.80 97.80 97.80 97.80 97.40 96.40 97.80 96.00 96.00 96.00 96.80 96.80 96.80 98.80 97.40 98.80 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 S3 S4 S2 S3 S3 S4 | Para- meter α β_1 β_2 β_1 β_2 β_2 β_3 β_1 β_2 β_1 β_2 β_1 β_2 β_2 β_3 | Autes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0370 0.2271 0.5537 0.8422 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.20 95.60 93.80 97.20 95.40 95.40 95.40 95.40 94.40 73.60 94.40 73.60 94.40 73.60 94.40 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 0.2524 0.428 0.974 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 95.20 97.00 96.00 95.20 97.00 96.40 84.80 93.80 95.20 97.20 96.20 97.20 95.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.3707 0.8655 1.1308 0.3707 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6230 0.2737 0.4820 1.0262 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.80 91.20 73.80 71.40 68.20 91.00 91.60 90.90 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.6091 0.9682 0.1071 0.2802 0.5042 | CP 97.20 95.80 96.60 97.80 97.80 97.80 97.20 97.60 97.80 97.40 96.40 97.80 96.00 96.80 96.80 96.80 96.80 97.40 96.80 95.80 96.80 97.40 98.80 97.40 98.00 98.00 98.00 98.00 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 S4 S2 S3 S4 | Para- meter α β_1 β_2 β_2 β_2 β_3 β_3 β_3 β_4 β_3 β_4 | Autes of an me Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0370 0.2271 0.5537 0.8433 0.1954 | CP 95.60 93.20 97.20 96.40 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 98.40 97.20 95.60 93.80 97.20 95.40 95.40 95.40 94.40 73.60 94.40 95.60 94.40 96.60 96.60 94.40 97.60 94.40 94.40 96.60 96.60 97.60 94.40 96.60 94.40 94.40 96.60 96.60 97.60 94.40 94.40 96.60 96.60 97.60 94.40 94.40 96.60 96.60 96.60 97.60 94.40 96.60 96.60 96.60 97.60 94.40 96.60 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 0.2524 0.4428 0.9974 0.1920 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 95.00 93.80 96.60 95.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.3707 0.8655 1.1388 0.3707 0.8657 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6230 0.2737 0.4820 1.0362 0.2132 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.20 73.80 71.40 68.20 91.00 91.60 90.80 92.90 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.10111 0.3403 0.4752 0.3706 0.6091 0.9682 0.1071 0.2802 0.5042 | CP 97.20 95.80 96.60 97.00 95.80 97.80 99.00 97.20 97.20 97.60 97.80 96.40 97.80 96.40 97.80 96.00 96.80 96.80 96.80 95.80 95.80 95.80 98.80 97.40 |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 S4 S5 S4 S5 | Para- meter α β_1 β_2 β_2 β_3 β_1 β_2 β_3 β_1 β_2 β_3 | Altes of all life Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0370 0.2271 0.5537 0.8433 0.1954 0.6773 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.20 95.60 95.60 93.80 97.20 95.40 95.40 95.40 95.40 95.40 95.40 95.40 95.40 95.40 95.60 100.00 94.40 95.60 100.00 94.40 95.60 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 0.2524 0.4428 0.9974 0.1989 0.6514 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 93.30 96.60 95.20 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.1868 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6230 0.2737 0.4820 1.0362 0.2132 0.6433 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.20 73.80 71.40 68.20 91.00 91.60 90.80 92.00 93.60 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.6091 0.9682 0.1071 0.2802 0.5042 0.1004 0.3459 | CP 97.20 95.80 96.60 97.00 95.80 97.00 98.00 97.20 97.20 97.60 97.80 98.80 97.40 96.40 97.80 96.00 96.00 96.00 96.80 95.80 97.80 95.80 97.20 95.80 95.80 97.80 96.00 96.00 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 96.80 97.80 97.80 97.80 97.80 97.80 97.80 97.80 97.80 97.80 96.80 97.80 97.80 97.80 96.80 97.80 98.80 97.80 98.00 98.00 98.00 98.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 99.00 90. |
| (b): h_{113} Sch. k = 40 S1 S2 S3 S4 S5 k = 80 S1 S2 S3 S4 S5 S4 S5 | Para- meter α β_1 β_2 β_3 | Altes of all life Asy-CI AIL 0.2571 1.1874 1.4842 0.3654 0.7829 1.0750 0.1489 1.1358 1.3939 0.5140 0.8420 1.0654 0.3365 0.8161 1.0490 0.1785 0.9059 1.0778 0.1919 0.6575 0.9632 0.1156 0.8339 1.0370 0.2271 0.5537 0.8433 0.1954 0.6773 0.8739 | CP 95.60 93.20 97.20 96.40 97.20 93.60 74.80 86.40 86.20 95.00 97.20 96.60 96.60 96.60 96.60 98.40 97.20 95.60 93.80 97.20 95.40 95.40 95.40 95.40 95.40 95.40 95.60 100.00 94.40 95.60 100.00 94.40 95.60 | 0.2580 1.5905 1.4902 0.4155 0.7918 1.3966 0.6842 1.9073 2.5150 0.6605 1.1020 1.5627 0.3421 0.8136 1.1107 0.1789 0.9747 1.0522 0.2068 0.6066 1.0831 0.6452 1.2869 2.2267 0.2524 0.4428 0.9974 0.1989 0.6514 0.9031 | CP 95.00 96.00 96.80 96.80 96.80 96.80 95.80 77.60 61.20 95.60 93.30 94.00 96.60 96.60 96.60 95.20 97.00 95.20 97.00 95.20 97.00 95.20 97.00 96.40 84.80 93.60 96.20 97.20 95.80 95.40 97.80 95.40 97.80 | Boot-T AlL 0.2809 1.6074 1.5374 0.3981 0.8538 1.3954 0.3785 2.2704 2.9084 5.0848 0.8383 1.1308 0.3707 0.8655 1.1388 0.1868 1.0584 1.0584 1.1423 0.2142 0.6576 1.1341 0.4267 1.7003 2.6230 0.2737 0.4820 1.0362 0.2132 0.6943 0.9452 | CP 88.80 90.40 90.00 88.40 91.00 90.60 75.00 52.00 37.80 0.00 90.70 93.80 89.60 92.80 93.40 91.80 89.60 88.80 92.00 91.80 91.20 73.80 71.40 68.20 91.00 91.60 90.80 92.00 93.60 92.00 93.60 92.20 | HPD AIL 0.1235 0.5227 0.3864 0.1042 0.3377 0.5072 0.4173 0.8486 1.1603 0.0959 0.2994 0.3966 0.1045 0.3204 0.4265 0.1023 0.4158 0.4082 0.1111 0.3403 0.4752 0.3706 0.6091 0.9682 0.1071 0.2802 0.5042 0.1071 0.2802 0.5042 0.1004 0.3459 0.4574 | CP 97.20 95.80 96.60 97.00 95.80 97.80 97.80 97.80 97.20 97.60 97.80 97.40 96.40 97.80 96.00 96.00 96.00 96.00 96.80 96.80 98.80 97.40 96.80 96.80 98.80 97.40 96.40 |

(continued on next page)

| Table 3 | (con | tinue | d) |
|---------|-------|-------|----|
| Iupic 3 | 10011 | unuc | u, |

| Sch. | Para- | Asy-CI | CD | Boot-P | CD | Boot-T | CD | HPD | CD |
|----------------|---------------|--------------------|-------------------|----------------------|------------------|--------|-------|--------|-------|
| | meter | AIL | CP | AIL | CP | AIL | CP | AIL | CP |
| (c): AILs | and CP(%) val | lues of all method | ls underJP–IIC fo | or BIIID at $m = 12$ | 20 and $n = 120$ | | | | |
| Sch. | Para- | Asy-CI | - | Boot-P | | Boot-T | | HPD | |
| | meter | AIL | CP | AIL | CP | AIL | CP | AIL | СР |
| k = 80 | | | | | | | | | |
| \$1 | α | 0.1771 | 96.20 | 0.1811 | 95.80 | 0.1889 | 93.60 | 0.1099 | 97.20 |
| | β_1 | 0.8613 | 95.20 | 1.0049 | 97.40 | 1.0492 | 93.00 | 0.4700 | 95.80 |
| | β_2 | 1.0569 | 96.60 | 1.0187 | 96.00 | 1.0999 | 91.20 | 0.3918 | 98.20 |
| 52 | α | 0.2202 | 95.80 | 0.2418 | 97.00 | 0.2549 | 92.00 | 0.1114 | 96.00 |
| | β_1 | 0.5462 | 98.60 | 0.5037 | 96.80 | 0.5438 | 94.20 | 0.3334 | 98.20 |
| | β_2 | 0.8296 | 94.60 | 1.0196 | 96.40 | 1.0578 | 93.40 | 0.5286 | 96.80 |
| 53 | α | 0.1007 | 75.40 | 0.6427 | 96.40 | 0.3702 | 75.80 | 0.4286 | 97.80 |
| | β_1 | 0.8345 | 76.80 | 2.1430 | 47.80 | 1.9855 | 43.40 | 1.0027 | 98.80 |
| | β_2 | 1.0150 | 73.80 | 3.0245 | 57.80 | 2.7551 | 35.00 | 1.3374 | 97.80 |
| 54 | α | 0.2621 | 94.60 | 0.2750 | 95.20 | 0.2930 | 89.00 | 0.1179 | 96.80 |
| | β_1 | 0.5718 | 99.40 | 0.5199 | 96.60 | 0.5443 | 92.80 | 0.3005 | 96.40 |
| | β_2 | 0.7355 | 97.40 | 0.8037 | 97.20 | 0.8086 | 94.80 | 0.4343 | 97.60 |
| \$5 | α | 0.2091 | 96.20 | 0.2144 | 96.00 | 0.2262 | 92.40 | 0.1101 | 96.40 |
| | β_1 | 0.6019 | 97.20 | 0.5871 | 95.40 | 0.6262 | 93.00 | 0.3384 | 97.00 |
| | β_2 | 0.7741 | 98.60 | 0.7960 | 98.00 | 0.8150 | 95.80 | 0.4468 | 99.00 |
| <i>k</i> = 120 | | | | | | | | | |
| S1 | α | 0.1453 | 97.60 | 0.1458 | 97.20 | 0.1515 | 94.20 | 0.2135 | 97.80 |
| | β_1 | 0.7301 | 97.40 | 0.7882 | 97.60 | 0.8278 | 93.40 | 0.5553 | 97.30 |
| | β_2 | 0.8719 | 97.20 | 0.8481 | 96.60 | 0.8943 | 91.40 | 0.4660 | 97.50 |
| 52 | α | 0.1570 | 96.80 | 0.1638 | 95.20 | 0.1727 | 91.00 | 0.1010 | 97.60 |
| | β_1 | 0.5330 | 98.20 | 0.4887 | 97.00 | 0.5148 | 94.00 | 0.3376 | 96.80 |
| | β_2 | 0.7867 | 95.20 | 0.8636 | 96.40 | 0.8998 | 91.80 | 0.5056 | 97.20 |
| 53 | α | 0.0865 | 77.40 | 0.6233 | 95.80 | 0.4016 | 77.60 | 0.4479 | 97.00 |
| | β_1 | 0.7004 | 90.80 | 2.0098 | 75.80 | 1.6471 | 68.00 | 6.6680 | 95.10 |
| | β_2 | 0.8649 | 82.40 | 2.9424 | 92.20 | 2.6217 | 62.00 | 4.4758 | 95.10 |
| 54 | α | 0.1793 | 95.60 | 0.1965 | 95.80 | 0.2063 | 94.00 | 0.1019 | 98.20 |
| | β_1 | 0.4509 | 99.40 | 0.3573 | 97.60 | 0.3778 | 95.20 | 0.2318 | 98.00 |
| | β_2 | 0.6866 | 96.00 | 0.7924 | 97.00 | 0.8171 | 95.40 | 0.4404 | 97.80 |
| S5 | α | 0.1567 | 97.20 | 0.1602 | 97.20 | 0.1667 | 94.00 | 0.1010 | 97.40 |
| | β_1 | 0.5545 | 98.40 | 0.5316 | 96.80 | 0.5628 | 94.40 | 0.3555 | 99.60 |
| | β_2 | 0.7112 | 96.60 | 0.7326 | 96.00 | 0.7606 | 92.20 | 0.4616 | 98.40 |

Table 4

Strength data measured for single carbon fibers and impregnated 1000-carbon fiber tows.

Data set I: W

1.312,1.314,1.479,1.552,1.700,1.803,1.861,1.865,1.944,1.958,1.966,1.997,2.00,2.021,2.027,2.055,2.063,2.098,2.140,2.179,2.224,2.240,2.253,2.270,2.272, 2.274,2.301,2.301,2.359,2.382,2.426,2.434,2.435,2.478,2.490,2.511,2.514,2.535,2.554,2.566,2.570,2.586,2.629,2.633,2.642,2.648,2.684,2.697, 2.726,2.770,2.773,2.800,2.809,2.818,2.821,2.848,2.880,2.954,3.012,3.067,3.084,3.090,3.096,3.128,3.233,3.433,3.585,3.585. Data set II: Z

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.997, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020

Bootstrap-T approaches. We also investigated Bayesian estimators for both symmetric and asymmetric loss functions, including SELF, LiLF, MELF, and DeLF. The Markov chain Monte Carlo method has been used to compute the Bayes estimates and the related credible intervals. We carry out in-depth simulation tests to assess the estimators' performance. Our simulation results indicate that Bayes estimates based on DeLF are better than MLEs. In light of this, the Bayesian method is superior to the ML method. The bias performance of the BEs employing MCMC under DeLF is typically superior to that of the BEs under SELF, LiLF, and MELF. The Boot-P approach is superior to the Boot-T method for obtaining confidence intervals. As compared to the Asy-CI, Boot-P, and Boot-T, the HPD credible interval has the smallest interval length. The present work can also be extended to design of optimal progressive censoring sampling plan, and other censoring schemes can also be considered.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.



Fig. 1. Convergence of MCMC estimates using MH algorithm for JP-IIC scheme (S1).

 Table 5

 Removal patterns of units in various censoring schemes.

| - | | • | | |
|--------------|----|---------------------------|--------------------------|---------------------------------|
| (<i>m</i> , | | Censoring Schemes | | |
| n) | k | <i>S</i> 1 | \$2 | \$3 |
| (69,63) | 20 | $(112, 0 \times {}^{19})$ | $(5 \times {}^{19}, 17)$ | $(0 \times {}^{19}, 112)$ |
| | 40 | $(92, 0 \times {}^{39})$ | $(2 \times {}^{39}, 14)$ | $(0 \times {}^{39}, 92)$ |
| | 60 | $(72, 0 \times {}^{59})$ | $(1 \times {}^{59}, 13)$ | (0 \times ⁵⁹ , 72) |
| | | | | |

Here, $(1 \times {}^{5}, 0)$, for example, means that the censoring scheme employed is (1, 1, 1, 1, 1, 0).

CRediT authorship contribution statement

Amal S. Hassan: Conceptualization, Methodology, Formal analysis, Investigation, Writing – review & editing, Visualization, Supervision. **E.A. Elsherpieny:** Conceptualization, Methodology, Supervision. **Wesal.E. Aghel:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Writing – review & editing.

| Table 6 | | | | | | |
|------------|-------------|---------------|-------------|----------------|--------------------|--|
| MLEs, BEs, | and St.Er f | for real data | set based o | on different s | schemes of JP–IIC. | |

| | Para | - MLE | | MCMC-SEL | F | MCMC-LiLl | F | MCMC-Del | F | MCMC-ME | LF |
|------------|-----------|-------------|---------|----------|---------|-----------|---------|----------|---------|----------|----------|
| Sc | h.mete | er Estimate | St.Er | Estimate | St.Er | Estimate | St.Er | Estimate | St.Er | Estimate | St.Er |
| k : | = 20 | | | | | | | | | | |
| S 1 | α | 3.6103 | 0.4590 | 3.6518 | 0.3832 | 3.6167 | 7.0368 | 3.5774 | 1.8091 | 3.6920 | 7.6078 |
| | β_1 | 31.6661 | 15.5309 | 29.0037 | 4.4511 | 25.4283 | 23.5337 | 27.7178 | 13.7752 | 29.6867 | 60.4628 |
| | β_2 | 32.3911 | 10.1761 | 35.4538 | 9.3707 | 25.6598 | 23.7641 | 30.6347 | 15.2873 | 37.9303 | 73.8167 |
| S2 | α | 3.1858 | 0.4510 | 3.1626 | 0.1735 | 3.1551 | 8.0512 | 3.1436 | 1.5648 | 3.1721 | 6.3542 |
| | β_1 | 22.7117 | 9.3229 | 23.5667 | 4.3406 | 19.7805 | 18.6144 | 21.8288 | 10.6017 | 24.3661 | 46.2492 |
| | β_2 | 15.6761 | 4.2108 | 14.3962 | 1.4290 | 13.9099 | 14.5542 | 14.1123 | 6.9944 | 14.5380 | 29.0295 |
| \$3 | α | 3.2596 | 0.4779 | 3.5193 | 0.2449 | 3.5046 | 7.7430 | 3.4861 | 1.7506 | 3.5364 | 7.1315 |
| | β_1 | 12.8881 | 8.5927 | 13.9612 | 1.2284 | 13.5524 | 14.1634 | 13.7265 | 6.6250 | 14.0693 | 27.3948 |
| | β_2 | 15.2676 | 4.4749 | 17.1882 | 1.4545 | 16.6919 | 17.3522 | 16.9445 | 8.4503 | 17.3112 | 34.7174 |
| k = | = 40 | | | | | | | | | | |
| S 1 | α | 3.2820 | 0.3271 | 3.0839 | 0.2421 | 3.0693 | 7.2886 | 3.0454 | 1.5077 | 3.1029 | 6.1795 |
| | β_1 | 39.1132 | 16.9310 | 38.2264 | 8.7288 | 27.8068 | 25.4700 | 34.0289 | 16.3062 | 40.2193 | 77.8869 |
| | β_2 | 25.0145 | 6.4048 | 20.2287 | 3.7081 | 17.9751 | 17.6509 | 18.9976 | 9.8056 | 20.9084 | 42.0755 |
| S 2 | α | 3.3688 | 0.3358 | 3.4064 | 0.1728 | 3.3988 | 8.2455 | 3.3879 | 1.6594 | 3.4152 | 6.7247 |
| | β_1 | 34.1195 | 11.2328 | 35.1609 | 4.2007 | 32.6967 | 32.3161 | 34.2737 | 17.7891 | 35.6627 | 73.7586 |
| | β_2 | 19.1834 | 4.4333 | 19.9250 | 2.1597 | 18.7904 | 18.4881 | 19.4388 | 9.5239 | 20.1590 | 39.7271 |
| \$3 | α | 4.5393 | 0.4499 | 4.5002 | 0.2365 | 4.4860 | 8.7145 | 4.4745 | 2.2109 | 4.5126 | 8.9218 |
| | β_1 | 35.5419 | 15.7404 | 35.5530 | 5.7438 | 28.8807 | 26.4282 | 33.5001 | 15.9297 | 36.4808 | 69.9895 |
| | β_2 | 27.1410 | 7.5590 | 26.6407 | 2.8031 | 25.0690 | 24.8540 | 26.0655 | 13.1160 | 26.9356 | 53.9722 |
| k = | = 60 | | | | | | | | | | |
| \$1 | α | 4.2314 | 0.3397 | 4.5378 | 0.2796 | 4.5184 | 8.4696 | 4.5036 | 2.2553 | 4.5551 | 9.1020 |
| | β_1 | 72.3768 | 26.4601 | 105.4081 | 29.3904 | 71.4381 | 68.9882 | 91.2807 | 47.5875 | 113.6019 | 225.7353 |
| | β_2 | 40.4562 | 10.1515 | 50.1352 | 12.0631 | 36.5719 | 34.7222 | 44.4102 | 22.2444 | 53.0374 | 102.3557 |
| S 2 | α | 3.7189 | 0.3007 | 3.4997 | 0.1479 | 3.4943 | 8.7195 | 3.4874 | 1.7462 | 3.5060 | 7.0426 |
| | β_1 | 44.1152 | 12.5906 | 37.0987 | 5.2820 | 33.9953 | 33.6252 | 35.8697 | 19.0028 | 37.8507 | 80.4539 |
| | β_2 | 24.3808 | 5.2322 | 20.7519 | 1.7047 | 20.0883 | 20.3721 | 20.4779 | 10.2285 | 20.8919 | 42.1475 |
| \$3 | α | 5.2310 | 0.4333 | 4.7585 | 0.2833 | 4.7385 | 8.6621 | 4.7249 | 2.3638 | 4.7753 | 9.5463 |
| | β_1 | 74.9334 | 30.5517 | 55.9383 | 12.3844 | 42.4541 | 40.3994 | 50.7808 | 25.7430 | 58.6798 | 116.0031 |
| | β_2 | 35.3717 | 9.4492 | 22.4411 | 3.5367 | 20.4239 | 20.1390 | 21.4523 | 11.0836 | 22.9984 | 47.2459 |

| Table 7 |
|---|
| Associated interval estimates for ML and Bayesian for real data set based on different schemes of JP-IIC. |

| | | Para- | Asy-CI (MLE) | | HPD (Bayes) | |
|----|------------|-----------|--------------|----------|-------------|----------|
| k | Sch. | meter | LB | UB | LB | UB |
| 20 | <i>S</i> 1 | α | 2.7456 | 4.5418 | 2.9414 | 4.3781 |
| | | β_1 | 11.9437 | 82.2102 | 23.3788 | 39.4288 |
| | | β_2 | 17.5511 | 59.6376 | 20.5844 | 52.5693 |
| | S2 | α | 2.3428 | 4.1047 | 2.7579 | 3.4385 |
| | | β_1 | 10.4102 | 51.8329 | 15.8505 | 29.5066 |
| | | β_2 | 9.3891 | 26.7917 | 11.9801 | 17.3979 |
| | \$3 | α | 2.3631 | 4.2261 | 3.0735 | 4.0075 |
| | | β_1 | 3.5825 | 42.2954 | 11.3195 | 15.9439 |
| | | β_2 | 8.6945 | 27.2475 | 13.8705 | 19.8405 |
| 40 | S1 | α | 2.6605 | 3.9393 | 2.6122 | 3.4947 |
| | | β_1 | 16.7258 | 91.1989 | 23.0837 | 53.2539 |
| | | β_2 | 15.1682 | 41.2470 | 15.2323 | 26.8684 |
| | S2 | α | 2.7314 | 4.0448 | 3.0885 | 3.7307 |
| | | β_1 | 18.1234 | 65.6700 | 29.1067 | 42.7882 |
| | | β_2 | 12.2473 | 30.2145 | 15.7543 | 23.4250 |
| | \$3 | α | 3.6835 | 5.4440 | 4.0516 | 4.8712 |
| | | β_1 | 14.7528 | 83.8819 | 24.2193 | 44.5962 |
| | | β_2 | 15.6894 | 46.6297 | 21.8820 | 31.3155 |
| 60 | S1 | α | 3.5827 | 4.9122 | 4.0654 | 4.9989 |
| | | β_1 | 35.5163 | 148.4712 | 64.7688 | 153.6393 |
| | | β_2 | 24.6792 | 65.8405 | 31.0230 | 69.6114 |
| | \$2 | α | 3.1445 | 4.3217 | 3.2259 | 3.7825 |
| | | β_1 | 25.4281 | 77.7927 | 30.5063 | 48.6125 |
| | | β_2 | 16.0284 | 37.0981 | 17.6606 | 24.5698 |
| | \$3 | α | 4.4078 | 6.1061 | 4.2702 | 5.3625 |
| | | β_1 | 33.7937 | 167.2960 | 38.0413 | 85.4039 |
| | | β_2 | 20.8362 | 59.3184 | 17.2260 | 34.5112 |

Algorithm 1

The algorithm of the Boot-P method.

Step 1: Generate two random samples, which are from BIIID $(\hat{\beta}_1, \alpha)$ and BIIID $(\hat{\beta}_2, \alpha)$ respectively, and use the JP–IIC scheme to get the observed data. **Step 2:** Calculate the ML estimates (MLEs), say $\hat{\phi} = (\hat{\beta}_1, \hat{\beta}_2, \hat{\alpha})$.

Step 3: Use $\hat{\phi}$ to generate two new samples, respectively. **Step 4:** Get new MLEs $\hat{\phi}^{(i)} = (\hat{\beta}_1^{(i)}, \hat{\beta}_2^{(i)}, \hat{a}^{(i)})$ **Step 5:** Repeat steps 3 and 4 for N times. **Step 6:** Get the results $(\hat{\phi}^{(1)}), \dots, (\hat{\phi}^{(N)})$. **Step 7:** Sort $\hat{\phi}^{(1)}, \dots, \hat{\phi}^{(N)}$ in ascending order as $(\hat{\phi}_{(1)}, \dots, \hat{\phi}_{(N)})$. **Step 8:** The 100(1 – ε)% Boot-P CIs for (ϕ) are $[\hat{\phi}_{(lb)}, \hat{\phi}_{(hb)}]$ here, $lb = [(\varepsilon/2)N]$, $hb = [((1 - \varepsilon)/2)N]$. Here [x] means the largest integer not exceeding x.

Algorithm 2

The algorithm of the Boot-T method.

Step 1 to 5 are the same as those in Algorithm 1. Step 6: Get the results $((\hat{\phi}^{(1)}), \dots, (\hat{\phi}^{(N)}))$ and $(Cov^{(1)}, Cov^{(2)}, \dots, Cov^{(N)})$ where $Cov^{(i)}$ is the covariance, which is given by: $Cov^{(i)} = I_0^{-1}(\hat{\phi}^{(i)})$. Step 7: $Var(\hat{\phi}^{(i)})$ which are diagonals of $Cov^{(i)}$ can be obtained. Define $T_j^{(i)} = \frac{\hat{\phi}_j^{(0)} - \hat{\phi}_j}{\sqrt{var(\hat{\phi}^{(i)})}}$. Step 8: $Sort(T_1^{(1)}, \dots, T_1^{(N)}), (T_2^{(1)}, \dots, T_2^{(N)}), (T_{\phi}^{(1)}, \dots, T_{\phi}^{(N)})$ in ascending order as $(T_{1(1)}, \dots, T_{1(N)}), (T_{2(1)}, \dots, T_{2(N)}), (T_{\phi(1)}, \dots, T_{\phi(N)})$. Step 9: The $100(1 - \varepsilon)$ % Boot-T CI for ϕ is given by:

Appendix

The elements of FIM are obtained follows

$$\begin{split} \frac{\partial^2 l^*}{\partial \beta_1^2} &= \frac{-k_1}{\beta_1^2} - \sum_{i=1}^k \frac{s_i (D_i(\alpha))^{\beta_1} [ln(D_i(\alpha))]^2}{[(D_i(\alpha))^{\beta_1} - 1]^2}, \\ \frac{\partial^2 l^*}{\partial \beta_2^2} &= \frac{-k_2}{\beta_2^2} - \sum_{i=1}^k \frac{t_i (D_i(\alpha))^{\beta_2} [ln D_i(\alpha)]^2}{[D_i(\alpha)^{\beta_2} - 1]^2}, \\ \frac{\partial l^*}{\partial \alpha^2} &= \frac{-k}{\alpha^2} + \sum_{i=1}^k \frac{t_i (D_i(\alpha))^{\beta_2} [ln D_i(\alpha)]^2}{(1 + w_i^{\alpha})^2} - \sum_{i=1}^k \frac{s_i \beta_1^2 (D_i(\alpha))^{\beta_1 - 2} (w_i)^{-2\alpha} (lnw_i)^2}{[(D_i(\alpha))^{\beta_1 - 1}]^2} \\ &\quad - \sum_{i=1}^k \frac{s_i (h(w_i)^2 w_i^{-\alpha} (w_i^{-\alpha} (D_i(\alpha))^{-2} - (D_i(\alpha))^{-1}]}{[(D_i(\alpha))^{\beta_1 - 1}]} - \sum_{i=1}^k \frac{t_i \beta_2^2 (D_i(\alpha))^{\beta_2 - 2} (w_i)^{-2\alpha} (lnw_i)^2}{[(D_i(\alpha))^{\beta_2 - 1}]^2} \\ &\quad - \sum_{i=1}^k \frac{s_i (h(w_i)^2 w_i^{-\alpha} (w_i^{-\alpha} (D_i(\alpha))^{-2} - (D_i(\alpha))^{-1}]}{[(D_i(\alpha))^{\beta_1 - 1}]}, \\ \frac{\partial^2 l^*}{\partial \alpha \partial \beta_1} &= -\sum_{i=1}^k \frac{z_i \ln w_i}{1 + w_i^{\alpha}} - \sum_{i=1}^k \frac{s_i ln w_i}{(1 + w_i^{\alpha}) [(D_i(\alpha))^{\beta_1} - 1]} + \sum_{i=1}^k \frac{s_i \beta_1 (D_i(\alpha))^{\beta_1 - 1} w_i^{-\alpha} (lnw_i) ln D_i(\alpha)}{[(D_i(\alpha))^{\beta_1} - 1]^2}, \\ \frac{\partial^2 l^*}{\partial \alpha \partial \beta_2} &= -\sum_{i=1}^k \frac{(1 - z_i) \ln w_i}{1 + w_i^{\alpha}} - \sum_{i=1}^k \frac{t_i ln w_i}{(1 + w_i^{\alpha}) [(D_i(\alpha))^{\beta_2} - 1]} + \sum_{i=1}^k \frac{t_i \beta_2 (D_i(\alpha))^{\beta_2 - 1} w_i^{-\alpha} lnw_i ln D_i(\alpha)}{[(D_i(\alpha))^{\beta_2} - 1]^2}, \\ \frac{\partial^2 l^*}{\partial \alpha \partial \beta_2} &= -\sum_{i=1}^k \frac{(1 - z_i) \ln w_i}{1 + w_i^{\alpha}} - \sum_{i=1}^k \frac{t_i ln w_i}{(1 + w_i^{\alpha}) [(D_i(\alpha))^{\beta_2} - 1]} + \sum_{i=1}^k \frac{t_i \beta_2 (D_i(\alpha))^{\beta_2 - 1} w_i^{-\alpha} lnw_i ln D_i(\alpha)}{[(D_i(\alpha))^{\beta_2} - 1]^2}, \\ \frac{\partial^2 l^*}{\partial \alpha \partial \beta_2} &= -\sum_{i=1}^k \frac{(1 - z_i) \ln w_i}{1 + w_i^{\alpha}} - \sum_{i=1}^k \frac{t_i ln w_i}{(1 + w_i^{\alpha}) [(D_i(\alpha))^{\beta_2} - 1]} + \sum_{i=1}^k \frac{t_i \beta_2 (D_i(\alpha))^{\beta_2 - 1} w_i^{-\alpha} lnw_i ln D_i(\alpha)}{[(D_i(\alpha))^{\beta_2} - 1]^2}, \\ \frac{\partial^2 l^*}{\partial \alpha \partial \beta_2} &= -\sum_{i=1}^k \frac{(1 - z_i) \ln w_i}{1 + w_i^{\alpha}} - \sum_{i=1}^k \frac{t_i ln w_i}{(1 + w_i^{\alpha}) [(D_i(\alpha))^{\beta_2} - 1]} + \sum_{i=1}^k \frac{t_i \beta_2 (D_i(\alpha))^{\beta_2 - 1} w_i^{-\alpha} lnw_i ln D_i(\alpha)}{[(D_i(\alpha))^{\beta_2 - 1}]^2} \\ \frac{\partial^2 l^*}{(D_i(\alpha))^{\beta_2 - 1}} &= 0. \\ \end{array}$$

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