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Statistical inference of the Burr Type III distribution under joint progressively Type-II censoring

Amal S. Hassan [a], E.A. Elsherpieny [a], Wesal.E. Aghel [b,*]

[a] Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt
[b] Faculty Sciences Zawia University Libya

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The joint censoring technique is essential when the study's objective is to assess the relative benefits of products in relation to their service times. To reduce the cost and duration of the experiment, progressive censoring has gained a lot of attention in recent years. This article examines the statistical inference for the Burr Type III distribution using a joint progressive Type II censoring method on two samples. For model parameters, both the maximum likelihood and Bayesian methods are considered. Next, approximate confidence intervals are obtained based on the observed information matrix. Confidence intervals are also obtained using the procedures of Bootstrap-P and Bootstrap-T. Bayesian estimators are provided for symmetric and asymmetric loss functions. The Bayesian estimators cannot be produced in closed forms; hence, we compute the Bayesian estimators and the related credible intervals using the Markov chain Monte Carlo method. To evaluate the performance of the estimators, we conduct comprehensive simulation experiments. Finally, for purposes of illustration, we analyze two real data sets.

Introduction

Burr Type III distribution

Burr [1] established twelve distinct classifications of cumulative distribution functions that provide a range of density arrangements. The main goal of selecting one of these distribution types is to make the mathematical analysis it undergoes as simple as possible while still producing a fair approximation. Among these distributions, the Burr Type III distribution (BIIID) can accommodate different hazard lifetime data, so it has received considerable attention in the recent past. This distribution has been widely used in numerous fields of science with different parameterizations under other names. In studies of income, wage, and wealth distributions, it is referred to as the Dagum distribution [2]. It is referred to as the inverse Burr distribution in actuarial literature [3] and the Kappa distribution in meteorological literature [4]. The BIIID has several applications in statistical modeling fields. It has also been employed in reliability theory [5]. Gove et al. [6] fitted BIIID to data related to forestry. Lindsay et al. [7] employed BIIID as an alternative to the Weibull distribution to simulate

* Corresponding author.
E-mail addresses: amal52_soliman@cu.edu.eg (A.S. Hassan), Elsayed.Elsherpieny@cu.edu.eg (E.A. Elsherpieny), 12422022546163@pg.cu.edu.eg (Wesal.E. Aghel).

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the diameter distributions of forest stands. BIIID was used by Shao [8] to model the NOEC toxicity data. Given that the BIIID can approximate a number of conventional lifetime models, including Weibull, gamma, and lognormal (see Zoraghi et al. [9]), it makes sense to model failure data with this distribution. Therefore, the BIIID is crucial to lifetime analysis.

The lifetime X of the product has BIIID if the cumulative distribution function, probability distribution function, and survival function for $x > 0$, have the following specifications:

$$F(x;\alpha,\beta) = \left[1 + x^{-\alpha}\right]^{-\beta}, \tag{1}$$

$$f(x;\alpha,\beta) = \alpha\beta x^{-(\alpha+1)}\left[1 + x^{-\alpha}\right]^{-(\beta+1)}, \tag{2}$$

and,

$$\bar{F}(x;\alpha,\beta) = 1 - [1 + x^{-\alpha}]^{-\beta}, \tag{3}$$

where $\alpha > 0$, and $\beta > 0$ are shape parameters. Some researchers offered some of the most significant studies about the BIIID. Altindag et al. [10] presented the estimation and prediction problems for the BIIID with Type II censored samples. The statistical inference of a BIIID was established by Panahi [11] using the unified hybrid censored sample. Gamchi et al. [12] conducted research on the estimation and prediction problems for the BIIID under progressive Type II hybrid censored data. A maximum likelihood (ML) estimation of the lifetime performance index for BIIID, based on progressive censoring, was taken into consideration by Hassan et al. [13]. Based on unified progressive hybrid censoring, Dutta and Kayal [14] presented estimation and prediction for the BIIID.

The joint progressive Type-II censoring scheme

The most common censorship techniques employed are Types I and II. A Type I censoring approach is one in which the observations are terminated at a predetermined time and the failure timings are recorded. When a sufficient and predetermined number of units fail, the observations are terminated, which is the definition of a Type II censoring method. Although the lifetimes of the tested units are relatively long, neither of these two censoring strategies performs well in this situation. Later, further censorship plans were put forth; the most well-liked and attractive of these are progressive censoring (PC) plans, which eliminate test units each time they fail rather than simply the last time. Information on the PC plans was provided by Balakrishnan and Aggarwala [15]. The two types of PC are progressive Type-I and progressive Type II. The PC Type I and Type II for some lifetime distributions have been researched by many academics [16,17,18,19,20,21].

The aforementioned censoring techniques are all applied to one-sample issues. In real life, we encounter and must take into account two or more samples from various assembly lines. The joint progressive Type II censoring (JP−IIC) scheme has drawn a lot of attention recently and is particularly helpful in comparing the lives of goods from various manufacturing lines. To analyze two populations from various exponential distributions, the JP−IIC scheme was first introduced by Balakrishnan and Rasouli [22]. For the JP−IIC method, Parsi et al. [23] explored the conditional ML and the interval estimators of the Weibull distribution. The Bayes estimator was produced by Doostparast et al. [24], when data were sampled using the JP−IIC scheme from a general class of distributions. Shafay et al. [25] addressed Bayesian inference based on a JP−IIC sample from two exponential populations. Balakrishnan et al. [26] considered a JP−IIC sample arising from k independent exponential populations. The procedure of estimating lifetime using multiple exponential and Weibull distributions was investigated by Mondal and Kundu [27,28]. Abo-Kasem [29] discussed inferences for two Rayleigh populations based on JP−IIC data. Krishna and Goel [30] discussed inferences for two Lindley populations, based on JP−IIC data. Chen and Gui [31] provided statistical inference of the generalized inverted exponential distribution via JP−IIC. Fan and Gui [32] studied the inference of an inverted exponential Rayleigh distribution under JP−IIC.

Motivated by the various applications of BIIID in many fields, in this study, we use a JP−IIC strategy to construct statistical inferences and evaluate two independent samples from BIIID. Point and interval estimators are obtained by using Bayesian and ML estimation procedures. On the basis of the observed information matrix, asymptotic confidence intervals (Asy-CIs) are then calculated. The CIs are computed using the Bootstrap-P and Bootstrap-T techniques. For both shape parameters, a gamma prior is assumed. Using the Metropolis-Hastings (MH) method, it is possible to obtain the Bayes estimates and credible intervals for the informative prior, under the squared error loss function (SELF), linear exponential loss function (LiLF), minimum expected loss function (MELF), and Degroot loss function (DeLF). To assess the effectiveness of various approaches, Monte Carlo simulation and actual data analysis are used.

The remainder of the paper is set up as follows. The model is described in Section 2 along with the ML estimators and the Asy-CIs of the BIIID parameters. Bootstrap-P, Bootstrap-T CIs and highest posterior density (HPD) credible intervals are provided in Section 3. Section 4 examines Bayesian estimation with gamma priors under some loss functions. Section 5 uses a simulation study and actual data sets as examples. Bootstrap-P, as well as Bootstrap-T CIs are provided in Section 3.

Estimating the maximum likelihood using the model

Assume that the lifetime of m units of product A, W_1, W_2,..., W_m are identically independent distributed (iid) random variables (RVs) possessing BIIID (α, β_1). The same is true for the lifetime of n units of product B, Z_1, Z_2,..., Z_n, which are iid RVs possessing BIIID (α, β_2).

The first step is to organize $N = m + n$ RVs in combined order, represented by $W_1 \le W_2 \le \ldots \le W_k$, then the JP−IIC implementation for the next two samples. N units are initially subjected to a lifetime experiment, and when the first failure occurs (either at W or Z), R_1 live units are subtracted from the remaining $N-1$ live units. When the second component (W or Z) fails, R_2 live units are subtracted from the remaining N, R_1, and R_2 live units, and so on. The remaining live $R_k = N - r - R_1 - R_2 - \ldots - R_{k-1}$ units are pulled from the test upon the occurrence of the k-th failure, which may come from W or Z.

In this case, the total number of failures k and the censoring strategy $R=(R_1,R_2,\ldots,R_k)$ are pre-fixed before the experiment is run, where $R_i = S_i+T_i$ and S_i and T_i are unknown RVs denoting the number of units removed from the A and B populations, respectively, at the time of the ith failure. As a result, the collected data are represented as (W,Z,S), where $W=(W_1,W_2,\ldots,W_k)$ with $1 \le k \le N$and $Z=(Z_1,Z_2,\ldots,Z_k)$ are defined as:

$$Z_i = \begin{cases} 1; & if\ W_i \in A \text{ population} \\ 0; & 0.w. \end{cases}$$

In other words, Z_i only accepts one of two values, 1 or 0, depending on whether W_i is a W failure or a Z failure. We further divide the censoring scheme $\underline{R}=(R_1, R_2,\ldots,R_k)$ into $S + T=(S_1, S_2,\ldots,S_k) + (T_1,T_2,\ldots,T_k)$. For $R_1=R_2=\ldots=R_{k-1} = 0$, $R_k=N-k$, $S_k= m - k_1$ and $T_k= n - k_2$ the results are very consistent with joint Type II.

Here $k_1 = \sum_{i=1}^{k} z_i$, this is case refers to the number of failures from line A.

Similarly, $k_2 = \sum_{i=1}^{k} (1 - z_i) = k - k_1$ represents the quantity of failures from line B.

The likelihood function (LF), based on (1), (2), (3) for Z and W, in case of JP−IIC scheme can be written as

$$L(\phi, data) = C \prod_{i=1}^{k} \left[\{\bar{F}(w_i)\}^{s_i} \{\bar{G}(w_i)\}^{t_i} \{f(w_i)\}^{z_i} \{g(w_i)\}^{(1-z_i)} \right], \tag{4}$$

where $0 \le w_1 \le w_2 \le \ldots \le w_k, \phi = (\beta_1, \beta_2, \alpha), \bar{F}(.) = 1 - F(.), \bar{G}(.) = 1 - G(.)$,and

$$C = \prod_{j=1}^{k} \left[z_j \left(m - \sum_{i=1}^{j-1} s_i - \sum_{i=1}^{j-1} z_i \right) + (1 - z_j) \left(n - \sum_{i=1}^{j-1} (R_i - s_i) - \sum_{i=1}^{j-1} (1 - z_i) \right) \right]$$
$$\times \prod_{j=1}^{k-1} \binom{m - \sum_{i=1}^{j-1} s_i - \sum_{i=1}^{j-1} z_i}{s_j} \binom{n - \sum_{i=1}^{j-1} (R_i - s_i) - \sum_{i=1}^{j-1} (1 - z_i)}{t_j} \binom{m - j - n - \sum_{i=1}^{j-1} R_i}{R_j}^{-1}.$$

The LF (4) can be written as:

$$L(\phi, data) = C\alpha^k \beta_1^{k_1} \beta_2^{k_2} \prod_{i=1}^{k} w_i^{-(\alpha+1)} D_i(\alpha)^{-[\beta_1 z_i + \beta_2 (1-z_i) + 1]} \left[1 - (D_i(\alpha))^{-\beta_1} \right]^{s_i} \left[1 - (D_i(\alpha))^{-\beta_2} \right]^{t_i}, \tag{5}$$

where$D_i(\alpha) = (1 + w_i^{-\alpha})$. The log-LF of (5), denoted by l^*, is written as

$$l^* = k\,ln\,\alpha + k_1 ln\beta_1 + k_2 ln\beta_2 - (\alpha + 1) \sum_{i=1}^{k} lnw_i - \sum_{i=1}^{k} [\beta_1 z_i + \beta_2 (1 - z_i) + 1]\, ln\, D_i(\alpha)$$
$$+ \sum_{i=1}^{k} \left[s_i ln\,(1 - D_i(\alpha)^{-\beta_1}) + t_i\, ln\,\,(1 - D_i(\alpha)^{-\beta_2}) \right].$$

The first partial derivative of β_1, β_2 and$\alpha$are obtained, respectively, as follows:

$$\frac{\partial l^*}{\partial \beta_1} = \frac{k_1}{\beta_1} - \sum_{i=1}^{k} z_i ln(D_i(\alpha)) + \sum_{i=1}^{k} \frac{s_i ln(D_i(\alpha))}{(D_i(\alpha))^{\beta_1} - 1}, \tag{6}$$

$$\frac{\partial l^*}{\partial \beta_2} = \frac{k_2}{\beta_2} - \sum_{i=1}^{k} (1 - z_i)\, ln(D_i(\alpha)) + \sum_{i=1}^{k} \frac{t_i ln(D_i(\alpha))}{(D_i(\alpha))^{\beta_2} - 1}, \tag{7}$$

$$\frac{\partial l^*}{\partial \alpha} = \frac{k}{\alpha} - \sum_{i=1}^{k} ln\,w_i - \sum_{i=1}^{k} \frac{[\beta_1 z_i + \beta_2 (1-z_i) + 1] ln\,w_i}{1 + w_i^{\alpha}} - \sum_{i=1}^{k} \frac{s_i \beta_1 (D_i(\alpha))^{-1} w_i^{-\alpha} ln\,w_i}{[(D_i(\alpha))^{\beta_1} - 1]}$$
$$- \sum_{i=1}^{k} \frac{t_i \beta_2 (D_i(\alpha))^{-1} w_i^{-\alpha} ln\,w_i}{\left[(D_i(\alpha))^{\beta_2} - 1 \right]}. \tag{8}$$

The ML estimators of β_1, β_2 and$\alpha$are obtained by settling (6)–(8) with zero and solving numerically via R-statistical programming language.

Furthermore, for evaluating estimated variance-covariance matrix (VCM) and related asymptotic CIs of ML estimators, with JP−IIC data, the observed Fisher information matrix (FIM) is defined as:

$$\hat{\hat{I}}(\phi) = \begin{bmatrix} \frac{-\partial^2 l^*}{\partial \beta_1^2} & \frac{\partial^2 l^*}{\partial \beta_1 \beta_2} & \frac{-\partial^2 l^*}{\partial \beta_1 \partial \alpha} \\ \frac{\partial^2 l^*}{\partial \beta_2 \beta_1} & \frac{\partial^2 l^*}{\partial \beta_2^2} & \frac{-\partial^2 l^*}{\partial \beta_2 \partial \alpha} \\ \frac{-\partial^2 l^*}{\partial \alpha \partial \beta_1} & \frac{-\partial^2 l^*}{\partial \alpha \partial \beta_2} & \frac{-\partial^2 l^*}{\partial \alpha^2} \end{bmatrix}_{\substack{\beta_1 = \hat{\hat{\beta}}_1 \\ \beta_2 = \hat{\hat{\beta}}_2 \\ \alpha = \hat{\hat{\alpha}}}} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{\substack{\beta_1 = \hat{\hat{\beta}}_1 \\ \beta_2 = \hat{\hat{\beta}}_2 \\ \alpha = \hat{\hat{\alpha}}}} = \begin{bmatrix} \hat{\hat{I}}_{11} & \hat{\hat{I}}_{12} & \hat{\hat{I}}_{13} \\ \hat{\hat{I}}_{21} & \hat{\hat{I}}_{22} & \hat{\hat{I}}_{23} \\ \hat{\hat{I}}_{31} & \hat{\hat{I}}_{32} & \hat{\hat{I}}_{33} \end{bmatrix}$$

Note that the expressions for second-order partial derivatives are provided in Appendix. The $(1-\varepsilon)$ 100% Asy-CIs for$\phi = (\beta_1, \beta_2, \alpha)$ by using the approximated standard normal distribution are given by

$$\hat{\hat{\phi}} \pm Z^*_{\varepsilon/2}\sqrt{\widehat{\mathrm{var}}(\hat{\hat{\phi}})},$$

where $Z^*_{\varepsilon/2}$denoted the upper $\varepsilon/2$ percent point of standard normal distribution and $\widehat{\mathrm{var}}(\hat{\hat{\phi}})$ is the estimated variance.

Bootstrap methods

The construction of CIs is done in this part using the Bootstrap methodology. Bootstrap-P (Boot-P) and Bootstrap-T (Boot-T) approaches' algorithms are provided in Algorithms 1 and 2, respectively.

$$\left[\hat{\hat{\phi}} - \sqrt{Var\left(\hat{\hat{\phi}}\right)}\, T_{\phi(hb)}, \hat{\hat{\phi}} - \sqrt{Var\left(\hat{\hat{\phi}}\right)}\, T_{\phi(Ib)}\right],$$

where $Ib = [\frac{\varepsilon}{2}N], hb = [\frac{1-\varepsilon}{2}N]$. Here $[x]$ means the largest integer not exceeding x.

Bayesian estimation

Bayes estimate takes into account the previous knowledge of life factors, in contrast to classical statistics. As a result, Bayesian estimation considers both the available data and the prior probability to infer the relevant parameters.

According to different gamma distributions, that β_1, β_2 and α are independent, where

$$\pi_i(\beta_i) = \frac{b_i^{a_i}}{\Gamma(a_i)}\beta_i^{a_i-1}e^{-b_i\beta_i}, \qquad a_i,\ \ b_i,\ \beta_i > 0,\ \ i = 1, 2,$$

$$\pi_3(\alpha) = \frac{d^c}{\Gamma(c)}\alpha^{c-1}e^{-d\alpha}, \qquad \alpha > 0, c, d > 0,$$

where $a_i, b_i, c, d, i = 1, 2$are the hyper-parameters that contain the prior information. The joint prior distribution can be written as.

$$\pi_0(\phi) \propto \beta_1^{a_1-1}\beta_2^{a_2-1}\alpha^{c-1}e^{-(b_1\beta_1+b_2\beta_2+d\alpha)}.$$

The joint posterior probability distribution is

$$\pi(\phi|data) = \frac{\pi_0(\phi)L(\phi, data)}{\int_0^\infty\int_0^\infty\int_0^\infty \pi_0(\phi)L(\phi, data)d\beta_1 d\beta_2 d\alpha}. \tag{9}$$

The denominator of (9) is a function of the observed data. Thus, $L(\phi, data)$ and$\pi(\phi|data)$have a coefficient-proportional relationship. Therefore, the joint posterior probability distribution is

$$\pi(\phi|data) \propto \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}e^{-(b_1\beta_1+b_2\beta_2+d\alpha)} \times \exp-\left(\beta_1\sum_{i=1}^{k}\psi_{1i}(\alpha, \beta_1) + \beta_2\sum_{i=1}^{k}\psi_{2i}(\alpha, \beta_2) + \alpha\sum_{i=1}^{k}\psi_{3i}(\alpha)\right), \tag{10}$$

where

$$\psi_{1i}(\alpha, \beta_1) = z_i \ln D_i(\alpha) - s_i \ln(1-(D_i(\alpha))^{-\beta_1}),$$
$$\psi_{2i}(\alpha, \beta_2) = (1-z_i)\ \ln\ D_i(\alpha) - t_i \ln(1-(D_i(\alpha))^{-\beta_2}),$$
$$\psi_{3i}(\alpha) = \ln w_i + \ln D_i(\alpha).$$

Hence, the marginal posterior distributions of β_1, β_2 and αtake the following forms:

$$\pi^*(\beta_1|\beta_2, \alpha, data) = \int_0^\infty\int_0^\infty \pi(\phi|data)d\beta_2 d\alpha$$
$$= N\beta_1^{k_1+a_1-1}\int_0^\infty\int_0^\infty \beta_2^{k_2+a_2-1}\alpha^{k+c-1}\, e^{-\left(\beta_1\left(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^k \psi_{3i}(\alpha)+d\right)\right)} d\beta_2 d\alpha,$$
$$\pi^{**}(\beta_2|\beta_1, \alpha, data) = \int_0^\infty\int_0^\infty \pi(\phi|data)d\beta_1 d\alpha$$
$$= N\beta_2^{k_2+a_2-1}\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\alpha^{k+c-1} e^{-\left(\beta_1\left(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^k \psi_{3i}(\alpha)+d\right)\right)} d\beta_1 d\alpha,$$
$$\pi^{**}(\beta_2|\beta_1, \alpha, data) = \int_0^\infty\int_0^\infty \pi(\phi|data)d\beta_1 d\alpha$$
$$= N\beta_2^{k_2+a_2-1}\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\alpha^{k+c-1} e^{-\left(\beta_1\left(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^k \psi_{3i}(\alpha)+d\right)\right)} d\beta_1 d\alpha,$$

where, $N^{-1} = \int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}\, e^{-(\beta_1(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1)+\beta_2(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2)+\alpha(\sum_{i=1}^k \psi_{3i}(\alpha)+d))} d\beta_1 d\beta_2 d\alpha.$

In the following, the Bayesian estimators are provided under different loss functions.

Loss functions

Here the Bayesian estimators of $\phi = (\beta_1, \beta_2, \alpha)$are all obtained under symmetric and asymmetric loss functions.

(1) Squared Error loss function

One of the helpful symmetric loss functions seen in nature, a quadratic or SELF, prioritizes both over and under estimation equally. A definition of the SELF is

$$L_{SELF}(\hat{\hat{\phi}}, \phi) = (\hat{\hat{\phi}} - \phi)^2.$$

Therefore, the Bayesian estimators of β_1, β_2,and α under SELF, say $\hat{\hat{\beta}}_{1(SE)}, \hat{\hat{\beta}}_{2(SE)}$ and $\hat{\hat{\alpha}}_{(SE)}$ are obtained as a posterior mean as follows

$$\hat{\hat{\beta}}_{1(SE)} = E(\beta_1|\underline{w}) = N\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2\right)\right)} \times e^{-\alpha\left(\sum_{i=1}^k \psi_{3i}(\alpha)+d\right)} d\beta_1 d\beta_2 d\alpha, \tag{11}$$

$$\hat{\hat{\beta}}_{2(SE)} = E(\beta_2|\underline{w}) = N\int_0^\infty\int_0^\infty\int_0^\infty \beta_2^{k_2+a_2}\beta_1^{k_1+a_1-1}\alpha^{k+c-1}\, e^{-\left(\beta_1\left(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2\right)\right)} \times e^{-\alpha\left(\sum_{i=1}^k \psi_{3i}(\alpha)+d\right)} d\beta_1 d\beta_2 d\alpha, \tag{12}$$

$$\hat{\hat{\alpha}}_{(SE)} = E(\alpha|\underline{w}) = N\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{k+c}\beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}e^{-\left(\beta_1\left(\sum_{i=1}^k \psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^k \psi_{2i}(\alpha,\beta_2)+b_2\right)\right)} \times e^{-\alpha\left(\sum_{i=1}^k \psi_{3i}(\alpha)+d\right)} d\beta_1 d\beta_2 d\alpha. \tag{13}$$

(1) Linear - Exponential loss function

The LiLF is the most universally used asymmetric loss function. The asymmetric loss function is considered more comprehensive in many respects. The LiLF is given below:

$$l_{LiLF}(\phi, \hat{\hat{\phi}}) = e^{h(\hat{\hat{\phi}}-\phi)} - h(\hat{\hat{\phi}} - \phi) - 1, h \neq 0,$$

where $\hat{\hat{\phi}}$is an estimate of $\phi$and h is the real number and stands for the sign, which presents the asymmetry. The corresponding Bayesian estimate $\hat{\hat{\phi}}_{LiLF}$ of ϕ can be derived from

$$\hat{\hat{\phi}}_{LiLF} = -\frac{1}{h}ln[E(e^{-h\phi}|\underline{w})].$$

Hence, the Bayesian estimators of β_1, β_2,and α under LiLF, say $\hat{\hat{\beta}}_{1(LiLF)}$, $\hat{\hat{\beta}}_{2(LiLF)}$ and $\hat{\hat{\alpha}}_{(LiLF)}$ are obtained as follows:

$$\hat{\hat{\beta}}_{1(LiLF)} = \frac{-1}{h} ln\left[N\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\alpha^{k+c-1}\beta_2^{k_2+a_2-1} \times e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1+h\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha\right], \tag{14}$$

$$\hat{\hat{\beta}}_{2(LiLF)} = \frac{-1}{h} ln\left[N\int_0^\infty\int_0^\infty\int_0^\infty \alpha^{k+c-1}\beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1} \times e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2+h\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha\right], \tag{15}$$

$$\hat{\hat{\alpha}}_{(LiLF)} = \frac{-1}{h} ln\left[N\int_0^\infty\int_0^\infty\int_0^\infty \beta_2^{k_2+a_2-1}\alpha^{k+c-1}\beta_1^{k_1+a_1-1} \times e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d+h\right)\right)}d\beta_1 d\beta_2 d\alpha\right]. \tag{16}$$

(1) Minimum expected loss function

The MELF was developed in Tummala and Sathe [33], which is defined by:

$$L_{MELF}(\hat{\hat{\phi}}, \phi) = \frac{(\hat{\hat{\phi}} - \phi)^2}{\phi^2},$$

where, $\hat{\hat{\phi}}_{MELF}$ is an estimator of ϕ. Hence, the Bayesian estimators of $\beta_1, \beta_2$and α under MELF, $\hat{\hat{\beta}}_{1(MELF)}, \hat{\hat{\beta}}_{2(MELF)}$ and $\hat{\hat{\alpha}}_{(MELF)}$, are derived as follows:

$$\hat{\hat{\beta}}_{1(MELF)} = \frac{E(\beta_1^{-1}|\underline{w})}{E(\beta_1^{-2}|\underline{w})} = \left[\frac{\int_0^\infty \beta_1^{-1}\pi^*(\beta_1|\beta_2,\alpha,data)d\beta_1}{\int_0^\infty \beta_1^{-2}\pi^*(\beta_1|\beta_2,\alpha,data)d\beta_1}\right],$$

$$\hat{\hat{\beta}}_{1(MELF)} = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-2}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-3}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}\right], \tag{17}$$

$$\hat{\hat{\beta}}_{2(MELF)} = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-2}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-3}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}\right], \tag{18}$$

$$\hat{\hat{\alpha}}_{(MELF)} = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}\alpha^{k+c-2}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}\alpha^{k+c-3}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}\right]. \tag{19}$$

(1) Degroot loss function

The DeLF was provided in Degroot [34], as follows:

$$L_{DeLF}(\hat{\hat{\phi}}, \phi) = \frac{(\hat{\hat{\phi}} - \phi)^2}{\phi^2},$$

where, $\hat{\hat{\phi}}_{DeLF}$ is an estimator of ϕ.Hence, the Bayesian estimators of β_1, β_2 andα under DeLF, $\hat{\hat{\beta}}_{1(DeLF)}, \hat{\hat{\beta}}_{2(DeLF)}$,and $\hat{\hat{\alpha}}_{(DeLF)}$ are derived as follows:

$$\hat{\hat{\beta}}_{1(DeLF)} = \frac{E(\beta_1^2|\underline{w})}{E(\beta_1|\underline{w})} = \left[\frac{\int_0^\infty \beta_1^2 \pi^*(\beta_1|\beta_2, \alpha, data)d\beta_1}{\int_0^\infty \beta_1 \pi^*(\beta_1|\beta_2, \alpha, data)d\beta_1}\right],$$

$$\hat{\hat{\beta}}_{1(DeLF)} = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1+1}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1}\beta_2^{k_2+a_2-1}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}\right], \quad (20)$$

$$\hat{\hat{\beta}}_{2(DeLF)} = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2+1}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2}\alpha^{k+c-1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}\right], \quad (21)$$

$$\hat{\hat{\alpha}}_{(DeLF)} = \left[\frac{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}\alpha^{k+c+1}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k_1+a_1-1}\beta_2^{k_2+a_2-1}\alpha^{k+c}e^{-\left(\beta_1\left(\sum_{i=1}^{k}\psi_{1i}(\alpha,\beta_1)+b_1\right)+\beta_2\left(\sum_{i=1}^{k}\psi_{2i}(\alpha,\beta_2)+b_2\right)+\alpha\left(\sum_{i=1}^{k}\psi_{3i}(\alpha)+d\right)\right)}d\beta_1 d\beta_2 d\alpha}\right]. \quad (22)$$

Integrals (11) to (22) are difficult to obtain; therefore, the MH algorithm is employed to generate MCMC samples from posterior density functions (10). After acquiring Markov chain Monte Carlo (MCMC) samples from the posterior distribution, we can get the Bayes estimate of $\beta_1, \beta_2$and α.

Highest Posterior Density Credible Interval (HPD)

An approach provided by Chen and Shao [35] is utilized to create the HPD credible intervals forϕ.The MCMC samples$\phi_1, \phi_2, ..., \phi_M$ are taken into consideration for generating the $(1-\varepsilon)$ 100% symmetric HPD credible intervals forϕ. These samples are arranged in ascending order$\phi_{(1)}, \phi_{(2)}, ..., \phi_{(M)}$, and has an HPD credible interval of $[\phi_{[(M(\varepsilon/2))]}, \phi_{[(M(1-\varepsilon/2))]}]$.

Simulation study and data analysis

Analyzing the effectiveness of the different estimating methods described in the parts above is the goal of this section. A simulation study is used to assess the statistical performances of the different estimates given a JP−IIC scheme for BIIID. A real dataset is used for illustration purposes in order to investigate the behavior of the recommended approaches. The R statistical programming language has been used to do the calculations. With the help of the bbmle and HPDInterval packages, MLEs and HPD intervals are calculated in R.

Simulation study

A Monte Carlo simulation analysis is used in this section, using the JP−IIC scheme for BIIID, to assess the effectiveness of the ML and Bayesian estimation methods. Based on the subsequent hypotheses, 1000 observations for the MLEs are obtained from BIIID:

1 Assume the following selected values of parameters,α=0.5,β_1=1.5,β_2=1.75
2 The sum of sample sizes ($N = n + m$) of the two samples are given as: N = 80, 160, and 240, where $n = m$.
3 Removed items R_i are assumed at different sample sizes N and number of stages k.
4 Five different censorning schemes, namely S_1, S_2, S_3, S_4, and S_5 are selected as shown in Table 1.

The MLEs and related 95% Asy-CI are produced based on the generated data. When deriving MLEs, be aware that the initial assumed values are regarded as true parameter values. Also, Boot- P and Boot-T are computed.

We compute Bayesian estimates (BEs) using gamma (informative prior (IP)) for the Bayesian estimation method. As historical samples, we construct 500 completed samples of size 60 each from the BIIID, then determine the values of the hyper-parameter given below as:

$$a_1 = 91.08, b_1 = 179.09, a_2 = 50.84, b_2 = 33.44, c = 59.06, d = 33.16.$$

Table 1
Patterns for removing items in test at different stage numbers.

(n, m)	k	Censoring Schemes				
		S1	S2	S3	S4	S5
(40,40)	20	$(60, 0 \times {}^{19})$	$(30, 0 \times {}^{18}, 30)$	$(0 \times {}^{9},30,30, 0 \times {}^{9})$	$(0 \times {}^{19}, 60)$	$(3 \times {}^{20})$
	40	$(40, 0 \times {}^{39})$	$(20, 0 \times {}^{38}, 20)$	$(0 \times {}^{19},20,20, 0 \times {}^{19})$	$(0 \times {}^{39}, 40)$	$(1 \times {}^{40})$
(80,80)	40	$(120, 0 \times {}^{39})$	$(60, 0 \times {}^{38}, 60)$	$(0 \times {}^{19},60,60, 0 \times {}^{19})$	$(0 \times {}^{39}, 120)$	$(3 \times {}^{40})$
	80	$(80, 0 \times {}^{79})$	$(40, 0 \times {}^{78}, 40)$	$(0 \times {}^{39},40,40, 0 \times {}^{39})$	$(0 \times {}^{79}, 80)$	$(1 \times {}^{80})$
(120,120)	80	$(160, 0 \times {}^{79})$	$(80, 0 \times {}^{78}, 80)$	$(0 \times {}^{39},80,80, 0 \times {}^{39})$	$(0 \times {}^{79}, 160)$	$(2 \times {}^{80})$
	120	$(120, 0 \times {}^{119})$	$(60, 0 \times {}^{118}, 60)$	$(0 \times {}^{59},60,60, 0 \times {}^{59})$	$(0 \times {}^{119}, 120)$	$(1 \times {}^{120})$

Here, $(5 \times {}^{3},0)$, for example, means that the censoring scheme employed is (5,5,5,0).

To analyze the necessary estimates, such IP values are fed in. The MLEs are used as starting guess values when the MH algorithm is implemented, along with the associated VCM S_ϕ of $(\ln(\hat{\hat{\alpha}}), \ln(\hat{\hat{\beta}}_1), \ln(\hat{\hat{\beta}}_2))$. In the end, 2000 burn-in samples were deleted from the total of 10,000 generated samples by the posterior density, and produced BEs under different loss functions, namely: SELF, LiLF at $h = 0.5$, MELF, and DeLF. Also, HPD interval estimates have been computed according to the technique of Chen and Shao [35].

All the average estimates for methods are reported in Table 2(a,b,c) for total sample sizes $N = 80$, 160, and 240, respectively. Further, the first column denotes the average estimates (AVEs), and, in the second column, related mean squared errors (MSEs). For CIs, we have Asy-CI for MLEs and HPD for BEs based on MCMC, which is reported in Table 3(a,b,c) for given parameter values and N. Further, the first column represents average interval lengths (AILs), and, in the second column, related coverage probabilities (CPs).

From the tabulated results, one can conclude that:

- As n and m increase, the accuracy of the estimated values increases.
- Scheme S_3 usually gives the highest MSE values over the S_2 and S_4 schemes, which sometimes give peaks in MSE values over the average estimated parameters.
- The ML method gives larger MSE values over the estimated parameters compared to the Bayesian method.
- In approximately most situations, the MSEs of all parameter estimates based on Scheme S_1 usually give the smallest values compared to the other schemes.
- Estimates of $\beta_2$usually provide the largest MSE values, and the range of AVE values is [1.64 – 2.86].
- From Table 2(a, b, c), we can find that the MLEs perform better in terms of the MSE as the value of k increases. The MSEs of αestimates are always much smaller than those of β_1 and β_2 estimates which means αestimates are better. This is reasonable because α is the same in the two populations.
- The BEs using MCMC for different schemes under DeLF provide better estimates in terms of MSE than the others under MELF, LiLF, and SELF, for total sample sizes $N = 80$, 160, and 240, respectively (see Table 2(a, b, c)).
- The BEs have good bias and MSE under SELF, LiLF, DeLF, and MELF compared to MLEs. Nonetheless, the BEs using MCMC under DeLF frequently perform better than those under SELF, LiLF, and MELF in terms of bias.
- The BEs of all parameters under DeLF are preferable to the others in almost all situations.
- The quality of Bayesian and ML methods based on MSE is listed in the following order:

  DeLF→MLE→MELF→LiLF→ SELF in most situations.

- The coverage probability for the parameters in the case of interval estimation is 95% of their nominal values.
- Table 3 (a, b, c) display the AIL and CP of confidence intervals for all methods. The contrast between Boot-P and Boot−T indicates that the AILs of Boot-T are wider than those of Boot-P. Therefore, the Boot-P method is more appropriate to get the confidence intervals than the Boot-T.
- The AIL of the Asy-CI is larger than the HPD credible intervals.
- The quality of CI methods, based on the average shortest interval, is listed in the following order:

  The HPD credible interval (Good)→Asy-CI→Boot-P→Boot-T (worst), in approximately most situations.

- The convergence of MCMC estimation in the case of scheme S_1 of JP-IIC can be shown for α, β_1 and and is represented in Fig. 1. This figure showed a scatter plot, histogram, and cumulative mean of posterior samples for each estimated parameter, which showed the normality of the generated posterior samples.

Data analysis

The strength values provided by Badar and Priest [36] are analysed in this sub-section. For single carbon fibres and impregnated 1000-carbon fiber tows, the strength measurements are represented using GPA units. Tension tests on single fibres were conducted at gage lengths of 1, 10, 20, and 50 mm. At gage lengths of 20, 50, 150, and 300 mm, 1000-fiber impregnated tows were tested.

Table 2
(a): AVE values and MSEs of both estimation methods under JP−IIC for BIIID at $m = 40$ and $n = 40$.

Sch.	Para-meter	MLE AVE	MSE	MCMC-SELF AVE	MSE	MCMC-LiLF AVE	MSE	MCMC-DeLF AVE	MSE	MCMC-MELF AVE	MSE
$k = 20$											
s_1	α	0.5400	0.0100	0.5107	0.0008	0.5102	0.0008	0.5028	0.0007	0.5146	0.0009
	β_1	1.7166	0.5471	1.5213	0.0158	1.5141	0.0154	1.4838	0.0154	1.5402	0.0170
	β_2	2.0563	0.4919	1.7877	0.0093	1.7770	0.0084	1.7395	0.0078	1.8120	0.0118
s_2	α	0.6997	0.4295	0.5120	0.0012	0.5153	0.0028	0.5044	0.0006	0.5164	0.0014
	β_1	1.0724	0.2796	1.3866	0.0251	1.3803	0.0267	1.3497	0.0386	1.4054	0.0199
	β_2	1.9536	0.3725	1.8484	0.0257	1.8369	0.0240	1.7987	0.0347	1.8756	0.0484
s_3	α	0.3241	0.0795	0.3752	0.0284	0.3749	0.0285	0.3687	0.0299	0.3785	0.0278
	β_1	2.0727	0.8031	1.6136	0.0395	1.6064	0.0376	1.5774	0.0321	1.6318	0.0442
	β_2	2.6880	1.7897	1.9889	0.1198	1.9781	0.1137	1.9452	0.0993	2.0107	0.1314
s_4	α	0.6949	0.2886	0.5109	0.0006	0.5104	0.0006	0.5017	0.0004	0.5151	0.0007
	β_1	1.1633	0.1995	1.4133	0.0174	1.4443	0.6053	1.4257	0.9752	1.4298	0.0146
	β_2	1.7463	0.2019	1.8095	0.0225	1.8008	0.0213	1.7675	0.0259	1.8292	0.0242
s_5	α	0.5775	0.0295	0.5081	0.0007	0.5076	0.0007	0.4996	0.0006	0.5123	0.0008
	β_1	1.2864	0.1317	1.4327	0.0104	1.4266	0.0112	1.3985	0.0160	1.4499	0.0085
	β_2	1.9355	0.1993	1.8276	0.0166	1.8185	0.0151	1.7877	0.0116	1.8476	0.0204
$k = 40$											
s_1	α	0.5210	0.0047	0.5100	0.0009	0.5096	0.0008	0.5037	0.0008	0.5132	0.0009
	β_1	1.5964	0.1537	1.5185	0.0135	1.5110	0.0131	1.4790	0.0134	1.5384	0.0148
	β_2	1.8701	0.2024	1.7661	0.0100	1.7568	0.0096	1.7238	0.0100	1.7873	0.0114
s_2	α	0.5645	0.0120	0.5180	0.0011	0.5176	0.0011	0.5113	0.0009	0.5213	0.0013
	β_1	1.1287	0.1967	1.3594	0.0292	1.3532	0.0309	1.3230	0.0405	1.3777	0.0247
	β_2	2.2373	0.4578	1.9056	0.0372	1.8965	0.0341	1.8673	0.0261	1.9248	0.0438
s_3	α	0.3500	0.0525	0.3943	0.0201	0.3940	0.0201	0.3896	0.0208	0.3966	0.0197
	β_1	1.6468	0.1808	1.5276	0.0187	1.5205	0.0181	1.4901	0.0175	1.5465	0.0203
	β_2	2.3043	0.7391	1.9248	0.0799	1.9154	0.0761	1.8860	0.0669	1.9442	0.0876
s_4	α	0.5954	0.0202	0.5204	0.0012	0.5199	0.0012	0.5132	0.0009	0.5240	0.0013
	β_1	0.8691	0.4253	1.2547	0.0663	1.2492	0.0690	1.2193	0.0846	1.2724	0.0581
	β_2	2.2315	0.3705	1.9640	0.0598	1.9556	0.0560	1.9296	0.0457	1.9812	0.0677
s_5	α	0.5367	0.0066	0.5125	0.0008	0.5121	0.0008	0.5058	0.0006	0.5159	0.0009
	β_1	1.2751	0.1103	1.4071	0.0175	1.4009	0.0185	1.3716	0.0250	1.4249	0.0147
	β_2	1.9731	0.1713	1.8324	0.0207	1.8241	0.0192	1.7961	0.0155	1.8506	0.0243

(b): AVE values and MSEs of both estimation methods under JP−IIC for BIIID at $m = 80$ and $n = 80$

Sch.	Para-meter	MLE AVE	MSE	MCMC-SELF AVE	MSE	MCMC-LiLF AVE	MSE	MCMC-DeLF AVE	MSE	MCMC-MELF AVE	MSE
$k = 40$											
s_1	α	0.5306	0.0061	0.5152	0.0011	0.5148	0.0011	0.5091	0.0009	0.5182	0.0012
	β_1	1.6106	0.2116	1.5374	0.0237	1.5304	0.0230	1.5010	0.0222	1.5556	0.0255
	β_2	1.8909	0.1637	1.7831	0.0107	1.7741	0.0100	1.7428	0.0094	1.8034	0.0126
s_2	α	0.5928	0.0214	0.5197	0.0010	0.5193	0.0010	0.5126	0.0007	0.5233	0.0011
	β_1	1.0886	0.2141	1.3232	0.0406	1.3181	0.0423	1.2924	0.0521	1.3386	0.0356
	β_2	2.0169	0.2000	1.9079	0.0424	1.9005	0.0399	1.8769	0.0332	1.9235	0.0478
s_3	α	0.2796	0.0868	0.3199	0.0485	0.3197	0.0486	0.3157	0.0498	0.3219	0.0480
	β_1	2.1925	0.9583	1.7861	0.1635	1.7789	0.1586	1.7540	0.1453	1.8021	0.1734
	β_2	2.8350	2.1002	2.2034	0.3639	2.1939	0.3536	2.1695	0.3323	2.2204	0.3806
s_4	α	0.6055	0.0451	0.5151	0.0008	0.5269	0.0585	0.5062	0.0012	0.5188	0.0010
	β_1	1.2113	0.1439	1.4272	0.1226	1.3723	0.0280	1.3476	0.0386	1.3950	0.0201
	β_2	1.8028	0.1125	1.8372	0.0262	1.8296	0.0259	1.8055	0.0292	1.8568	0.0277
s_5	α	0.5468	0.0103	0.5120	0.0008	0.5115	0.0008	0.5053	0.0007	0.5153	0.0009
	β_1	1.2562	0.1005	1.3823	0.0222	1.3774	0.0233	1.3538	0.0294	1.3966	0.0192
	β_2	1.9001	0.1028	1.8457	0.0222	1.8388	0.0208	1.8160	0.0170	1.8606	0.0255
$k = 80$											
s_1	α	0.5119	0.0025	0.5086	0.0008	0.5084	0.0008	0.5042	0.0008	0.5109	0.0009
	β_1	1.5589	0.0811	1.5192	0.0156	1.5133	0.0152	1.4884	0.0149	1.5346	0.0166
	β_2	1.8190	0.0844	1.7687	0.0126	1.7612	0.0121	1.7349	0.0120	1.7856	0.0137
s_2	α	0.5372	0.0043	0.5163	0.0010	0.5160	0.0010	0.5117	0.0009	0.5187	0.0011
	β_1	1.1192	0.1698	1.2927	0.0511	1.2884	0.0528	1.2659	0.0627	1.3062	0.0458
	β_2	2.1577	0.2460	1.9586	0.0593	1.9517	0.0563	1.9303	0.0479	1.9727	0.0657
s_3	α	0.3186	0.0604	0.3472	0.0369	0.3470	0.0369	0.3443	0.0375	0.3486	0.0366
	β_1	1.6737	0.1789	1.5723	0.0415	1.5668	0.0404	1.5444	0.0377	1.5862	0.0440
	β_2	2.3554	0.7823	2.0577	0.2011	2.0501	0.1954	2.0284	0.1825	2.0724	0.2110
s_4	α	0.5784	0.0104	0.5290	0.0016	0.5287	0.0015	0.5237	0.0013	0.5317	0.0017
	β_1	0.8685	0.4109	1.1487	0.1284	1.1450	0.1309	1.1231	0.1468	1.1615	0.1197
	β_2	2.2189	0.2929	2.0441	0.1046	2.0380	0.1008	2.0200	0.0906	2.0562	0.1120
s_5	α	0.5245	0.0035	0.5121	0.0009	0.5118	0.0009	0.5073	0.0008	0.5145	0.0010
	β_1	1.2528	0.0879	1.3579	0.0290	1.3535	0.0302	1.3320	0.0368	1.3709	0.0256
	β_2	1.9862	0.1104	1.8817	0.0314	1.8754	0.0296	1.8552	0.0247	1.8950	0.0352

(*continued on next page*)

Table 2 (*continued*)

Sch.	Para-meter	MLE AVE	MSE	MCMC-SELF AVE	MSE	MCMC-LiLF AVE	MSE	MCMC-DeLF AVE	MSE	MCMC-MELF AVE	MSE
(c): AVE values and MSEs of both estimation methods under JP−IIC for BIIID at m = 120 and n = 120											
Sch.	Para-meter	MLE AVE	MSE	MCMC-SELF AVE	MSE	MCMC-LiLF AVE	MSE	MCMC-DeLF AVE	MSE	MCMC-MELF AVE	MSE
$k = 80$											
$s1$	α	0.5131	0.0023	0.5086	0.0009	0.5083	0.0009	0.5041	0.0008	0.5109	0.0010
	β_1	1.5375	0.0678	1.5165	0.0150	1.5110	0.0147	1.4875	0.0146	1.5311	0.0158
	β_2	1.8249	0.0826	1.7817	0.0160	1.7737	0.0152	1.7458	0.0143	1.7998	0.0178
$s2$	α	0.5729	0.0093	0.5262	0.0015	0.5258	0.0015	0.5208	0.0013	0.5289	0.0017
	β_1	0.9688	0.2975	1.2011	0.0951	1.1976	0.0972	1.1777	0.1095	1.2128	0.0883
	β_2	2.1801	0.2559	2.0447	0.1075	2.0383	0.1036	2.0197	0.0930	2.0573	0.1153
$s3$	α	0.2740	0.0852	0.2925	0.0620	0.2924	0.0620	0.2900	0.0626	0.2937	0.0617
	β_1	2.0391	0.6401	1.7934	0.1869	1.7879	0.1829	1.7689	0.1723	1.8056	0.1946
	β_2	2.6805	1.5748	2.3152	0.5640	2.3069	0.5525	2.2868	0.5304	2.3294	0.5814
$s4$	α	0.5546	0.0089	0.5158	0.0011	0.5154	0.0011	0.5097	0.0009	0.5188	0.0012
	β_1	1.1414	0.1468	1.2861	0.0517	1.2827	0.0531	1.2652	0.0609	1.2965	0.0474
	β_2	1.9550	0.0832	1.9298	0.0463	1.9247	0.0443	1.9084	0.0389	1.9406	0.0504
$s5$	α	0.5272	0.0038	0.5103	0.0008	0.5100	0.0008	0.5052	0.0007	0.5129	0.0009
	β_1	1.2459	0.0873	1.3459	0.0315	1.3422	0.0326	1.3243	0.0384	1.3567	0.0284
	β_2	1.9570	0.0790	1.9036	0.0359	1.8980	0.0341	1.8798	0.0290	1.9155	0.0399
$k = 120$											
$s1$	α	0.5089	0.0015	0.5096	0.0031	0.5090	0.0029	0.5024	0.0025	0.5138	0.0040
	β_1	1.5242	0.0418	1.7048	6.5233	1.6436	2.6488	1.5129	1.1320	1.5571	0.4332
	β_2	1.7906	0.0558	1.7846	1.1381	1.7818	1.1264	1.7497	2.1769	1.9310	5.1967
$s2$	α	0.5384	0.0035	0.5204	0.0012	0.5201	0.0012	0.5166	0.0010	0.5223	0.0013
	β_1	1.1034	0.1721	1.2572	0.0648	1.2541	0.0663	1.2373	0.0748	1.2672	0.0601
	β_2	2.1590	0.2198	2.0054	0.0815	1.9996	0.0784	1.9824	0.0698	2.0170	0.0877
$s3$	α	0.2942	0.0671	0.3084	0.0548	0.3083	0.0548	0.3048	0.0561	0.3102	0.0541
	β_1	1.7362	0.1967	2.4507	2.3879	2.2881	1.7116	1.7015	9.0081	2.2377	1.4598
	β_2	2.4258	0.8419	2.8923	2.8592	2.8627	1.7030	2.4717	1.9209	2.8544	2.2357
$s4$	α	0.5624	0.0068	0.5272	0.0016	0.5269	0.0016	0.5229	0.0014	0.5293	0.0017
	β_1	0.8672	0.4080	1.1049	0.1598	1.1023	0.1618	1.0861	0.1749	1.1143	0.1525
	β_2	2.2228	0.2629	2.0952	0.1342	2.0902	0.1306	2.0763	0.1212	2.1046	0.1409
$s5$	α	0.5167	0.0020	0.5099	0.0008	0.5096	0.0008	0.5061	0.0007	0.5118	0.0009
	β_1	1.2611	0.0739	1.3437	0.0308	1.3406	0.0318	1.3249	0.0370	1.3532	0.0280
	β_2	1.9837	0.0928	1.9040	0.0384	1.8990	0.0367	1.8829	0.0321	1.9145	0.0419

To illustrate the findings of the paper, we will be considering the single fibers of 20 mm (Data Set I) and 10 mm (Data Set II) in gage length, with sample sizes, $m = 69$ and $n = 63$, respectively, as seen in Table 4.

We fit BIIID to each sample using the Kolmogorov- Smirnov (K-S) test statistics, where the K-S distance between the empirical and the fitted for the first population (W) is 0.1219 and its p-value is 0.2349 where $\hat{\hat{\alpha}} = 4.2441$and $\hat{\hat{\beta}}_1 = 26.4297$. Also, for the second population (Z) the K-S distance is 0.1384 and its p-value is 0.1266 where $\hat{\hat{\alpha}} = 5.6704$ and $\hat{\hat{\beta}}_2 = 19.0345$ which indicate that this distribution can be considered an adequate model for the given two data sets (W and Z).

From the original data, one can generate, e.g., three JP−IIC schemes with different numbers of stages $k = (20, 40, 60)$ and the removed items R_j are assumed as given in the following table (Table 5).

We compute the MLEs of $\alpha, \beta_1,$ and β_2 together with the associated 95% Asy-CI estimates. The BE is computed using the MH algorithm, where$a_1 = b_1 = a_2 = b_2 = c = d = 0$. While generating samples from the posterior distribution utilizing the MH algorithm, initial values of $(\alpha, \beta_1, \beta_2)$ are considered as $(\alpha^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}) = (\hat{\hat{\alpha}}, \hat{\hat{\beta}}_1, \hat{\hat{\beta}}_2)$, where $\hat{\hat{\alpha}}, \hat{\hat{\beta}}_1$ and $\hat{\hat{\beta}}_2$ are the MLEs of the parameters α, β_1 andβ_2 respectively. Thus, we considered the VCM $S_\phi$of $(\ln(\hat{\hat{\alpha}}), \ln(\hat{\hat{\beta}}_1), \ln(\hat{\hat{\beta}}_2))$. Eventually, 2000 burn-in samples are terminated from the entire 10,000 samples generated by the posterior density, and the technique is adopted to produce BEs: SELF, LiLF at $h = 0.5$, MELF, and DeLF. Also, HPD interval estimates have been computed according to the technique of Chen and Shao [35].

All values of MLEs and BEs utilizing MCMC under different proposed loss functions employing the MH algorithm, as well as the related standard error (St.Er) are illustrated in Table 6. Also, Asy-CI and associated HPD intervals are calculated based on the SELF in Table 7. Here, LB denotes the lower bound, and UB denotes the upper bound.

Conclusion

In this study, we used a JP-IIC approach to analyze two samples for the Burr IIID. When estimating model parameters, both the Bayesian and maximum likelihood techniques are taken into consideration. The asymptotic confidence intervals are then calculated using the observed information matrix. Confidence intervals can also be obtained using the Bootstrap-P and

Table 3
(a): AILs and CP(%) values of all methods under JP−IIC for BIIID at $m = 40$ and $n = 40$.

Sch.	Parameter	Asy-CI AIL	CP	Boot-P AIL	CP	Boot-T AIL	CP	HPD AIL	CP
$k = 20$									
s_1	α	0.3726	97.60	0.3962	98.20	0.4129	92.40	0.1052	97.80
	β_1	1.7058	91.00	2.7233	96.80	2.2108	89.00	0.5460	96.00
	β_2	2.2839	96.40	2.4704	97.00	2.1607	88.40	0.3628	97.00
s_2	α	0.7624	96.60	1.0662	97.60	0.5230	90.00	0.0843	97.60
	β_1	1.1025	98.00	1.4191	96.40	1.2372	90.10	0.3569	96.40
	β_2	1.5273	93.30	2.5880	97.00	2.0066	91.10	0.4798	96.60
s_3	α	0.2518	73.60	0.7860	96.40	0.4352	71.40	0.3653	99.00
	β_1	1.5701	93.60	1.9486	79.80	2.2473	61.80	0.5871	97.80
	β_2	1.9309	93.20	2.8134	89.40	2.9992	88.00	0.8510	98.00
s_4	α	0.7849	96.60	1.3088	97.70	0.5430	91.50	0.0802	97.20
	β_1	1.0728	98.60	1.7103	97.50	1.2389	96.80	0.3147	97.50
	β_2	1.3686	95.20	2.4790	95.20	1.8351	95.90	0.4504	97.30
s_5	α	0.5213	95.40	0.5945	96.60	0.5210	91.70	0.0807	98.40
	β_1	1.1851	99.20	1.2299	96.00	1.3022	89.80	0.3349	98.40
	β_2	1.5178	97.60	1.7583	96.80	1.7084	91.20	0.4480	97.00
$k = 40$									
s_1	α	0.2569	97.60	0.2603	97.60	0.2747	93.40	0.1220	97.80
	β_1	1.2791	95.60	1.4814	96.40	1.5445	90.40	0.4199	96.80
	β_2	1.5441	95.20	1.5524	95.20	1.5746	88.00	0.3801	98.00
s_2	α	0.2859	94.80	0.3164	95.60	0.3380	90.20	0.1295	96.80
	β_1	0.9286	96.60	0.8840	95.00	0.9716	88.40	0.3412	96.20
	β_2	1.4205	94.20	1.5903	94.60	1.7277	87.60	0.4757	97.40
s_3	α	0.1809	72.00	0.6944	96.80	0.4681	72.80	0.3416	98.80
	β_1	1.1757	98.00	1.3828	90.40	1.7846	76.60	0.4707	97.60
	β_2	1.4486	95.00	2.3093	96.40	2.6511	76.60	0.7503	98.40
s_4	α	0.3346	95.40	0.3995	97.40	0.4141	89.80	0.1200	96.60
	β_1	0.7875	99.00	0.6486	96.80	0.7354	89.60	0.2870	98.00
	β_2	1.2110	95.00	1.4990	96.40	1.5820	92.20	0.4501	97.60
s_5	α	0.2853	96.80	0.2872	96.60	0.3126	92.00	0.1167	97.60
	β_1	0.9752	97.60	0.9350	96.80	1.0195	90.60	0.3629	97.20
	β_2	1.2429	96.60	1.3077	96.60	1.3596	91.00	0.4616	96.80

(b): AILs and CP(%) values of all methods under JP−IIC for BIIID at $m = 80$ and $n = 80$

Sch.	Parameter	Asy-CI AIL	CP	Boot-P AIL	CP	Boot-T AIL	CP	HPD AIL	CP
$k = 40$									
s_1	α	0.2571	95.60	0.2580	95.00	0.2809	88.80	0.1235	97.20
	β_1	1.1874	93.20	1.5905	96.00	1.6074	90.40	0.5227	95.80
	β_2	1.4842	97.20	1.4902	96.80	1.5374	90.00	0.3864	96.60
s_2	α	0.3654	96.40	0.4155	96.80	0.3981	88.40	0.1042	98.60
	β_1	0.7829	97.20	0.7918	94.80	0.8538	91.00	0.3377	97.00
	β_2	1.0750	93.60	1.3966	96.00	1.3954	90.60	0.5072	95.80
s_3	α	0.1489	74.80	0.6842	95.80	0.3785	75.00	0.4173	97.80
	β_1	1.1358	86.40	1.9073	77.60	2.2704	52.00	0.8486	99.00
	β_2	1.3939	86.20	2.5150	61.20	2.9084	37.80	1.1603	98.00
s_4	α	0.5140	95.00	0.6605	95.60	5.0848	0.00	0.0959	97.20
	β_1	0.8420	97.20	1.1020	93.30	0.8383	90.70	0.2994	97.60
	β_2	1.0654	96.60	1.5627	94.00	1.1308	93.80	0.3966	97.80
s_5	α	0.3365	96.60	0.3421	96.60	0.3707	89.60	0.1045	98.80
	β_1	0.8161	98.40	0.8136	96.80	0.8655	92.80	0.3204	97.40
	β_2	1.0490	97.00	1.1107	96.60	1.1388	93.40	0.4265	96.40
$k = 80$									
s_1	α	0.1785	95.60	0.1789	95.00	0.1868	91.80	0.1023	97.80
	β_1	0.9059	93.80	0.9747	93.80	1.0584	89.60	0.4158	96.00
	β_2	1.0778	97.20	1.0522	95.20	1.1423	88.80	0.4082	96.00
s_2	α	0.1919	95.40	0.2068	97.00	0.2142	92.00	0.1111	97.60
	β_1	0.6575	98.00	0.6066	96.00	0.6576	91.80	0.3403	96.80
	β_2	0.9632	94.40	1.0831	95.20	1.1341	91.20	0.4752	96.80
s_3	α	0.1156	73.60	0.6452	96.40	0.4267	73.80	0.3706	98.80
	β_1	0.8339	94.80	1.2869	84.80	1.7003	71.40	0.6091	97.40
	β_2	1.0370	86.60	2.2267	93.60	2.6230	68.20	0.9682	98.00
s_4	α	0.2271	95.60	0.2524	96.20	0.2737	91.00	0.1071	98.20
	β_1	0.5537	100.00	0.4428	97.20	0.4820	91.60	0.2802	98.00
	β_2	0.8433	94.40	0.9974	95.80	1.0362	90.80	0.5042	99.00
s_5	α	0.1954	96.00	0.1989	95.40	0.2132	92.00	0.1004	96.60
	β_1	0.6773	99.00	0.6514	97.80	0.6943	93.60	0.3459	96.20
	β_2	0.8739	96.60	0.9031	95.60	0.9452	92.20	0.4574	96.40

(*continued on next page*)

Table 3 (*continued*)

Sch.	Para-meter	Asy-CI AIL	CP	Boot-P AIL	CP	Boot-T AIL	CP	HPD AIL	CP
(c): AILs and CP(%) values of all methods underJP−IIC for BIIID at $m = 120$ and $n = 120$									
Sch.	Para-meter	Asy-CI AIL	CP	Boot-P AIL	CP	Boot-T AIL	CP	HPD AIL	CP
$k = 80$									
s_1	α	0.1771	96.20	0.1811	95.80	0.1889	93.60	0.1099	97.20
	β_1	0.8613	95.20	1.0049	97.40	1.0492	93.00	0.4700	95.80
	β_2	1.0569	96.60	1.0187	96.00	1.0999	91.20	0.3918	98.20
s_2	α	0.2202	95.80	0.2418	97.00	0.2549	92.00	0.1114	96.00
	β_1	0.5462	98.60	0.5037	96.80	0.5438	94.20	0.3334	98.20
	β_2	0.8296	94.60	1.0196	96.40	1.0578	93.40	0.5286	96.80
s_3	α	0.1007	75.40	0.6427	96.40	0.3702	75.80	0.4286	97.80
	β_1	0.8345	76.80	2.1430	47.80	1.9855	43.40	1.0027	98.80
	β_2	1.0150	73.80	3.0245	57.80	2.7551	35.00	1.3374	97.80
s_4	α	0.2621	94.60	0.2750	95.20	0.2930	89.00	0.1179	96.80
	β_1	0.5718	99.40	0.5199	96.60	0.5443	92.80	0.3005	96.40
	β_2	0.7355	97.40	0.8037	97.20	0.8086	94.80	0.4343	97.60
s_5	α	0.2091	96.20	0.2144	96.00	0.2262	92.40	0.1101	96.40
	β_1	0.6019	97.20	0.5871	95.40	0.6262	93.00	0.3384	97.00
	β_2	0.7741	98.60	0.7960	98.00	0.8150	95.80	0.4468	99.00
$k = 120$									
s_1	α	0.1453	97.60	0.1458	97.20	0.1515	94.20	0.2135	97.80
	β_1	0.7301	97.40	0.7882	97.60	0.8278	93.40	0.5553	97.30
	β_2	0.8719	97.20	0.8481	96.60	0.8943	91.40	0.4660	97.50
s_2	α	0.1570	96.80	0.1638	95.20	0.1727	91.00	0.1010	97.60
	β_1	0.5330	98.20	0.4887	97.00	0.5148	94.00	0.3376	96.80
	β_2	0.7867	95.20	0.8636	96.40	0.8998	91.80	0.5056	97.20
s_3	α	0.0865	77.40	0.6233	95.80	0.4016	77.60	0.4479	97.00
	β_1	0.7004	90.80	2.0098	75.80	1.6471	68.00	6.6680	95.10
	β_2	0.8649	82.40	2.9424	92.20	2.6217	62.00	4.4758	95.10
s_4	α	0.1793	95.60	0.1965	95.80	0.2063	94.00	0.1019	98.20
	β_1	0.4509	99.40	0.3573	97.60	0.3778	95.20	0.2318	98.00
	β_2	0.6866	96.00	0.7924	97.00	0.8171	95.40	0.4404	97.80
s_5	α	0.1567	97.20	0.1602	97.20	0.1667	94.00	0.1010	97.40
	β_1	0.5545	98.40	0.5316	96.80	0.5628	94.40	0.3555	99.60
	β_2	0.7112	96.60	0.7326	96.00	0.7606	92.20	0.4616	98.40

Table 4
Strength data measured for single carbon fibers and impregnated 1000-carbon fiber tows.

Data set I: W
1.312,1.314,1.479,1.552,1.700,1.803,1.861,1.865,1.944,1.958,1.966,1.997,2.00,2.021,2.027,2.055,2.063,2.098,2.140,2.179,2.224,2.240,2.253,2.270,2.272, 2.274,2.301,2.301,2.359,2.382,2.382,2.426,2.434,2.435,2.478,2.490,2.511,2.514,2.535,2.554,2.566,2.570,2.586,2.629,2.633,2.642,2.648,2.684,2.697, 2.726,2.770,2.773,2.800,2.809,2.818,2.821,2.848,2.880,2.954,3.012,3.067,3.084,3.090,3.096,3.128,3.233,3.433,3.585,3.585.
Data set II: Z
1.901,2.132,2.203,2.228,2.257,2.350,2.361,2.396,2.397,2.445,2.454,2.474,2.518,2.522,2.525,2.532,2.575,2.614,2.616,2.618,2.624,2.659,2.675,2.738,2.740, 2.856,2.917,2.928,2.937,2.937,2.977,2.996,3.030,3.125,3.139,3.145,3.220,3.223,3.235,3.243,3.264,3.272,3.294,3.332,3.346,3.377,3.408,3.435,3.493,3.501, 3.537,3.554,3.562,3.628,3.852,3.871,3.886,3.971,4.024,4.027,4.225,4.395,5.020

Bootstrap-T approaches. We also investigated Bayesian estimators for both symmetric and asymmetric loss functions, including SELF, LiLF, MELF, and DeLF. The Markov chain Monte Carlo method has been used to compute the Bayes estimates and the related credible intervals. We carry out in-depth simulation tests to assess the estimators' performance. Our simulation results indicate that Bayes estimates based on DeLF are better than MLEs. In light of this, the Bayesian method is superior to the ML method. The bias performance of the BEs employing MCMC under DeLF is typically superior to that of the BEs under SELF, LiLF, and MELF. The Boot-P approach is superior to the Boot-T method for obtaining confidence intervals. As compared to the Asy-CI, Boot-P, and Boot-T, the HPD credible interval has the smallest interval length. The present work can also be extended to design of optimal progressive censoring sampling plan, and other censoring schemes can also be considered.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

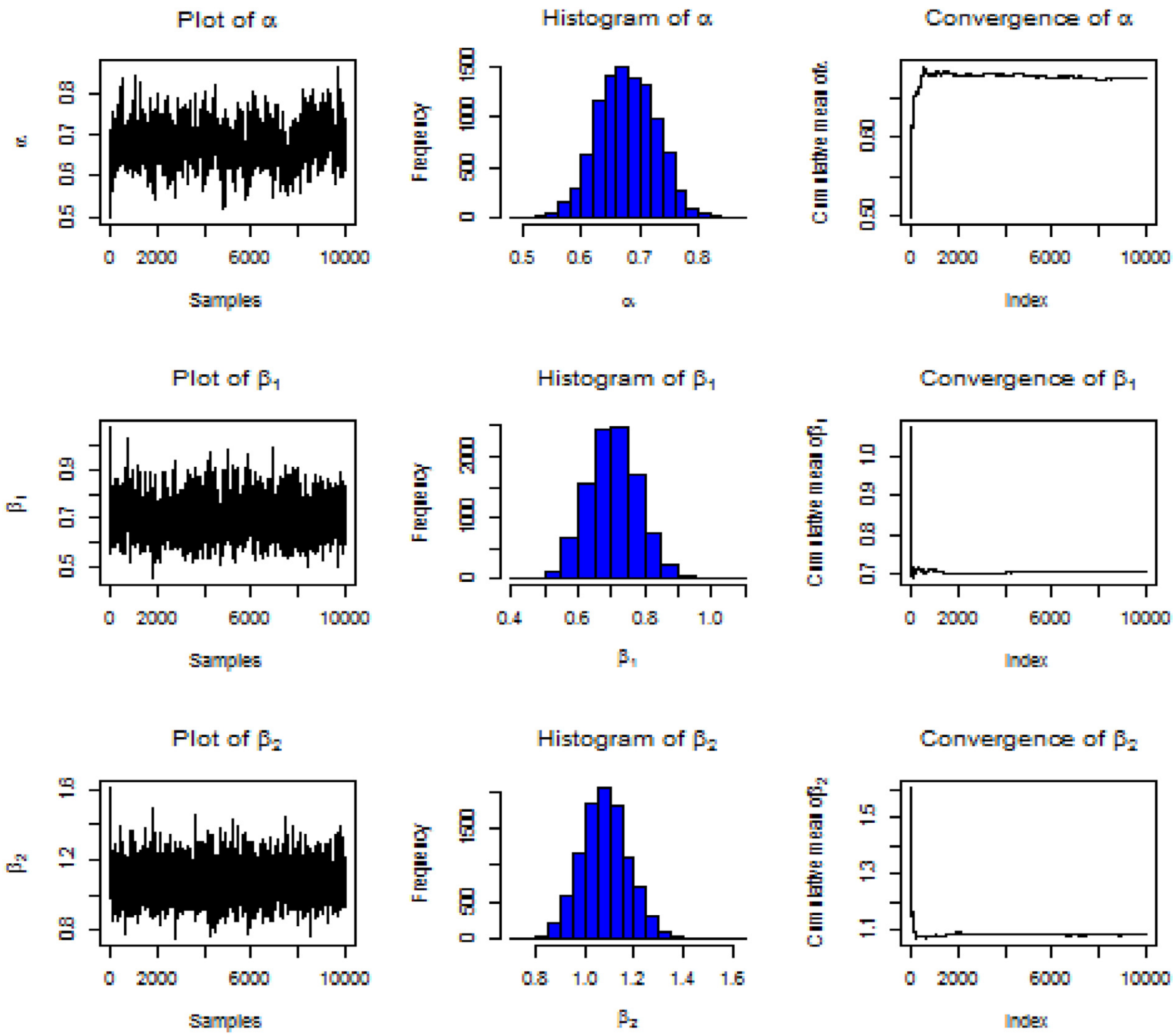

Fig. 1. Convergence of MCMC estimates using MH algorithm for JP−IIC scheme (S_1).

Table 5
Removal patterns of units in various censoring schemes.

(*m*, *n*)	*k*	Censoring Schemes *s1*	*s2*	*s3*
(69,63)	20	(112, 0 × ^{19})	(5 × ^{19}, 17)	(0 × ^{19}, 112)
	40	(92, 0 × ^{39})	(2 × ^{39}, 14)	(0 × ^{39}, 92)
	60	(72, 0 × ^{59})	(1 × ^{59}, 13)	(0 × ^{59}, 72)

Here, (1 × ^{5}, 0), for example, means that the censoring scheme employed is (1, 1, 1, 1, 1, 0).

CRediT authorship contribution statement

Amal S. Hassan: Conceptualization, Methodology, Formal analysis, Investigation, Writing – review & editing, Visualization, Supervision. **E.A. Elsherpieny:** Conceptualization, Methodology, Supervision. **Wesal.E. Aghel:** Conceptualization, Methodology, Software, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Writing – review & editing.

Table 6
MLEs, BEs, and St.Er for real data set based on different schemes of JP−IIC.

Sch.	Para-meter	MLE Estimate	St.Er	MCMC-SELF Estimate	St.Er	MCMC-LiLF Estimate	St.Er	MCMC-DeLF Estimate	St.Er	MCMC-MELF Estimate	St.Er
$k=20$											
s_1	α	3.6103	0.4590	3.6518	0.3832	3.6167	7.0368	3.5774	1.8091	3.6920	7.6078
	β_1	31.6661	15.5309	29.0037	4.4511	25.4283	23.5337	27.7178	13.7752	29.6867	60.4628
	β_2	32.3911	10.1761	35.4538	9.3707	25.6598	23.7641	30.6347	15.2873	37.9303	73.8167
s_2	α	3.1858	0.4510	3.1626	0.1735	3.1551	8.0512	3.1436	1.5648	3.1721	6.3542
	β_1	22.7117	9.3229	23.5667	4.3406	19.7805	18.6144	21.8288	10.6017	24.3661	46.2492
	β_2	15.6761	4.2108	14.3962	1.4290	13.9099	14.5542	14.1123	6.9944	14.5380	29.0295
s_3	α	3.2596	0.4779	3.5193	0.2449	3.5046	7.7430	3.4861	1.7506	3.5364	7.1315
	β_1	12.8881	8.5927	13.9612	1.2284	13.5524	14.1634	13.7265	6.6250	14.0693	27.3948
	β_2	15.2676	4.4749	17.1882	1.4545	16.6919	17.3522	16.9445	8.4503	17.3112	34.7174
$k=40$											
s_1	α	3.2820	0.3271	3.0839	0.2421	3.0693	7.2886	3.0454	1.5077	3.1029	6.1795
	β_1	39.1132	16.9310	38.2264	8.7288	27.8068	25.4700	34.0289	16.3062	40.2193	77.8869
	β_2	25.0145	6.4048	20.2287	3.7081	17.9751	17.6509	18.9976	9.8056	20.9084	42.0755
s_2	α	3.3688	0.3358	3.4064	0.1728	3.3988	8.2455	3.3879	1.6594	3.4152	6.7247
	β_1	34.1195	11.2328	35.1609	4.2007	32.6967	32.3161	34.2737	17.7891	35.6627	73.7586
	β_2	19.1834	4.4333	19.9250	2.1597	18.7904	18.4881	19.4388	9.5239	20.1590	39.7271
s_3	α	4.5393	0.4499	4.5002	0.2365	4.4860	8.7145	4.4745	2.2109	4.5126	8.9218
	β_1	35.5419	15.7404	35.5530	5.7438	28.8807	26.4282	33.5001	15.9297	36.4808	69.9895
	β_2	27.1410	7.5590	26.6407	2.8031	25.0690	24.8540	26.0655	13.1160	26.9356	53.9722
$k=60$											
s_1	α	4.2314	0.3397	4.5378	0.2796	4.5184	8.4696	4.5036	2.2553	4.5551	9.1020
	β_1	72.3768	26.4601	105.4081	29.3904	71.4381	68.9882	91.2807	47.5875	113.6019	225.7353
	β_2	40.4562	10.1515	50.1352	12.0631	36.5719	34.7222	44.4102	22.2444	53.0374	102.3557
s_2	α	3.7189	0.3007	3.4997	0.1479	3.4943	8.7195	3.4874	1.7462	3.5060	7.0426
	β_1	44.1152	12.5906	37.0987	5.2820	33.9953	33.6252	35.8697	19.0028	37.8507	80.4539
	β_2	24.3808	5.2322	20.7519	1.7047	20.0883	20.3721	20.4779	10.2285	20.8919	42.1475
s_3	α	5.2310	0.4333	4.7585	0.2833	4.7385	8.6621	4.7249	2.3638	4.7753	9.5463
	β_1	74.9334	30.5517	55.9383	12.3844	42.4541	40.3994	50.7808	25.7430	58.6798	116.0031
	β_2	35.3717	9.4492	22.4411	3.5367	20.4239	20.1390	21.4523	11.0836	22.9984	47.2459

Table 7
Associated interval estimates for ML and Bayesian for real data set based on different schemes of JP−IIC.

k	Sch.	Para-meter	Asy-CI (MLE) LB	UB	HPD (Bayes) LB	UB
20	s_1	α	2.7456	4.5418	2.9414	4.3781
		β_1	11.9437	82.2102	23.3788	39.4288
		β_2	17.5511	59.6376	20.5844	52.5693
	s_2	α	2.3428	4.1047	2.7579	3.4385
		β_1	10.4102	51.8329	15.8505	29.5066
		β_2	9.3891	26.7917	11.9801	17.3979
	s_3	α	2.3631	4.2261	3.0735	4.0075
		β_1	3.5825	42.2954	11.3195	15.9439
		β_2	8.6945	27.2475	13.8705	19.8405
40	s_1	α	2.6605	3.9393	2.6122	3.4947
		β_1	16.7258	91.1989	23.0837	53.2539
		β_2	15.1682	41.2470	15.2323	26.8684
	s_2	α	2.7314	4.0448	3.0885	3.7307
		β_1	18.1234	65.6700	29.1067	42.7882
		β_2	12.2473	30.2145	15.7543	23.4250
	s_3	α	3.6835	5.4440	4.0516	4.8712
		β_1	14.7528	83.8819	24.2193	44.5962
		β_2	15.6894	46.6297	21.8820	31.3155
60	s_1	α	3.5827	4.9122	4.0654	4.9989
		β_1	35.5163	148.4712	64.7688	153.6393
		β_2	24.6792	65.8405	31.0230	69.6114
	s_2	α	3.1445	4.3217	3.2259	3.7825
		β_1	25.4281	77.7927	30.5063	48.6125
		β_2	16.0284	37.0981	17.6606	24.5698
	s_3	α	4.4078	6.1061	4.2702	5.3625
		β_1	33.7937	167.2960	38.0413	85.4039
		β_2	20.8362	59.3184	17.2260	34.5112

Algorithm 1
The algorithm of the Boot-P method.

Step 1: Generate two random samples, which are from BIIID (β_1, α) and BIIID (β_2, α) respectively, and use the JP−IIC scheme to get the observed data.

Step 2: Calculate the ML estimates (MLEs), say $\hat{\hat{\phi}} = (\hat{\hat{\beta}}_1, \hat{\hat{\beta}}_2, \hat{\hat{\alpha}})$.
Step 3: Use $\hat{\hat{\phi}}$to generate two new samples, respectively.
Step 4: Get new MLEs$\hat{\hat{\phi}}^{(i)} = (\hat{\hat{\beta}}_1^{(i)}, \hat{\hat{\beta}}_2^{(i)}, \hat{\hat{\alpha}}^{(i)})$
Step 5: Repeat steps 3 and 4 for N times.
Step 6: Get the results $((\hat{\hat{\phi}}^{(1)}), \ldots, (\hat{\hat{\phi}}^{(N)}))$.
Step 7: Sort $(\hat{\hat{\phi}}^{(1)}, \ldots, \hat{\hat{\phi}}^{(N)})$in ascending order as$(\hat{\hat{\phi}}_{(1)}, \ldots, \hat{\hat{\phi}}_{(N)})$.
Step 8: The $100(1-\varepsilon)\%$ Boot-P CIs for (ϕ) are $[\hat{\hat{\phi}}_{(lb)}, \hat{\hat{\phi}}_{(hb)}]$ here,$lb = [(\varepsilon/2)N]$, $hb = [((1-\varepsilon)/2)N]$. Here $[x]$ means the largest integer not exceeding x.

Algorithm 2
The algorithm of the Boot-T method.

Steps 1 to 5 are the same as those in Algorithm 1.

Step 6: Get the results$((\hat{\hat{\phi}}^{(1)}), \ldots, (\hat{\hat{\phi}}^{(N)}))$and $(Cov^{(1)}, Cov^{(2)}, \ldots, Cov^{(N)})$where$Cov^{(i)}$ is the covariance, which is given by:
$Cov^{(i)} = I_o^{-1}(\hat{\hat{\phi}}^{(i)})$.
Step 7: $Var\ (\widehat{\phi}^{(i)})$which are diagonals of $Cov^{(i)}$ can be obtained. Define
$T_j^{(i)} = \frac{\hat{\hat{\phi}}_j^{(i)} - \hat{\hat{\phi}}_j}{\sqrt{Var(\hat{\hat{\phi}}_j^{(i)})}}$.
Step 8: Sort$(T_1^{(1)}, \ldots, T_1^{(N)})$, $(T_2^{(1)}, \ldots, T_2^{(N)})$, $(T_\phi^{(1)}, \ldots, T_\phi^{(N)})$in ascending order as
$(T_{1(1)}, \ldots, T_{1(N)}), (T_{2(1)}, \ldots, T_{2(N)}), (T_{\phi(1)}, \ldots, T_{\phi(N)})$.
Step 9: The $100(1-\varepsilon)\%$ Boot-T CI for$\phi$is given by:

Appendix

The elements of FIM are obtained follows

$$\frac{\partial^2 l^*}{\partial \beta_1^2} = \frac{-k_1}{\beta_1^2} - \sum_{i=1}^{k} \frac{s_i (D_i(\alpha))^{\beta_1} [ln(D_i(\alpha))]^2}{[(D_i(\alpha))^{\beta_1} - 1]^2},$$

$$\frac{\partial^2 l^*}{\partial \beta_2^2} = \frac{-k_2}{\beta_2^2} - \sum_{i=1}^{k} \frac{t_i (D_i(\alpha))^{\beta_2} [ln\, D_i(\alpha)]^2}{[D_i(\alpha)^{\beta_2} - 1]^2},$$

$$\begin{aligned}\frac{\partial l^*}{\partial \alpha^2} = \frac{-k}{\alpha^2} &+ \sum_{i=1}^{k} \frac{[z_i\beta_1 + \beta_2(1-z_i)+1]\, w_i^{\alpha} (ln\, w_i)^2}{(1+w_i^{\alpha})^2} - \sum_{i=1}^{k} \frac{s_i\beta_1^2 (D_i(\alpha))^{\beta_1-2} (w_i)^{-2\alpha} (ln\, w_i)^2}{[(D_i(\alpha))^{\beta_1}-1]^2} \\ &- \sum_{i=1}^{k} \frac{s_i\beta_1 (ln\, w_i)^2 w_i^{-\alpha} [w_i^{-\alpha} (D_i(\alpha))^{-2} - (D_i(\alpha))^{-1}]}{\left[(D_i(\alpha))^{\beta_1}-1\right]} - \sum_{i=1}^{k} \frac{t_i\beta_2^2 (D_i(\alpha))^{\beta_2-2} (w_i)^{-2\alpha} (ln\, w_i)^2}{[(D_i(\alpha))^{\beta_2}-1]^2} \\ &- \sum_{i=1}^{k} \frac{t_i\beta_2 (ln\, w_i)^2 w_i^{-\alpha} [w_i^{-\alpha} (D_i(\alpha))^{-2} - (D_i(\alpha))^{-1}]}{\left[(D_i(\alpha))^{\beta_2}-1\right]},\end{aligned}$$

$$\frac{\partial^2 l^*}{\partial \alpha \partial \beta_1} = -\sum_{i=1}^{k} \frac{z_i\, ln\, w_i}{1+w_i^{\alpha}} - \sum_{i=1}^{k} \frac{s_i ln\, w_i}{(1+w_i^{\alpha})\left[(D_i(\alpha))^{\beta_1} - 1\right]} + \sum_{i=1}^{k} \frac{s_i\beta_1 (D_i(\alpha))^{\beta_1-1} w_i^{-\alpha}\, (ln\, w_i) lnD_i(\alpha)}{\left[(D_i(\alpha))^{\beta_1} - 1\right]^2},$$

$$\frac{\partial^2 l^*}{\partial \alpha \partial \beta_2} = -\sum_{i=1}^{k} \frac{(1-z_i)\, ln\, w_i}{1+w_i^{\alpha}} - \sum_{i=1}^{k} \frac{t_i ln\, w_i}{(1+w_i^{\alpha})\left[(D_i(\alpha))^{\beta_2}-1\right]} + \sum_{i=1}^{k} \frac{t_i\beta_2 (D_i(\alpha))^{\beta_2-1} w_i^{-\alpha}\, ln\, w_i lnD_i(\alpha)}{\left[(D_i(\alpha))^{\beta_2}-1\right]^2}, \frac{\partial^2 l}{\partial \beta_1 \beta_2} = \frac{\partial^2 l}{\partial \beta_2 \beta_1} = 0 .$$

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