Stationary Distribution of $M/M/1$ Queue with Multiple vacations using Matrix Geometric Method

Ashour K. R. [a] **Assar S. M.** [b] **Shehata B. A.** [c]

Department of Mathematical Statistics, Institute of Statistical Studies and Research , Cairo University, Giza-Egypt

[a]**Karimrashad@cu.edu.eg** [b]**salwaassar@yahoo.com** [c]**btawafik534@yahoo.com**

Abstract

This paper analyze an $M/M/1$ queue with multiple vacations. We employ the Matrix Geometric Method to derive the stationary distribution of the model. Moreover, some performance measures was calculated. Also, a numerical illustration was given to compare the cases $M/M/1$ with and without vacation. The main objective of this paper is to show that The Matrix Geometric Method is easy to use for analyzing $M/M/1$ queue with vacation.

Keywords: Queues, Multiple vacations, Matrix Geometric Method, Markov chain.

1 Introduction

Queueing models are the mathematical modeling of waiting lines. Queueing models have great applications in computers, communication networks, telecommunications, manufacturing process, and even real life situations.

Queuing models with vacation is one of the important queuing models in which the server takes a vacation when the system becomes idle. A vacation is the time away from the primary service center. There are two basic types of vacation queuing systems, the single vacation and the multiple vacations. In the single vacation the server takes a vacation of a random duration when the queue is empty and at the end of the vacation the server returns to the queue. In the multiple vacations, if the server returns from a vacation and finds the queue empty, it immediately takes another vacation. Allowing servers to take vacations makes queueing models more realistic and flexible in studying the real world waiting line systems. The pioneering work in queues with vacation was done by Levy and Yechiali (1975) [9], which addressed the issue of utilizing the idle time in M/G/1 queue. A considerable number of works in this area were completed in the early 80's and surveyed by Doshi in (1986) [2]. Excellent surveys on the earlier works of vacation models have been reported by Takagi (1991) [12], Tian and Zhang (2006) [13], and Ke *et al.* (2010) [5].

Kumar *et al.* (2011) [6] used the probability generating functions to derive the stationary distribution of $M/M/1$ queuing model with multiple vacations and server break down. Kalidass and Ramanath (2014) [4] used the Kolmogrov equations and the laplace transforms to establish the $M/M/1$ queuing model with multiple vacations. Ibe and Isijola (2014) [3] also used the Kolmogrov equations in analyzing $M/M/1$ multiple vacations queuing systems with two types of vacations. Ammar (2015) [1] investigated the $M/M/1$ multiple vacations queuing systems with impatient behavior, he derived an explicit solution for the model by employed the generating functions a long with continued fractions and properties of the confluent hypergeometric function. Kv and Janani (2015) [7] used the laplace transforms and generating function techniques to obtain explicit expressions for the time-dependent system size probabilities. Sudhesh and Azhagappan (2016) [11] used the modified Bessel functions to drive the time-dependent system size probabilities, the time-dependent mean and variance.

Here, we interested to use Matrix Geometric Method to analyze $M/M/1$ queue with vacations. The method was first introduced by Neuts (1981) [10]. The method shows an effective and easier way to solve and find the stationary distribution of queueing systems rather than using traditional ways.

The paper is organized as follows. In Section 2, we state the model assumptions. Sections 3 is devoted to defining the Markov chain and the corresponding transition structure. Stability condition of the system is discussed in Section 4. In Section 5, we explicitly find the stationary distribution of the new model. In section 6 some performance measures of the model was calculated. A numerical illustration was given in section 7. Finally some concluding remarks are given.

2 Model Description

In this part of the paper we state and fully describe the model assumptions.

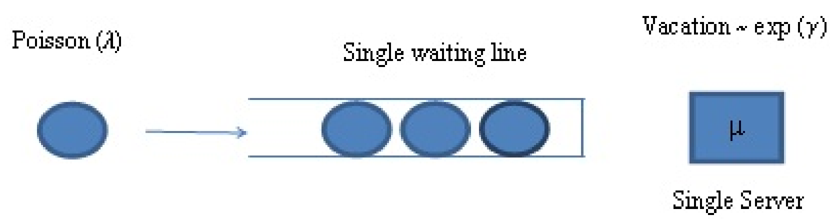

Figure 1: $M/M/1$ queue with multiple vacations.

1. We consider one server queueing system with unlimited single waiting line as shown in Figure 1.
2. Customers arrive at the system according to poisson process with rate λ.
3. Each customer was served with FCFS discipline.
4. The service distribution was exponential with service rate μ.
5. If the servers find the waiting line is empty at service completion instants, then it takes vacations.
6. The vacation duration of the server is exponentially distributed with mean $\frac{1}{\gamma}$, where the multiple vacation policy is assumed.
7. Through this paper the stability condition is assumed to be $\lambda < \mu$.
8. We will show that the queueing model is formulated by a Quasi Birth and Death (QBD) Markov chain.
9. Idle and Busy periods
    - Idle period : a period at which server is on vacation.
    - Busy period : at which the server is busy.
      we note that busy and idle periods occur alternatively so, any time t belongs to either one of these periods.

3 Markov Chain and Transition Rates of the Model

Assume the system first start at Idle period. So, for $t \geq 0$ we define :

- $X(t)$: The number of customers in the system.
- $Y(t)$: The state of the server.

Then, consider the two-dimensional stochastic process $\{(X(t), Y(t)); t \geq 0\}$, Where

$$Y(t) = \begin{cases} 0 & \text{if the server is on vacation} \\ 1 & \text{if server is busy} \end{cases}$$

Where, $X(t)$ is level process and $Y(t)$ is the background process. Obviously, the process $\{(X(t), Y(t)); t \geq 0\}$ is a continuous-time Markov chain with the following partitioned state space :

$$U = \bigcup_{l=0}^{\infty} U_l, \tag{1}$$

where

$$U_0 = \{0\} \times \{0\} = \{(0, 0)\} \tag{2}$$

and

$$U_r = \{r\} \times \{0, 1\} = \{(r, 0), (r, 1)\}, r \geq 1, \tag{3}$$

where the elements of the sets are arranged in lexicographical order.

Then the infinitesimal generator matrix is given by :

$$Q = \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & \cdots \\ B_{10} & A_1 & A_0 & 0 & 0 & \cdots \\ & A_2 & A_1 & A_0 & 0 & \cdots \\ & & A_2 & A_1 & A_0 & \cdots \\ & & \ddots & \ddots & \ddots & \end{pmatrix},$$

Which is tri-diagonal matrix. Hence $\{(X(t), Y(t)); t \geq 0\}$ is a (QBD) Markov chain.
where:

$$B_{00} = \begin{bmatrix} -(2\lambda + \gamma) \end{bmatrix}, \quad B_{01} = \begin{bmatrix} \lambda & \lambda + \gamma \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 0 \\ \mu \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mu \end{bmatrix}, \quad A_1 = \begin{bmatrix} -2(\lambda + \gamma) & \gamma \\ 0 & -(\lambda + \mu) \end{bmatrix}, \quad \textit{and} \quad A_0 = \begin{bmatrix} \lambda & \lambda + \gamma \\ 0 & \lambda \end{bmatrix}$$

4 Stability Condition

We now derive the condition for the system to reach the steady state. To accomplish this we define the matrix $A = A_0 + A_1 + A_2$ so:

$$A = \begin{bmatrix} -\lambda - 2\gamma & \lambda + 2\gamma \\ 0 & 0 \end{bmatrix}$$

Clearly, The Continuous Time Markov Chain with generator A is reducible with absorbing state $(1,1)$ and with stationary vector $x = (0,1)$.
Then the standard drift condition (See Latouche and Ramaswami (1999) [8]) which is necessary and sufficient condition for the stability of the Quasi Birth and Death (QBD) process is given by :

$$xA_0e < xA_2e \Rightarrow \lambda < \mu, \tag{4}$$

where e is a column vector with all it's element equal to one.

5 Stationary Distribution of The Model

To derive the stationary distribution of the model we consider the balance equations:

$$\pi Q = 0 \tag{5}$$

Where

$$\pi = (\pi_0, \pi_1, \pi_2, \ldots), \quad and \quad \pi_0 = \pi_{00}, \quad \pi_r = (\pi_{r0}, \pi_{r1}), \quad r \geq 1 \tag{6}$$

This leads to the following system of equations:

$$\pi_0 B_{00} + \pi_1 B_{10} = 0 \tag{7}$$

$$\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_2 = 0 \tag{8}$$

and

$$\pi_1 A_0 + \pi_2 A_1 + \pi_3 A_2 = 0 \tag{9}$$

Given the geometric relation:

$$\pi_j = \pi_1 R^{j-1} \qquad , j \geq 2 \tag{10}$$

Where R is the rate matrix, and for stability purposes the spectral radius of R must be less than one.
Using Equation(10) to substitute in Equation (9) and dividing by π_1 we can get :

$$A_0 + RA_1 + R^2 A_2 = 0, \tag{11}$$

where the matrices A_2, A_1 and A_0 are given from section 3.
By solving Equation(11) using Mathematica Package 9 the rate matrix R will be in the form:

$$R = \begin{bmatrix} \frac{\lambda}{2(\lambda+\gamma)} & \frac{\gamma\lambda+\gamma\mu+\lambda\mu}{2\mu(\lambda+\gamma)} \\ 0 & \frac{\lambda}{\mu} \end{bmatrix}$$

The Normalization condition is given by:

$$\sum_{j=0}^{\infty} \pi_j e = 1 \tag{12}$$

which gives:

$$\pi_0 e + \sum_{j=1}^{\infty} \pi_j e = 1 \tag{13}$$

using Equation (10) this leads to:

$$\pi_0 e + \pi_1 (I - R)^{-1} e = 1 \tag{14}$$

where e is column vector of 1's of appropriate order.
Since Equation (8) can be written as:

$$\pi_0 B_{01} + \pi_1(A_1 + RA_2) = 0 \tag{15}$$

One can use Equations (7),(14), and (15) to find π_0 and π_1, then using Equation (10) to find the stationary vector π.

6 Performance Measures of The System

First, we start by calculating the expected number of customer in the system (L_s) as follows :

$$L_s = E(n) = \sum_{n=0}^{\infty} n\pi_n e \tag{16}$$

$$L_s = \pi_1 e + 2\pi_2 e + 3\pi_3 e + 4\pi_4 e + \dots$$

Using Equation (10) to get :

$$L_s = \pi_1 e + 2\pi_1 Re + 3\pi_1 R^2 e + 4\pi_1 R^3 e + \dots$$

This gives:

$$L_s = \pi_1 (I - R)^{-2} e \tag{17}$$

Secondly, using Little's formula to get:

- The expected number of customer in queue : $L_q = L_s - (1 - \pi_0)$.
- The mean waiting time in the system : $W_s = \frac{L_s}{\lambda}$.
- The mean waiting time in the queue : $W_q = \frac{Lq}{\lambda}$.

7 Numerical Illustration

In this part of the paper we shall calculate L_s,Lq,W_s, and W_q for different values of $\lambda = 5, 10, 15$,$\mu = 20$, and $\gamma = 3$ in the vacation model and compare the results with the non-vacation model.

The following table illustrates the results:

	Vacation Model					Non-Vacation Model				
	$\mu = 20$,$\gamma = 3$					$\mu = 20$				
Arrival Rate	L_s	Lq	W_s	W_q	π_0	L_s	Lq	W_s	W_q	π_0
$\lambda = 5$	0.83717	0.19831	0.16743	0.03966	0.36	0.333'	0.0833	0.067	0.0167	0.75
$\lambda = 10$	1.53381	0.70837	0.15338	0.07083	0.17	1.000	0.500	0.100	0.050	0.50
$\lambda = 15$	3.35352	2.42779	0.22356	0.16185	0.07	3.000	2.250	0.200	0.150	0.25

Table 1: Numerical Results for Some Performance Measures

Concluding Remarks

In this paper, For an $M/M/1$ Markovian queueing system with multiple Vacations, the Matrix Geometric technique has been used to find the stationary distribution and to study the stationary queue length distribution along with its mean. Some numerical results have been obtained to display the effect of system parameters on the performance measures and to establish the validity of Little's formula with reasonable accuracy for the chosen parametric values for models with and without vacation.

References

[1] Ammar S. (2015) "Transient analysis of an M/M/1 queue with impatient behavior and multiple vacations". Applied Mathematics And Computation. 1,(260) : 97-105.

[2] Doshi B. (1986) "Queueing systems with vacations : a survey." Queueing systems. 1,(1) : 29-66.

[3] Ibe O., and Isijola O. (2014) "M/M/1 Multiple Vacation Queueing Systems with Differentiated Vacations. Modelling and Simulation in Engineering. 1,(2014) : 18-24.

[4] Kalidass K., and Ramanath K. (2014) "Transient analysis of an M/M/1 queue with multiple vacations". Pakistan Journal of Statistics and Operation Research. 10,(1): 121-130.

[5] Ke J., Wu C. , and Zhang Z. (2010) "Recent developments in vacation queueing models: a short survey". International Journal of Operations Research. 7,(4) : 3-8.

[6] Kumar J., Shinde V., and Singh B. (2011) "Analysis of the M/M/1 Queue Model with Multiple Vacations and Server Break-Down". Journal of International Academy Of Physical Sciences. 15,(3): 291-302.

[7] Kv V., and Janani B. (2015) "Transient Analysis of an M/M/1 queue with Multiple Exponential Vacation and N-Policy". Pakistan Journal of Statistics and Operation Research. 11,(4) : 587-600.

[8] Latouche G., and Ramswami V. (1999) "*Introduction to Matrix Analytic Methods in Stochastic Modeling*". Society for Industrial and Applied Mathematics. Philadelphia. Siam.

[9] Levy Y., and Yechiali U. (1975) "Utilization of idle time in an M/G/1 queueing system." Management Science. 22,(2) : 202-211.

[10] Neuts M. (1981) "*Matrix-geometric solutions in stochastic models: an algorithmic approach*". Johns Hopkins Series in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, Md, USA.

[11] Sudhesh R., and Azhagappan A. (2016) " Transient Analysis of M/M/1 Queue with Server Vacation Customers Impatient and a Waiting Server Timer". Asian Journal of Research in Social Sciences and Humanities. 6,(9) : 1096-1104.

[12] Takagi, H. (1991) "*Queueing analysis* ". North-Holland, Amsterdam

[13] Tian N-S. and Zhang G. (2006) "*Vacation Queueing Models : Theory and Applications*". Springer, NewYork.