Type II Half Logistic Family of Distributions with Applications

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Abstract

A new family of distributions called the type II half logistic is introduced and studied. Four new special models are presented. Some mathematical properties of the type II half logistic family are studied. Explicit expressions for the moments, probability weighted, quantile function, mean deviation, order statistics and Rényi entropy are investigated. Parameter estimates of the family are obtained based on maximum likelihood procedure. Two real data sets are employed to show the usefulness of the new family.

Keywords: Half logistic distribution; Order statistics; Maximum likelihood method.

1. Introduction

The most popular traditional distributions often do not characterize and do not predict most of the interesting data sets. Generated family of continuous distributions is a new improvement for creating and extending the usual classical distributions. The newly generated families have been broadly studied in several areas as well as yield more flexibility in applications. (Eugene et al. 2002) studied the beta-family of distributions. (Zografos and Balakrishnan, 2009) suggested a generated family using gamma distribution which is defined as follows

$$F_1(x) = \frac{1}{\Gamma(\delta)} \int_0^{-\log[1-G(x;\zeta)]} t^{\delta-1} e^{-t} dt, \quad (1)$$

Kumaraswamy generalized family provided by (Cordeiro and de Castro, 2011). Ristic and Balakrishnan, 2011) proposed an alternative gamma generator for any continuous distribution $G(x)$ which is defined as

$$F_2(x) = 1 - \frac{1}{\Gamma(\delta)} \int_0^{-\log G(x;\zeta)} t^{\delta-1} e^{-t} dt, \quad (2)$$

where $\Gamma(\delta) = \int_0^{\infty} t^{\delta-1} e^{-t} dt$ is the gamma function.

Further, some generated families were studied by several authors, for example, the kummer beta by (Pescim et al., 2012), exponentiated generalized class by (Cordeiro et al., 2013), Weibull-G by (Bourguignon et al., 2014), exponentiated half-logistic by (Cordeiro et al., 2014), the type I half-logistic by (Cordeiro et al., 2015), and the Kumaraswamy Weibull by (Hassan and Elgarhy, 2016).

In the current paper, we introduce a recently generated family of distributions using the half logistic distribution as a generator. This paper can be sorted as follows. In the next section, the type II half logistic- generated $(TIIHL-G)$ family is defined. Section 3 concerns with some general mathematical properties of the family. In Section 4, some new special models of the generated family are considered. In Section 5, estimation of the parameters of the family is implemented through maximum likelihood method. An illustrative purpose on the basis of real data is investigated, in Section 6. Finally, concluding remarks are handled in Section 7.

2. Type II Half Logistic Family

The half logistic distribution is a member of the family of logistic distributions which is introduced by (Balakrishnan, 1985) which has the following cumulative distribution function (cdf)

$$F(t)=\frac{1-e^{-\lambda t}}{1+e^{-\lambda t}}, \qquad t>0, \quad \lambda>0. \tag{3}$$

The associated probability density function (pdf) corresponding to (3) is as follows

$$f(t)=\frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2}. \tag{4}$$

On the basis of cdf (2) (see Ristic and Balakrishnan, 2011), we use the half logistic generator instead of gamma generator to obtain type II half logistic family which is denoted by $TIIHL-G$. Hence the cdf of $TIIHL-G$ family can be expressed as follows

$$F(x)=1-\int_0^{-\log G(x;\zeta)}\frac{2\lambda e^{-\lambda t}}{\left(1+e^{-\lambda t}\right)^2}dt=\frac{2\left[G(x;\zeta)\right]^{\lambda}}{1+\left[G(x;\zeta)\right]^{\lambda}}, x>0\ ,\ \lambda>0, \tag{5}$$

where, λ is a scale parameter and $G(x;\xi)$ is a baseline cdf, which depends on a parameter vector ζ. The distribution function (5) provides a broadly type II half logistic generated distributions. Therefore, the pdf of the type II half logistic generated family is as follows

$$f(x)=\frac{2\lambda g(x;\zeta)\left[G(x;\zeta)\right]^{\lambda-1}}{\left[1+\left[G(x;\zeta)\right]^{\lambda}\right]^2}, x>0\ ,\ \lambda>0. \tag{6}$$

Hereafter, a random variable X has pdf (6) will be denoted by $X\sim TIIHL-G.$

The survival function $\overline{F}(x)$ and hazard rate function $h(x)$ are, respectively, given by

$$\overline{F}(x)=\frac{1-[G(x;\zeta)]^{\lambda}}{1+[G(x;\zeta)]^{\lambda}},$$

and

$$h(x)=\frac{2\lambda g(x;\zeta)[G(x;\zeta)]^{\lambda-1}}{1-[G(x;\zeta)]^{2\lambda}}.$$

3. Some Statistical Properties

This section provides some statistical properties of $TIIHL-G$ family of distributions.

3.1 Quantile function

Let X denotes a random variable has the pdf (6), the quantile function, say $Q(u)$ of X is given by

$$Q(u)=G^{-1}\left[\frac{u}{2-u}\right]^{\frac{1}{\lambda}}, \tag{7}$$

where, u is a uniform distribution on the interval $(0,1)$ and $G^{-1}(.)$ is the inverse function of $G(.)$. In particular, the first quartile, the median, and the third quartile are obtained by putting $u=0.25, 0.5$ and 0.75, respectively, in (7).

3.2 Important representation

In this subsection, a useful expansion of the pdf and cdf for$TIIHL-G$ is provided.
Since the generalized binomial series is

$$(1+z)^{-\beta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta+i-1}{i}z^{i}, \tag{8}$$

for $|z|<1,$ and β is a positive real non integer. Then, by applying the binomial theorem (8) in (6), the density function of $TIIHL-G$ family becomes

$$f(x)=\sum_{i=0}^{\infty}2\lambda(-1)^{i}(i+1)g(x;\zeta)[G(x;\zeta)]^{\lambda(i+1)-1} \tag{9}$$

Hence, the pdf (9) can be written as follows

$$f(x)=\sum_{i=0}^{\infty}\eta_{i}g(x;\zeta)G(x;\zeta)^{\lambda(i+1)-1}, \tag{10}$$

where,

$$\eta_{i}=2\lambda(-1)^{i}(i+1),$$

Another formula can be extracted from pdf (9), which gives the following infinite linear combination

$$f(x)=\sum_{i=0}^{\infty}W_i\ h_{\lambda(i+1)}(x), \tag{11}$$

where, $W_i = 2(-1)^i$, and $h_a(x)=a\,g(x;\zeta)G(x;\zeta)^{a-1}$, is the exponentiated-generated (exp-G) density with power parameter a.

Further, an expansion for the $[F(x)]^h$ is derived, for h is integer, again, the binomial expansion is worked out.

$$[F(x)]^h=\sum_{j=0}^{h}(2\lambda)^h(-1)^j\binom{h+j-1}{j}G(x;\zeta)^{\lambda(j+h)}.$$

Again, the binomial expansion is applied to $G(x,\zeta)^{\lambda(j+h)}$ by adding and subtracting 1, then $[F(x)]^h$ can be expressed as follows

$$[F(x)]^h=\sum_{j=0}^{h}\sum_{k=0}^{\infty}(2\lambda)^h(-1)^j\binom{h+j-1}{j}\binom{\lambda(j+h)}{k}[1-G(x;\zeta)]^k.$$

For k is a real, then $[F(x)]^h$ is as follows

$$[F(x)]^h=\sum_{z=0}^{\infty}s_z G(x;\zeta)^z, \tag{12}$$

where,

$$s_z=\sum_{j=0}^{h}\sum_{k=0}^{\infty}(2\lambda)^h(-1)^j\binom{h+j-1}{j}\binom{\lambda(j+h)}{k}\binom{k}{z}.$$

3.3 The probability weighted moments

Class of moments, called the probability-weighted moments (PWMs), has been proposed by (Greenwood et al. (1979)). This class is used to derive estimators of the parameters and quantiles of distributions expressible in inverse form. For a random variable X, the PWMs, denoted by $\tau_{r,s}$, can be calculated through the following relation

$$\tau_{r,s}=E[X^r\,\mathrm{F}(x)^s]=\int_{-\infty}^{\infty}x^r\,\mathrm{f}(x)(\mathrm{F}(x))^s\,dx. \tag{13}$$

The PWMs of $TIIHL-G$ is obtained by substituting (10) and (12) into (13), and replacing h with s, as follows

$$\tau_{r,s}=\int_{-\infty}^{\infty}\sum_{i=0}^{\infty}\sum_{z=0}^{\infty}s_z\eta_i x^r g(x;\zeta)(\mathrm{G}(x;\zeta))^{z+\lambda(i+1)-1}dx.$$

Then,

$$\tau_{r,s}=\sum_{i=0}^{\infty}\sum_{z=0}^{\infty}s_z\eta_i\tau_{r,z+\lambda(i+1)-1},$$

where, $\tau_{r,z+\lambda(i+1)-1}=\int_{-\infty}^{\infty}x^r g(x;\zeta)(\mathrm{G}(x;\zeta))^{z+\lambda(i+1)-1}dx.$

Additionally; another formula will be yielded by using quantile function as follows

$$\tau_{r,s}=\int_0^1\sum_{i=0}^{\infty}\sum_{z=0}^{\infty}s_z\eta_i(Q_G(\mathrm{u}))^r u^{z+\lambda(i+1)-1}du.$$

3.4 Moments

Since the moments are necessary and important in any statistical analysis, especially in applications. Therefore, we derive the rth moment for the $TIIHL-G$ family. If X has the pdf (10), then rth moment is obtained as follows

$$\mu_r^{'}=\int_{-\infty}^{\infty}x^r f(x)dx=\int_{-\infty}^{\infty}\sum_{i=0}^{\infty}\eta_i x^r g(x;\zeta)\mathrm{G}(x;\zeta)^{\lambda(i+1)-1}dx.$$

Then,

$$\mu_r^{'}=\sum_{i=0}^{\infty}\eta_i\tau_{r,\lambda(i+1)-1},\frac{n!}{r!(n-r)!}$$

where, $\tau_{r,\lambda(i+1)-1}$ is the PWMs.

Further, another formula can be deduced, based on the parent quantile function, as follow;

$$\mu_r^{'}=\sum_{i=0}^{\infty}\eta_i\int_0^1(Q_G(\mathrm{u}))^r u^{\lambda(i+1)-1}du.$$

3.5 Moment generating function

For a random variable $X,$ it is known that, the moment generating function is defined as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{'}=\sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\frac{t^r}{r!}\eta_i\tau_{r,\lambda(i+1)-1}.$$

Additionally; different form will be yielded by using quantile function as follows;

$$M_u(t)=\sum_{i=0}^{\infty}\eta_i\int_0^1 e^{(\mathrm{t}Q_G(\mathrm{u}))}u^{\lambda(i+1)-1}du.$$

3.6 The mean deviation

In statistics, mean deviation about the mean and mean deviation about the median measure the amount of scattering in a population. For random variable X with pdf $f(x)$, cdf $F(x)$, the mean deviation about the mean and mean deviation about the median, are defined by

$$\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \qquad \text{and} \qquad \delta_2(X) = \mu - 2T(\mathrm{M}),$$

where, $\mu = E(X)$, M = Median (X), and $T(q) = \int_{-\infty}^{q} x f(x) dx$ which is the first incomplete moment.

Depending on the parent quantile function, additional form is obtained as follows;

$$T(q) = \sum_{i=0}^{\infty} \eta_i \int_0^q Q_G(\mathrm{u}) u^{\lambda(i+1)-1} du.$$

3.7 Order statistics

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $X_1, X_2, ..., X_n$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ the corresponding ordered random sample from a population of size n.

According to (David, 1981), the pdf of the kth order statistic, is defined as

$$f_{k:n}(x) = \frac{f(x)}{B(k, n-k+1)} \sum_{v=0}^{n-k} (-1)^v \binom{n-k}{v} F(x)^{v+k-1}, \tag{14}$$

where, $B(.,.)$ stands for beta function. The pdf of the kth order statistic for $TIIHL-G$ family is derived by substituting (10) and (12) in (14), replacing h with $v+k-1$,

$$f_{k:n}(x) = \frac{g(x;\zeta)}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i=0}^{\infty} \eta_i p_{z,v} \, \mathrm{G}(x;\zeta)^{z+\lambda(i+1)-1}, \tag{15}$$

where $p_{z,v} = \sum_{j=0}^{v+k-1} \sum_{k=0}^{\infty} (2\lambda)^{v+k-1} (-1)^j \binom{v+k+j-2}{j} \binom{\lambda(j+v+k-1)}{k} \binom{k}{z} (-1)^v \binom{n-k}{v}$,

$g(.)$ and $G(.)$ are the pdf and cdf of the $TIIHL-G$ family, respectively.

Further, the rth moment of kth order statistics for $TIIHL-G$ is defined family by:

$$E(\mathrm{X}_{k:n}{}^r) = \int_{-\infty}^{\infty} x^r f_{k:n}(x) dx. \tag{16}$$

By substituting (15) in (16), leads to

$$E(\mathrm{X}_{k:n}{}^r) = \frac{1}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i=0}^{\infty} \eta_i p_{z,v} \int_{-\infty}^{\infty} x^r g(x;\zeta) \mathrm{G}(x;\zeta)^{z+\lambda(i+1)-1} dx.$$

Then,

$$E(X_{k:n}{}^{r}) = \frac{1}{B(k,n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i=0}^{\infty} \eta_i \, p_{z,v} \, \tau_{r,z+\lambda(i+1)-1}.$$

3.8 Rényi entropy

The entropy of a random variable X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. As mentioned by (Renyi 1961), the Rényi entropy is defined by

$$I_{\delta}(X) = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f(x)^{\delta} dx, \quad \delta > 0 \text{ and } \delta \neq 1.$$

By applying the binomial theory (8) in the pdf (6), then the pdf $f(x)^{\delta}$ can be expressed as follows

$$f(x)^{\delta} = \sum_{i=0}^{\infty} t_i \, g(x;\zeta)^{\delta} \, \mathrm{G}(x;\zeta)^{\lambda(i+\delta)-\delta},$$

where

$$t_i = (2\lambda)^{\delta} (-1)^{i} \binom{2\delta + i - 1}{i}.$$

Therefore, the Rényi entropy of $TIIHL$ generated family of distributions is given by

$$I_{\delta}(X) = \frac{1}{1-\delta} \log \sum_{i=0}^{\infty} t_i \int_{-\infty}^{\infty} g(x;\zeta)^{\delta} \, \mathrm{G}(x;\zeta)^{\lambda(i+\delta)-\delta} dx.$$

4. Some special models

In this section, we define and describe four special models of the $TIIHL$ generated family namely, $TIIHL$-uniform, $TIIHL$ -BurrXII, $TIIHL$ -Weibull and $TIIHL$ - quasi Lindley.

4.1 TIIHL-uniform distribution

The pdf of type II half logistic- uniform ($TIIHLU$) is derived from (6), by taking as the following $G(x;\theta) = \frac{x}{\theta}$, and $g(x;\theta) = \frac{1}{\theta}$, $0 < x < \theta$,

$$f(x) = \frac{2\lambda\theta^{\lambda} x^{\lambda-1}}{\left(\theta^{\lambda} + x^{\lambda}\right)^{2}}, \qquad 0 < x < \theta.$$

The corresponding cdf takes the following form

$$F(x) = \frac{2x^{\lambda}}{\theta^{\lambda} + x^{\lambda}}.$$

Moreover, the survival and the hazard rate functions are given, respectively, as follows

$$\overline{F}(x) = \frac{\theta^{\lambda} - x^{\lambda}}{\theta^{\lambda} + x^{\lambda}},$$

and

$$h(x) = \frac{2\lambda\theta^{\lambda}x^{\lambda-1}}{\theta^{2\lambda} - x^{2\lambda}}.$$

The plots of pdf and hazard rate function for the $TIIHLU$ are showed in Figures 1 and 2 respectively.

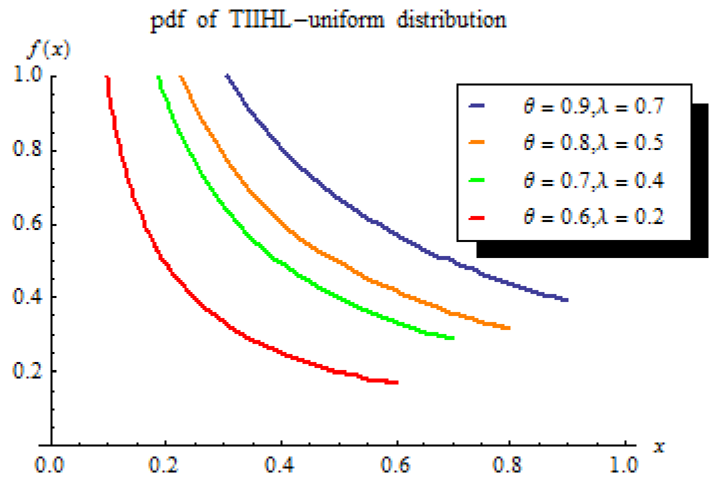

Figure 1. pdf of $TIIHLU$ distribution for different values of parameters

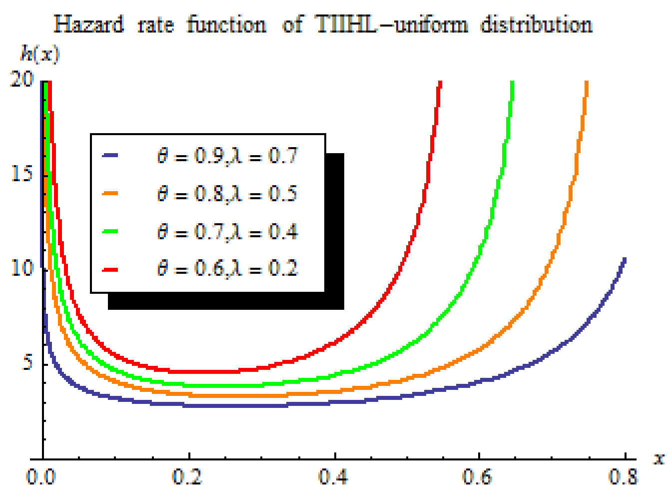

Figure 2. Hazard rate function of $TIIHLU$ distribution for different values of parameters

4.2 TIIHLBurrXII distribution

Let us consider the Burr XII distribution with probability density and distribution functions given, respectively by

$$g(x;c,\mu,\sigma)=c\sigma\mu^{-c}x^{c-1}[1+(\frac{x}{\mu})^{c}]^{-\sigma-1},\quad c,\mu,\sigma>0,$$

and,

$$G(x;c,\mu,\sigma)=1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}.$$

Then the *TIIHLBurrXII* distribution has the following cdf, pdf, survival and hazard rate functions

$$F(x)=\frac{2\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda}}{1+\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda}}\ ,\qquad \lambda,c,\mu,\sigma>0\quad,\quad x>0,$$

$$f(x)=\frac{2\lambda c\sigma\mu^{-c}x^{c-1}(1+(\frac{x}{\mu})^{c})^{-\sigma-1}\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda-1}}{\left[1+\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda}\right]^{2}},$$

$$\overline{F}(x)=\frac{1-\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda}}{1+\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda}},$$

and

$$h(x)=\frac{2\lambda c\sigma\mu^{-c}x^{c-1}(1+(\frac{x}{\mu})^{c})^{-\sigma-1}\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{\lambda-1}}{1-\left[1-[1+(\frac{x}{\mu})^{c}]^{-\sigma}\right]^{2\lambda}}.$$

The plots of pdf and hazard rate function for *TIIHLBurrXII* are displayed in Figures 3 and 4 respectively.

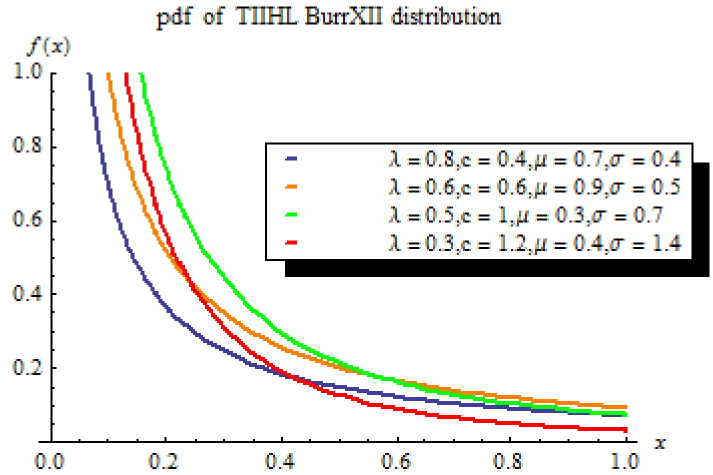

Figure 3. pdf of $TIIHLBurrXII$ distribution for different values of parameters

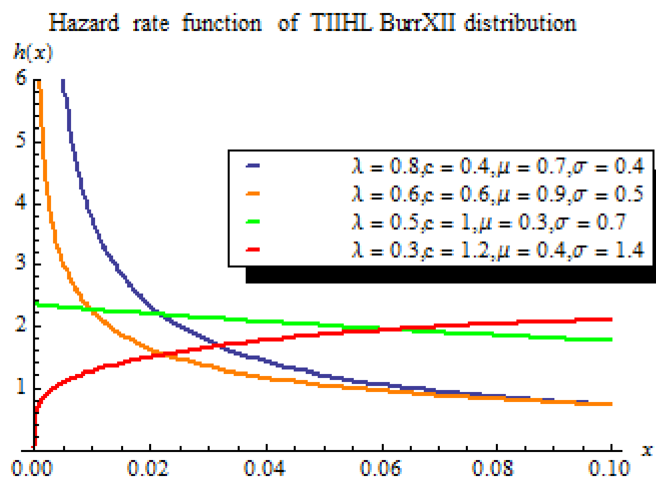

Figure 4. Hazard rate function of $TIIHLBurrXII$ distribution for different values of parameters

4.3 TIIHL-Weibull distribution

The cdf and pdf of TIIHL-Weibull $(TIIHLW)$ distribution are derived from (5) and (6) taking $G(x;\delta,\gamma)=1-e^{-\delta x^{\gamma}}$, as the following

$$F(x)=\frac{2\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}}{1+\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}} \qquad \lambda,\delta,\gamma>0 \quad , \quad x>0,$$

and,

$$f(x)=\frac{2\lambda\delta\gamma x^{\gamma-1}e^{-\delta x^{\gamma}}[1-e^{-\delta x^{\gamma}}]^{\lambda-1}}{\left[1+[1-e^{-\delta x^{\gamma}}]^{\lambda}\right]^{2}}.$$

Further, the survival and hazard rate functions are as follows

$$\overline{F}(x) = \frac{1-\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}}{1+\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}},$$

and

$$h(x) = \frac{2\lambda\delta\gamma x^{\gamma-1} e^{-\delta x^{\gamma}} [1-e^{-\delta x^{\gamma}}]^{\lambda-1}}{1-[1-e^{-\delta x^{\gamma}}]^{2\lambda}},$$

when $\gamma = 1$, we get $TIIHL-$exponential distribution. The plots of pdf and hazard rate function for the $TIIHLW$ are presented in Figures 5 and 6 respectively.

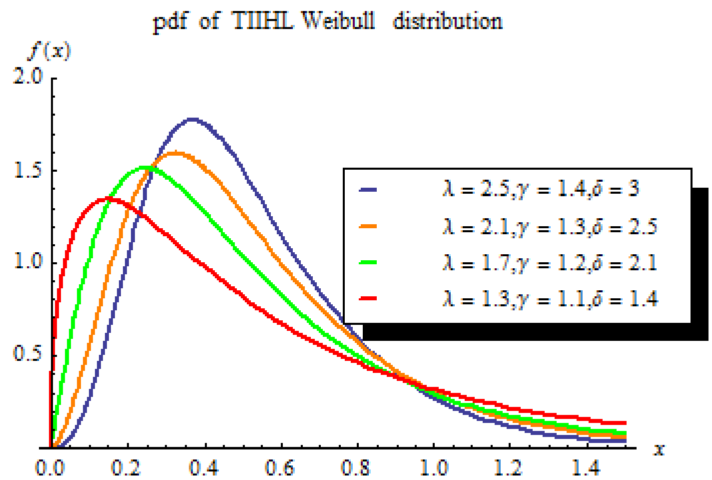

Figure 5. pdf of $TIIHLW$ distribution for different values of parameters

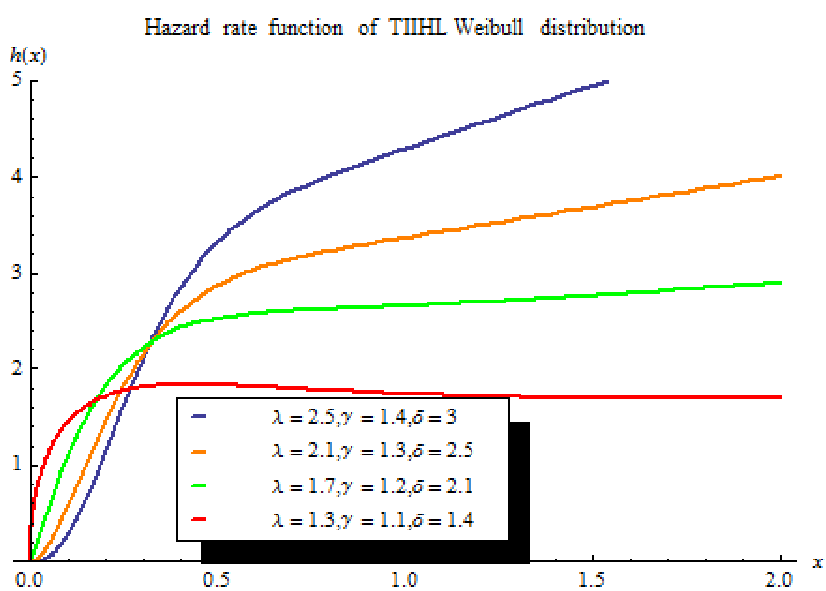

Figure 6. Hazard rate function of $TIIHLW$ distribution for different values of parameters

4.4 TIIHL- quasi Lindley distribution

The quasi Lindley distribution has been suggested by (Shanker and Mishra, 2013). The probability density and distribution functions of quasi Lindley distribution are given by;

$$G(x;\theta,p)=1-e^{-\theta x}[1+\frac{\theta x}{p+1}],\quad g(x;\theta,p)=\frac{\theta}{p+1}(p+\theta x)e^{-\theta x}$$

The cdf, pdf, survival and the hazard rate functions for TIIHL-quasi Lindley distribution $(TIIHLQL)$ are obtained from (5) and (6), respectively as

$$F(x)=\frac{2\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}}{1+\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}},\qquad \lambda,\theta>0\ ,p>-1,\ x>0,$$

$$f(x)=\frac{2\lambda\theta(p+\theta x)e^{-\theta x}\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda-1}}{(p+1)\left[1+\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}\right]^{2}},$$

$$\overline{F}(x)=\frac{1-\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}}{1+\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}},$$

and

$$h(x)=\frac{2\lambda\theta(p+\theta x)e^{-\theta x}\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda-1}}{(p+1)\left[1-\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{2\lambda}\right]}.$$

For $p=\theta$ the $TIIHL-$Lindley distribution is obtained. The plots of pdf and hazard rate function for the $TIIHLQL$ are given in Figures 7 and 8 respectively.

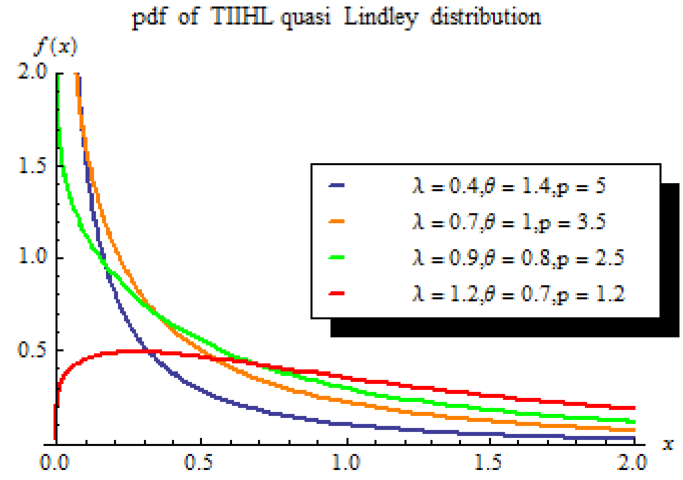

Figure 7. pdf of $TIIHLQL$ distribution for different values of parameters

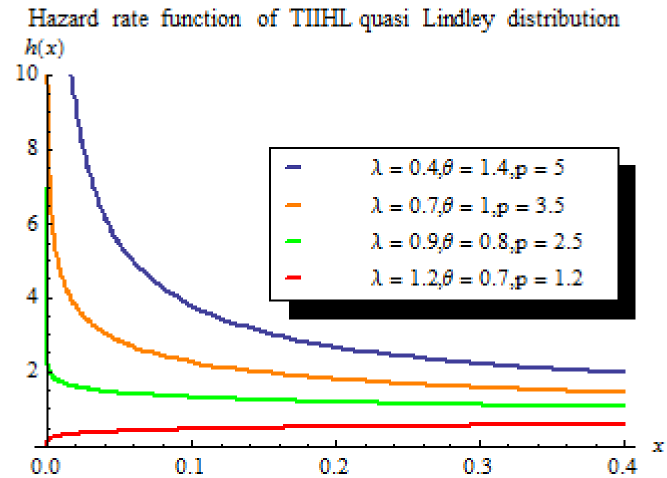

Figure 8. Hazard rate function of $TIIHLQL$ distribution for different values of parameters

5. Maximum Likelihood Method

This section deals with the maximum likelihood estimators of the unknown parameters for the $TIIHL-G$ family of distributions on the basis of complete samples. Let $X_1,...,X_n$ be the observed values from the $TIIHL-G$ family with set of parameter $\Phi=(\lambda,\zeta)^T$. The log-likelihood function for parameter vector $\Phi=(\lambda,\zeta)^T$ is obtained as follows

$$\ln L(\Phi)=n\ln 2\lambda+\sum_{i=1}^{n}\ln\left[g(x_i;\zeta)\right]+(\lambda-1)\sum_{i=1}^{n}\ln\left[G(x_i;\zeta)\right]-2\sum_{i=1}^{n}\ln[1+\left[G(x_i;\zeta)\right]^{\lambda}].$$

The elements of the score function $U(\Phi)=(U_{\lambda},\zeta_k)$ are given by

$$U_{\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left[G(x_i;\zeta)\right]-2\sum_{i=1}^{n}\frac{\left[G(x_i;\zeta)\right]^{\lambda}\ln\left[G(x_i;\zeta)\right]}{[1+\left[G(x_i;\zeta)\right]^{\lambda}]},$$

and

$$U_{\zeta_k}=\sum_{i=1}^{n}\frac{\partial g(x_i;\zeta)/\partial\zeta_k}{g(x_i;\zeta)}+(\lambda-1)\sum_{i=1}^{n}\frac{\partial G(x_i;\zeta)/\partial\zeta_k}{G(x_i;\zeta)}-2\lambda\sum_{i=1}^{n}\frac{[G(x_i;\zeta)]^{\lambda-1}\partial G(x_i;\zeta)/\partial\zeta_k}{1+\left[G(x_i;\zeta)\right]^{\lambda}}.$$

Setting U_{λ} and U_{ζ_k} equal to zero and solving these equations simultaneously yield the maximum likelihood estimate (MLE) $\hat{\Phi}=(\hat{\lambda},\hat{\zeta})$ of $\Phi=(\lambda,\zeta)^T$. These equations cannot be solved analytically and statistical software can be used to solve them numerically using iterative methods.

For interval estimation of the parameters, the $k\times k$ observed information matrix $I(\Phi)=\{I_{uv}\}$ for $(u,v=\lambda,\zeta_k)$ whose elements are given in Appendix. Under the regularity conditions, the known asymptotic properties of the maximum likelihood method ensure that: $\sqrt{n}(\hat{\Phi}-\Phi)\underset{\rightarrow}{d}N_2(0,I^{-1}(\Phi))$ as $n\rightarrow\infty$, where $\underset{\rightarrow}{d}$ means the convergence in distribution, with mean 0 and $k\times k$ covariance matrix $I^{-1}(\Phi)$ then, the $100(1-\alpha)\%$ confidence intervals for λ and ζ are given, respectively, as follows $\hat{\lambda}\pm z_{\frac{\alpha}{2}}\sqrt{\text{var}(\hat{\lambda})}$, $\hat{\zeta}\pm z_{\frac{\alpha}{2}}\sqrt{\text{var}(\hat{\zeta})}$ where $z_{\frac{\alpha}{2}}$ is the standard normal at $\frac{\alpha}{2}$, $\frac{\alpha}{2}$ is significance level and var(.)'s denote the diagonal elements of $I^{-1}(\Phi)$ corresponding to the model parameters.

6. Applications to Real Data

In this section, two real data sets are employed to illustrate the importance of the suggested $TIIHL-G$ family. Recently, considerable extensions of Weibull have been introduced, in the literature, by several authors, such as type I half logistic Weibull ($TIIHLW$) by (Corediro, et al. 2015), beta Weibull (BW) by (Lee et al., 2007) and Weibull Weibull (WW) by (Bourguignon et al., 2014). We fit $TIIHLW$ distribution to the two real data sets using MLEs and compared the suggested distribution with $TIHLW$, BW and WW distributions. The required numerical evaluations were implemented using MathCAD 14.

Data set 1 is obtained from (Hinkley, 1977), it consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data are as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Data set 2 represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by (Bjerkedal, 1960). The data are as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Various criteria are used to compare the four suggested models. The selected criteria are; $-2\ln L$, Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion ($CAIC$), Hannan-Quinn information criterion ($HQIC$), the Kolmogorov-Smirnov ($K-S$) and $p-value$ statistics. The formula for these criteria is as follows

$$AIC = 2k - 2\ln L\ ,\ BIC = k\ \ln(n) - 2\ln L,\ CAIC = AIC + \frac{2k\,(k+1)}{n-k-1},$$

$$HQIC = 2k\ \ln\left[\ln\left(n\right)\right] - 2\ln L, \qquad k-s = \sup_y \left[F_n(\mathrm{y}) - F(\mathrm{y})\right],$$

where k is the number of parameters in the statistical model, n is the sample size and $\ln L$ is the maximized value of the log- likelihood function under the considered model, $F_n(\mathrm{y})$ is the empirical distribution function and $F(\mathrm{y})$ denotes the cdf for each distribution.

The "best" distribution corresponds to the smallest values of $-2\ln L$, AIC, BIC, $CAIC$, $HQIC$, $K-S$ and the biggest value of $p-value$ criteria .

The following table shows the MLEs of the model parameters and its standard error (S.E) (in parentheses) for data set 1.

Table 1: The MLEs and S.E of the model parameters for data set 1

Model	**MLEs and S.E** (in parentheses)			
$TIIHLW(\lambda,\delta,\gamma)$	1.823 (0.42093)	0.451 (0.224)	1.465 (0.113)	-
$TIHLW(\lambda,\delta,\gamma)$	0.889 (0.1791)	0.626 (0.124)	1.532 (0.293)	-
$BW(a,b,\delta,\gamma)$	25.851 (1.533)	15.276 (0.787)	0.884 (0.201)	0.335 (0.027)
$WW(\alpha,\beta,\delta,\gamma)$	39.853 (0.414)	3.154 (0.518)	0.196 (0.102)	0.5 (0.072)

The variance covariance matrix $I(\hat{\theta})^{-1}$, where $\theta=(\lambda,\delta,\gamma)$, of the MLEs under the the type II half logistic Weibull distribution for data set 1 is computed as follows

$$\begin{pmatrix} 0.177 & 0.06 & -0.014 \\ 0.06 & 0.05 & -0.011 \\ -0.014 & -0.011 & 0.013 \end{pmatrix}.$$

Thus, the variances of the MLE of λ,δ and γ are $var(\hat{\lambda})=0.177, var(\hat{\delta})=0.05$ and $var(\hat{\gamma})=0.013$. Therefore, 95% confidence intervals for λ,δ and γ are $[0.998, 2.648], [0.012, 0.89]$ and $[1.244, 1.686]$ respectively.The following table gives the values of mesurments for the data set 1.

Table 2: Model Section Criteria for data set 1

Model	$-2\ln L$	AIC	BIC	$CAIC$	$HQIC$	$K-S$	$p-value$
$TIIHLW$	100.986	106.986	106.558	107.909	109.705	0.06239	0.99981
$TIHLW$	106.639	112.639	112.211	113.562	115.358	0.069	0.99881
BW	149.897	157.897	157.326	159.497	161.522	0.07958	0.9913
WW	138.194	146.194	145.623	147.794	149.819	0.07549	0.99554

Figures 9 and 10 provide the plots of estimated cumulative and estimated densities of the fitted $TIIHLW$, $TIHLW$, BW and WW models for the data set 1.

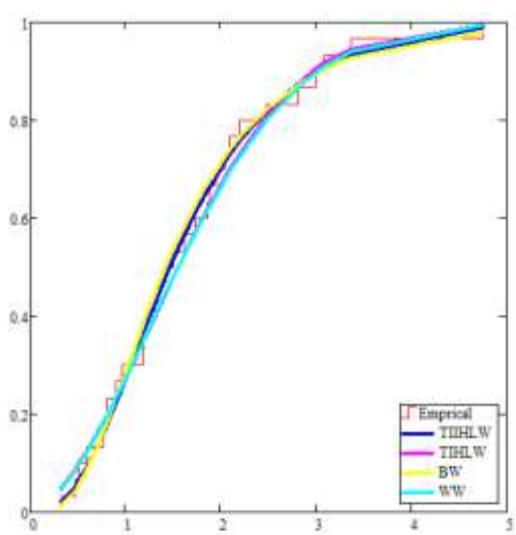

Figure 9. Estimated cumulative densities of the models for data set 1.

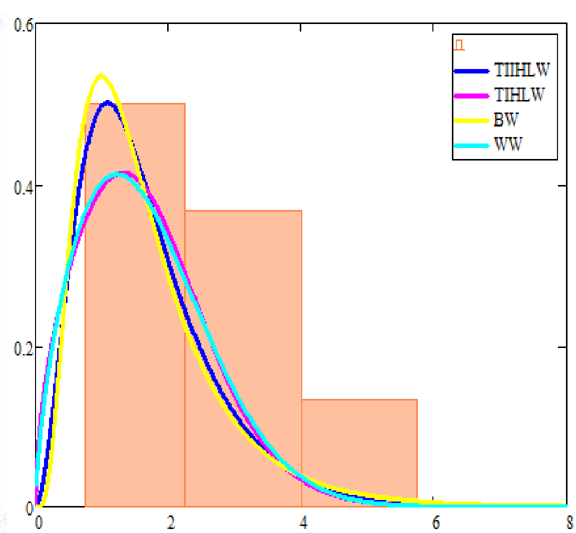

Figure 10. Estimated densities of the models for data set 1.

MLEs of the model parameters and its standard error (S.E) (in parentheses) for data set 2 are listed in the following table.

Table 3: The MLEs and S.E of the model parameters for data set 2.

Model	**MLEs and S.E** (in parentheses)			
$TIIHLW(\lambda,\delta,\gamma)$	2.033 (0.42795)	0.46 (0.189)	1.434 (0.082)	-
$TIHLW(\lambda,\delta,\gamma)$	0.952 (0.23859)	0.544 (0.136)	1.535 (0.035)	-
$BW(a,b,\delta,\gamma)$	31.789 (0.9)	16.887 (0.558)	0.937 (0.133)	0.307 (0.023)
$WW(\alpha,\beta,\delta,\gamma)$	48.725 (0.27)	2.947 (0.324)	0.162 (0.063)	0.546 (0.047)

The variance covariance matrix $I(\hat{\theta})^{-1}$, where $\theta=(\lambda,\delta,\gamma)$, of the MLEs under the the type II half logistic Weibull distribution for data set 2 is computed as

$$\begin{pmatrix} 0.183 & 0.054 & 0.005 \\ 0.054 & 0.036 & -9.73\times10^{-3} \\ 0.005 & -9.73\times10^{-3} & 6.7\times10^{-3} \end{pmatrix}.$$

Thus, the variances of the MLE of λ,δ and γ are $var(\hat{\lambda})=0.183, var(\hat{\delta})=0.036$ and $var(\hat{\gamma})=6.7\times10^{-3}$. Therefore, 95% confidence intervals for λ,δ and γ are $[1.194, 2.872]$, $[0.089, 0.83]$ and $[1.273, 1.595]$ respectively. The following table contains the values of mesurments for the data set 2.

Table 4: Model Section Criteria for data set 2

Model	$-2\ln L$	AIC	BIC	$CAIC$	$HQIC$	$K-S$	$p-value$
$TIIHLW$	235.199	241.199	240.771	241.552	243.919	0.08426	0.68621
$TIHLW$	354.017	360.017	359.589	360.37	362.736	0.113	0.31881
BW	327.349	335.349	334.779	335.946	338.975	0.09951	0.47392
WW	337.304	345.304	344.734	345.901	348.93	0.10975	0.35099

Figures 11 and 12 provide the plots of estimated cumulative and estimated densities of the fitted $TIIHLW$, $TIHLW$, BW and WW models for the data set 2.

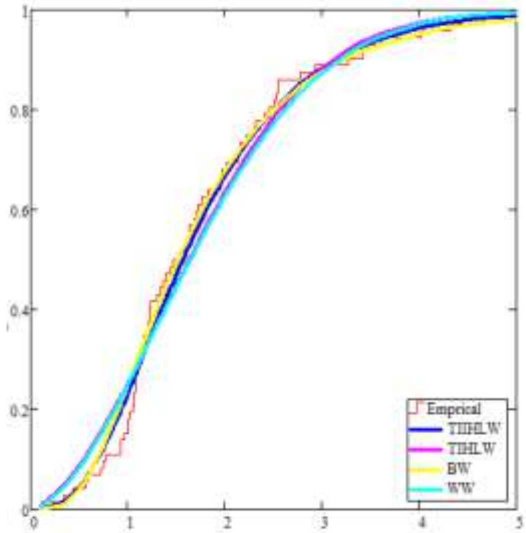

Figure 11. Estimated cumulative densities of the models for data set 2.

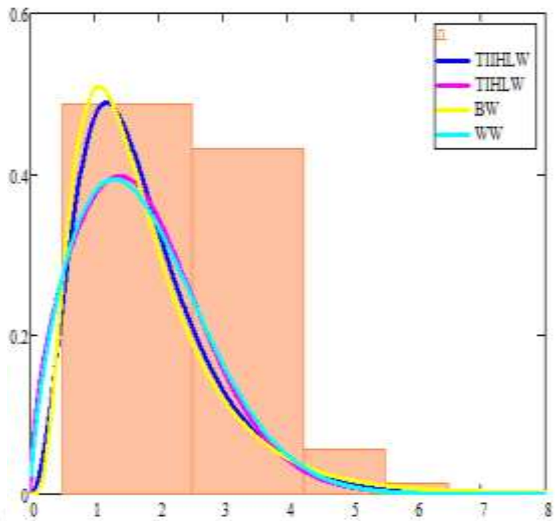

Figure 12. Estimated densities of the models for data set 2.

The values in Tables 2 and 4, indicate that the type II half logistic Weibull distribution is a strong competitor to other distributions used here for fitting data sets 1 and 2. A density plot compares the fitted densities of the models with the empirical histogram of the observed data (Figures 10 and 12). The fitted density for the type II half logistic Weibull model is the closest to the empirical histogram than the other fitted models.

7. Conclusion

In the present paper, the new type II half logistic generated family of distributions is proposed. More specifically, the type II half logistic generated family covers several new distributions. We wish a broadly statistical application in some area for this new generation. Some characteristics of the $TIIHL-G$, such as, expressions for the density function, moments, mean deviation, quantile function and order statistics are discussed. The maximum likelihood method is employed for estimating the model parameters. Type II half logistic uniform, type II half logistic Weibull, type II half logistic BurrXII and type II half logistic quasi Lindley selected models are provided. Applications to real data sets validate the priority of the new family.

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Appendix

The elements of the observed Fisher information matrix $I(\Phi)$, are given by

$$U_{\lambda\lambda}=\frac{-n}{\lambda^2}-2\sum_{i=1}^{n}\frac{\left[G(x_i;\zeta)\right]^{\lambda}\left[\ln[G(x_i;\zeta)]\right]^2}{\left[1+[G(x_i;\zeta)]^{\lambda}\right]^2},$$

$$U_{\lambda\zeta_k}=\sum_{i=1}^{n}\frac{\partial G(x_i;\zeta)/\partial\zeta_k}{G(x_i;\zeta)}-2\sum_{i=1}^{n}\frac{[G(x_i;\zeta)]^{\lambda-1}\partial G(x_i;\zeta)/\partial\zeta_k}{1+\left[G(x_i;\zeta)\right]^{\lambda}}$$
$$-2\lambda\sum_{i=1}^{n}\frac{[G(x_i;\zeta)]^{\lambda-1}\ln[G(x_i;\zeta)]\partial G(x_i;\zeta)/\partial\zeta_k}{\left[1+\left[G(x_i;\zeta)\right]^{\lambda}\right]^2},$$

and

$$U_{\zeta_k\zeta_l}=\sum_{i=1}^{n}\frac{g(x_i;\zeta)\left[\partial^2 g(x_i;\zeta)/\partial\zeta_k\partial\zeta_l\right]-\left[\partial g(x_i;\zeta)/\partial\zeta_k\right]\left[\partial g(x_i;\zeta)/\partial\zeta_k\right]}{\left[g(x_i,\zeta)\right]^2}$$
$$+(\lambda-1)\sum_{i=1}^{n}\frac{G(x_i;\zeta)\left[\partial^2 G(x_i;\zeta)/\partial\zeta_k\partial\zeta_l\right]-\left[\partial G(x_i;\zeta)/\partial\zeta_k\right]\left[\partial G(x_i;\zeta)/\partial\zeta_k\right]}{\left[G(x_i,\zeta)\right]^2}$$
$$-2\lambda\sum_{i=1}^{n}\frac{[G(x_i;\zeta)]^{\lambda-1}\ln\left[G(x_i;\zeta\right]\partial G(x_i;\zeta)/\partial\zeta_l\partial G(x_i;\zeta)/\partial\zeta_k}{\left[1+\left[G(x_i;\zeta)\right]^{\lambda}\right]^2}$$
$$-2\lambda\sum_{i=1}^{n}\frac{[G(x_i;\zeta)]^{\lambda-1}\partial^2 g(x_i;\zeta)/\partial\zeta_k\partial\zeta_l}{1+\left[G(x_i;\zeta)\right]^{\lambda}}.$$

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