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Statistical Inference of the Half Logistic Modified Kies Exponential Model with Modeling to Engineering Data

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Abstract: The half-logistic modified Kies exponential (HLMKEx) distribution is a novel three-parameter model that is introduced in the current work to expand the modified Kies exponential distribution and improve its flexibility in modeling real-world data. Due to its versatility, the density function of the HLMKEx distribution offers symmetrical, asymmetrical, unimodal, and reversed-J-shaped, as well as increasing, reversed-J shaped, and upside-down hazard rate forms. An infinite linear representation can be used to represent the HLMKEx density. The HLMKEx model's fundamental mathematical features are obtained, such as the quantile function, moments, incomplete moments, and moments of residuals. Additionally, some measures of uncertainty as well as stochastic ordering are derived. To estimate its parameters, eight estimation methods are used. With the use of detailed simulation data, we compare the performance of each estimating technique and obtain partial and total ranks for the accuracy measures of absolute bias, mean squared error, and mean absolute relative error. The simulation results demonstrate that, in contrast to other competing distributions, the proposed distribution can actually fit the data more accurately. Two actual data sets are investigated in the field of engineering to demonstrate the adaptability and application of the suggested distribution. The findings demonstrate that, in contrast to other competing distributions, the provided distribution can actually fit the data more accurately.

Keywords: half logistic family; modified Kies exponential distribution; symmetric; asymmetric; moments; entropy measures; maximum product spacing

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1. Motivation and Introduction

In the recent past, there has already been a lot of effort placed on creating more flexible distributions. Several generated (G) families of distributions have been built and studied over the past few decades to simulate real-world data in a number of practice areas, including engineering, economics, medical sciences, biological research, environmental studies, and insurance. In this context, by generalizing G families, many different types of distributions have been produced. These new families allow for additional variety because at least one shape parameter is combined with the baseline one; see Refs. [1–18]. Our interest here is with the type II half logistic (HL) G family prepared in Ref. [19], which has the following cumulative distribution function (CDF) and probability density function (PDF):

$$F(z) = \frac{2[G(z;\varsigma)]^{\eta}}{1+[G(z;\varsigma)]^{\eta}}, \qquad z \in R, \tag{1}$$

and

$$f(z) = \frac{2\,g(z;\varsigma)[\,G(z;\varsigma)\,]^{\eta-1}}{[1+[\,G(z;\varsigma)]^{\eta}]^{2}}, \quad z \in R, \tag{2}$$

where $\eta > 0$ is the shape parameter, $G(.)$ is a baseline CDF, and ς is the parameter vector. Equation (2) is easiest to solve when $G(.)$ and $g(.)$ have straightforward expressions. Several useful distributions have been provided using the HL-G family; for instance, the reader can refer to [20–24].

In a wide range of industries, the exponential (Ex) distribution is the most often used distribution for data analysis. The Ex distribution, however, can only be applied when the hazard rate is constant in many real-world scenarios. Its uni-modal density function and constant hazard rate function (HF), however, limit its uses and prevent it from being used to describe phenomena with decreasing, increasing, or bathtub-shaped hazard rates. As a result, the Ex distribution has been extended in the statistical literature in order to increase its validity and flexibility; see, for example, Refs. [25–30].

Recently, Ref. [31] proposed a new extension for the Ex distribution, known as the modified Kies-exponential (MKEx), with the following CDF and PDF:

$$G(z) = 1 - \exp\{-[\exp(\theta z) - 1]^{\gamma}\}, \quad z > 0, \tag{3}$$

and

$$g(z) = \gamma\,\theta \exp(\gamma\,\theta\, z)[1 - \exp(\theta\, z)]^{\gamma-1} \exp\{-[\exp(\theta z) - 1]^{\gamma}\}, \quad z > 0, \tag{4}$$

where $\theta, \gamma \in R^{+}$ are the scale and shape parameters, respectively. The MKEx distribution is increasing, decreasing, or bathtub HR-shaped. Reference [31] claimed that the MKEx distribution has a closed-form CDF and is very user friendly, making it suitable for usage in a variety of domains, such as survival analysis, biomedical investigations, reliability testing, and life testing. Reference [32] produced a new generalized form of MKEx distribution by using power transformation.

In this study, a new three-parameter model called the half logistic MKEx (HLMKEx) distribution will be introduced along with an examination of its statistical properties. Due to the following, we are motivated to suggest the HLMKEx distribution:

1. The proposed distribution is remarkably versatile, as evidenced by the variety of symmetrical and asymmetrical graphical shapes of the PDF and HF (Figure 1).

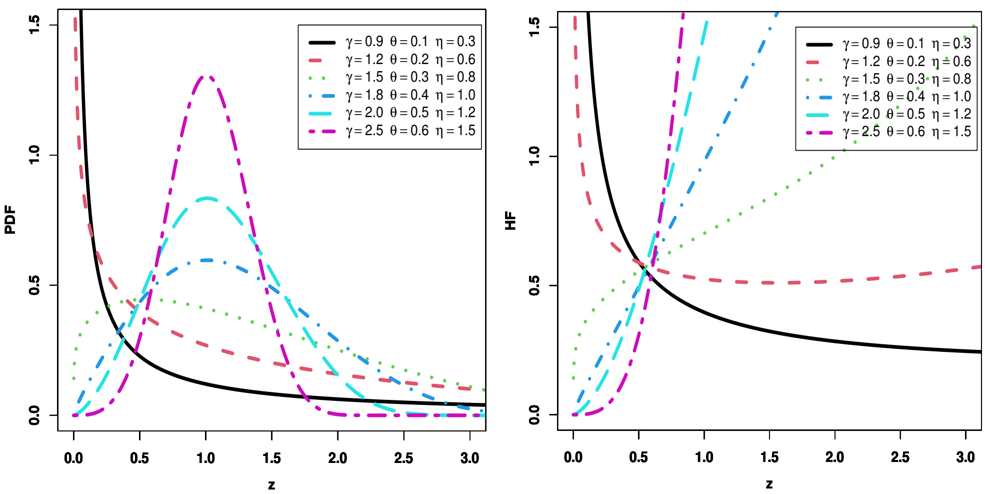

Figure 1. Plots of the PDF and HF for the HLMKEx model.

2. The closed-form expression of the CDF of this distribution makes it perfect for use in some areas, including engineering, reliability, life testing, survival analysis, and biomedical studies.
3. Some statistical properties, including linear representation of its PDF, moments, incomplete moments, moments of residual, entropy measures, and stochastic ordering, are explored.
4. To assess the behavior of the parameters, eight estimating techniques are recommended. The suggested methods are maximum likelihood (ML), least squares (LS), weighted LS (WLS), Anderson–Darling (AD), right-tail AD, percentiles, maximum product of the spacings (MS), and Cramér–von Mises (CM).
5. We conduct comprehensive simulation tests to examine the behaviors of various estimating methodologies using absolute bias (BIAS), mean squared error (MSE) and mean absolute relative errors (AREs) criteria because it is challenging to theoretically compare these behaviors.
6. Engineering real-world data sets are used to examine this new distribution's capabilities, demonstrating the model's usefulness in applications.

The layout of this article is as follows: In Section 2, a novel model that modifies the MKEx distribution is introduced. In Section 3, the HLMKEx distribution's basic characteristics are derived. Section 4 discusses eight distinct techniques for estimating model parameters. Section 5 performs a numerical investigation using Monte Carlo simulations. In Section 6, real-world data sets are numerically investigated, and Section 7 offers the findings.

2. Model Formulation

In this section, the CDF, and PDF of the HLMKEx distribution are defined by setting (3), (4) in (1) and (2). The HF, and quantile function (QF) of the HLMKEx distribution are also given.

Definition 1. *A random variable Z is assumed to have the HLMKEx distribution if its CDF is*

$$F(z;\omega) = \frac{2[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}}{1+[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}}, \qquad z,\omega>0, \tag{5}$$

where $\mathrm{K}(z,\theta,\gamma) = [\exp(\theta z)-1]^{\gamma}$, *and* $\omega \equiv (\eta,\theta,\gamma)$ *is the set of parameters. The PDF of the HLMKEx distribution is*

$$f(z;\omega) = \frac{2\eta\theta\gamma e^{\theta\gamma z}\left[1-e^{-\theta z}\right]^{\gamma-1}\exp\{-\mathrm{K}(z,\theta,\gamma)\}[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta-1}}{\left(1+[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}\right)^{2}}, \; z>0. \tag{6}$$

The survival, HF, and cumulative HF are as follows:

$$\bar{F}(z;\omega) = \frac{1-[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}}{1+[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}}, \qquad z>0,$$

$$h(z;\omega) = \frac{2\eta\theta\gamma e^{\theta\gamma z}\left[1-e^{-\theta\gamma z}\right]^{\gamma-1}\exp\{-\mathrm{K}(z,\theta,\gamma)\}[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta-1}}{1-[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{2\eta}},$$

and

$$H(z;\omega) = -\ln\left(\frac{1-[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}}{1+[1-\exp\{-\mathrm{K}(z,\theta,\gamma)\}]^{\eta}}\right).$$

Figure 1 shows the PDF and HF plots of the HLMKEx distribution for various parameter selections. The PDF shapes might be uni-modal, symmetrical, or asymmetrical (right-skewed). The shapes of the HF are J-shaped, increasing, and upside-down patterns.

The QF of the HLMKEx distribution is produced by inverting CDF (5) as follows:

$$z = \frac{1}{\theta}\ln\left[1 + \left(-\ln\left[1 - \left(\frac{u}{2-u}\right)^{\frac{1}{\eta}}\right]\right)^{\frac{1}{\gamma}}\right], \tag{7}$$

where u is a uniform distribution on (0, 1). We can find the median, upper, and lower quantiles by putting u = 0.5, 0.75, and 0.25 in (7), respectively.

3. Mathematical Properties

In this section, linear representation, moments and other associated measures, some entropy measures, and stochastic ordering (SO) of the HLMKEx distribution are all determined.

3.1. Linear Representation

Here, we express the HLMKEx PDF as a mixture of linear representation, which is helpful for examining the majority of the distribution's statistical characteristics.

Consider the following power series:

$$(1+t)^{-c} = \sum_{u_1=0}^{\infty} (-1)^{u_1} \binom{c+u_1-1}{u_1} t^{u_1}, \quad |t| < 1,\ c > 0, \tag{8}$$

and

$$(1-t)^{d-1} = \sum_{u_2=0}^{\infty} (-1)^{u_2} \binom{d-1}{u_2} t^{u_2}, \quad |t| < 1,\ d > 0, \tag{9}$$

Using Expansions (8) and (9) in (6) give

$$f(z;\omega) = \sum_{u_1,u_2=0}^{\infty} \psi_{u_1,u_2} e^{\theta\gamma z}\left(1 - e^{-\theta z}\right)^{\gamma-1} \exp\{-(u_2+1)\mathrm{K}(z,\theta,\gamma)\}, \tag{10}$$

where $\psi_{u_1,u_2} = (-1)^{u_1+u_2} 2(u_1+1)\gamma\eta\theta \binom{\eta(u_1+1)-1}{u_2}$. After using exponential and binomial expansions in (10), it can then be expressed by

$$f(z;\omega) = \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \Xi_{u_m,j_m} \exp[-\theta z(j_2 - \gamma(j_1+1))], \tag{11}$$

where $\Xi_{u_m,j_m} = \frac{(-1)^{j_1+j_2}\psi_{u_1,u_2}(u_2+1)^{j_1}}{j_1!} \binom{\gamma j_1 + \gamma - 1}{j_2}$, $m = 1, 2.$

3.2. Moments Measures

The nth moment of the HLMKEx distribution is obtained as follows:

$$\begin{aligned} \mu'_n &= \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \Xi_{u_m,j_m} \int_0^{\infty} z^n \exp -[\theta(j_2 - \gamma(j_1+1))z]dz \\ &= \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \Xi_{u_m,j_m} \frac{\Gamma(n+1)}{[\theta(j_2 - (\gamma j_1 + 1))]^{n+1}}, \end{aligned} \tag{12}$$

where $\Gamma(.)$ is the gamma function. Setting n = 1, 2, 3, and 4 in (12), the first four ordinary moments of Z are obtained.

The nth incomplete moment, say $\Im_n(x)$, of the HLMKEx distribution is given by

$$\begin{aligned}\Im_n(x) &= \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \Xi_{u_m,j_m} \int_0^x z^n \exp -[\theta(j_2 - \gamma(j_1+1))z]dz \\ &= \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \Xi_{u_m,j_m} \Gamma\left(\frac{n+1}{[\theta(j_2-\gamma(j_1+1))]^{n+1}}, \theta(j_2-\gamma(j_1+1))x \right),\end{aligned} \tag{13}$$

where $\Gamma(.,x)$ is the incomplete gamma (IG) function.

The Bonferroni and Lorenz curves are two examples of applications of the first incomplete moment for n = 1. These curves are frequently used in many different professions.

Tables 1 and 2 list some statistical computations of the first four moments—variance (V), coefficient of skewness (CS), coefficient of kurtosis (CK), and coefficient of variation (CV)—for the HLMKEx model for various values of the model parameters. According to the CS and CK values, we may deduce from these tables that the distribution is right skewed, platykurtic, and leptokurtic. The 3D plots of measures (μ'_1, V, CS, CK, CV, and index of dispersion (ID)) are shown in Figures 2 and 3 for further explanation.

Table 1. Results of μ'_1, μ'_2, μ'_3, μ'_4,V, CS, CK and CV for the HLMKEx model η = 1.2.

θ	γ	μ'_1	μ'_2	μ'_3	μ'_4	V	CS	CK	CV
0.5	0.4	0.157	0.171	0.272	0.525	0.146	3.578	17.7	2.439
	0.6	0.232	0.255	0.408	0.788	0.202	2.825	11.89	1.939
	0.8	0.303	0.339	0.543	1.05	0.247	2.371	9.054	1.64
	1.1	0.403	0.462	0.745	1.441	0.299	1.936	6.818	1.357
	1.3	0.465	0.542	0.878	1.702	0.326	1.734	5.942	1.226
	1.7	0.579	0.699	1.142	2.219	0.363	1.444	4.871	1.041
	1.9	0.631	0.774	1.272	2.476	0.376	1.336	4.529	0.972
	2.4	0.748	0.956	1.591	3.114	0.396	1.132	3.972	0.842
0.8	0.4	0.163	0.11	0.108	0.129	0.084	2.608	10.56	1.773
	0.6	0.236	0.164	0.162	0.194	0.108	2.03	7.33	1.395
	0.8	0.302	0.216	0.215	0.258	0.125	1.686	5.812	1.173
	1.1	0.388	0.291	0.294	0.353	0.14	1.364	4.669	0.966
	1.3	0.438	0.338	0.345	0.416	0.146	1.218	4.246	0.873
	1.7	0.525	0.428	0.444	0.54	0.152	1.016	3.759	0.741
	1.9	0.563	0.47	0.492	0.601	0.152	0.943	3.616	0.693
	2.4	0.645	0.567	0.609	0.751	0.151	0.812	3.399	0.602
1	0.4	0.176	0.1	0.081	0.079	0.069	2.146	7.849	1.496
	0.6	0.25	0.148	0.12	0.117	0.085	1.645	5.603	1.17
	0.8	0.314	0.193	0.159	0.156	0.095	1.351	4.585	0.981
	1.1	0.395	0.257	0.216	0.213	0.101	1.08	3.858	0.805
	1.3	0.441	0.297	0.253	0.251	0.103	0.961	3.605	0.726
	1.7	0.518	0.37	0.323	0.324	0.102	0.8	3.337	0.617
	1.9	0.55	0.403	0.356	0.36	0.101	0.744	3.266	0.577
	2.4	0.619	0.48	0.436	0.447	0.097	0.646	3.173	0.502
1.3	0.4	0.197	0.097	0.065	0.051	0.058	1.645	5.487	1.217
	0.6	0.273	0.141	0.096	0.076	0.066	1.224	4.111	0.944
	0.8	0.335	0.182	0.126	0.101	0.07	0.981	3.539	0.787
	1.1	0.41	0.238	0.169	0.137	0.07	0.765	3.181	0.643
	1.3	0.451	0.272	0.196	0.161	0.068	0.674	3.078	0.579
	1.7	0.517	0.331	0.247	0.206	0.064	0.557	2.997	0.491
	1.9	0.544	0.358	0.271	0.227	0.062	0.519	2.986	0.459
	2.4	0.6	0.417	0.326	0.279	0.057	0.458	2.992	0.4
1.5	0.4	0.212	0.098	0.06	0.043	0.053	1.397	4.544	1.085
	0.6	0.288	0.142	0.089	0.065	0.058	1.012	3.53	0.837
	0.8	0.35	0.181	0.116	0.085	0.059	0.794	3.143	0.696
	1.1	0.421	0.234	0.155	0.115	0.057	0.605	2.936	0.567
	1.3	0.459	0.265	0.178	0.134	0.055	0.528	2.893	0.51
	1.7	0.519	0.319	0.222	0.171	0.05	0.434	2.884	0.432
	1.9	0.543	0.343	0.243	0.188	0.048	0.406	2.895	0.403
	2.4	0.593	0.395	0.289	0.229	0.043	0.363	2.934	0.351

Table 2. Results of $\mu_1^{'}$, $\mu_2^{'}$, $\mu_3^{'}$, $\mu_4^{'}$, V, CS, CK and CV for the HLMKEx model η = 2.5.

θ	γ	$\mu_1^{'}$	$\mu_2^{'}$	$\mu_3^{'}$	$\mu_4^{'}$	V	CS	CK	CV
0.5	0.4	0.075	0.039	0.03	0.028	0.034	3.578	17.7	2.439
	0.6	0.111	0.059	0.045	0.042	0.046	2.825	11.888	1.939
	0.8	0.145	0.078	0.06	0.056	0.057	2.371	9.054	1.64
	1.1	0.194	0.106	0.082	0.077	0.069	1.936	6.818	1.357
	1.3	0.223	0.125	0.097	0.09	0.075	1.734	5.942	1.226
	1.7	0.278	0.161	0.126	0.118	0.084	1.444	4.871	1.041
	1.9	0.303	0.178	0.141	0.131	0.087	1.336	4.529	0.972
	2.4	0.359	0.22	0.176	0.165	0.091	1.132	3.972	0.842
0.8	0.4	0.078	0.025	0.012	0.0069	0.019	2.608	10.555	1.773
	0.6	0.113	0.038	0.018	0.01	0.025	2.03	7.33	1.395
	0.8	0.145	0.05	0.024	0.014	0.029	1.686	5.811	1.173
	1.1	0.186	0.067	0.032	0.019	0.032	1.364	4.669	0.966
	1.3	0.21	0.078	0.038	0.022	0.034	1.218	4.246	0.873
	1.7	0.252	0.099	0.049	0.029	0.035	1.016	3.759	0.741
	1.9	0.27	0.108	0.054	0.032	0.035	0.943	3.616	0.693
	2.4	0.31	0.131	0.067	0.04	0.035	0.812	3.399	0.602
1	0.4	0.084	0.023	0.0089	0.0042	0.016	2.146	7.856	1.496
	0.6	0.12	0.034	0.013	0.0062	0.02	1.645	5.603	1.17
	0.8	0.151	0.044	0.018	0.0083	0.022	1.351	4.585	0.981
	1.1	0.19	0.059	0.024	0.011	0.023	1.08	3.858	0.805
	1.3	0.212	0.068	0.028	0.013	0.024	0.961	3.605	0.727
	1.7	0.248	0.085	0.036	0.017	0.023	0.8	3.337	0.617
	1.9	0.264	0.093	0.039	0.019	0.023	0.744	3.266	0.577
	2.4	0.297	0.111	0.048	0.024	0.022	0.646	3.173	0.502
1.3	0.4	0.095	0.022	0.0071	0.0027	0.013	1.645	5.487	1.217
	0.6	0.131	0.032	0.011	0.004	0.015	1.224	4.111	0.944
	0.8	0.161	0.042	0.014	0.0054	0.016	0.981	3.539	0.787
	1.1	0.197	0.055	0.019	0.0073	0.016	0.765	3.181	0.643
	1.3	0.216	0.063	0.022	0.0085	0.016	0.674	3.078	0.579
	1.7	0.248	0.076	0.027	0.011	0.015	0.557	2.997	0.491
	1.9	0.261	0.083	0.03	0.012	0.014	0.519	2.986	0.459
	2.4	0.288	0.096	0.036	0.015	0.013	0.458	2.992	0.4
1.5	0.4	0.102	0.023	0.00665	0.00038	0.012	1.397	4.559	1.085
	0.6	0.138	0.033	0.0098	0.0034	0.013	1.012	3.53	0.837
	0.8	0.168	0.042	0.013	0.0045	0.014	0.794	3.143	0.696
	1.1	0.202	0.054	0.017	0.0061	0.013	0.605	2.936	0.567
	1.3	0.22	0.061	0.02	0.0071	0.013	0.528	2.893	0.51
	1.7	0.249	0.074	0.025	0.0091	0.012	0.434	2.884	0.432
	1.9	0.261	0.079	0.027	0.00997	0.011	0.406	2.895	0.403
	2.4	0.285	0.091	0.032	0.012	0.01	0.363	2.934	0.351

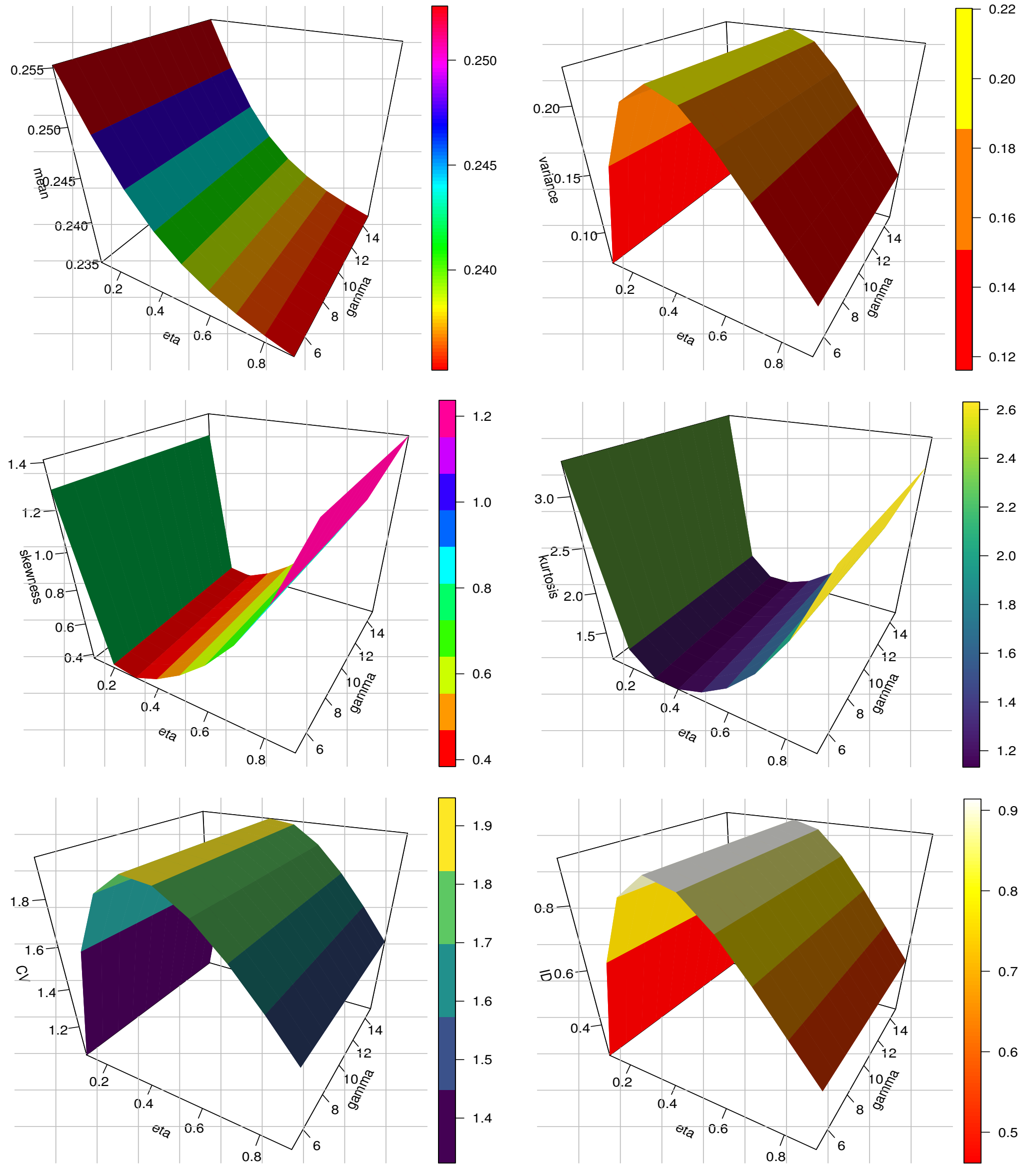

Figure 2. The 3D plots of mean, variance, SK, KU, CV and ID for the HLMKEx model at $\theta = 0.5$.

3.3. Moments of Residual Life Function

The kth moment of the residual life (MRL), represented by $\mathbb{Q}_k(x) = E\left[(Z-x)^k | Z > x\right]$, $k = 1, 2, \ldots$, uniquely determines the CDF $F(z)$. The kth MRL of Z is defined by

$$\mathbb{Q}_k(x) = \frac{1}{\bar{F}(x)} \int_x^\infty (Z-x)^k \, dF(z). \tag{14}$$

Using the binomial expansion in (14), then $\mathbb{Q}_k(x)$ of the HLMKEx distribution is given by

$$\begin{aligned}\mathbb{Q}_k(x) &= \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \frac{\Xi_{u_m,j_m}}{\bar{F}(x)} \int_x^\infty z^k \exp -[\theta(j_2 - \gamma(j_1+1))z] dz \\ &= \sum_{u_1,u_2,j_1,j_2=0}^{\infty} \frac{\Xi_{u_m,j_m}}{\bar{F}(x)} \frac{\Upsilon(k+1, \theta(j_2 - \gamma(j_1+1)))}{[\theta(j_2 - \gamma(j_1+1))]^{k+1}},\end{aligned}$$

where $\Upsilon(.,x)$ is the upper IG function.

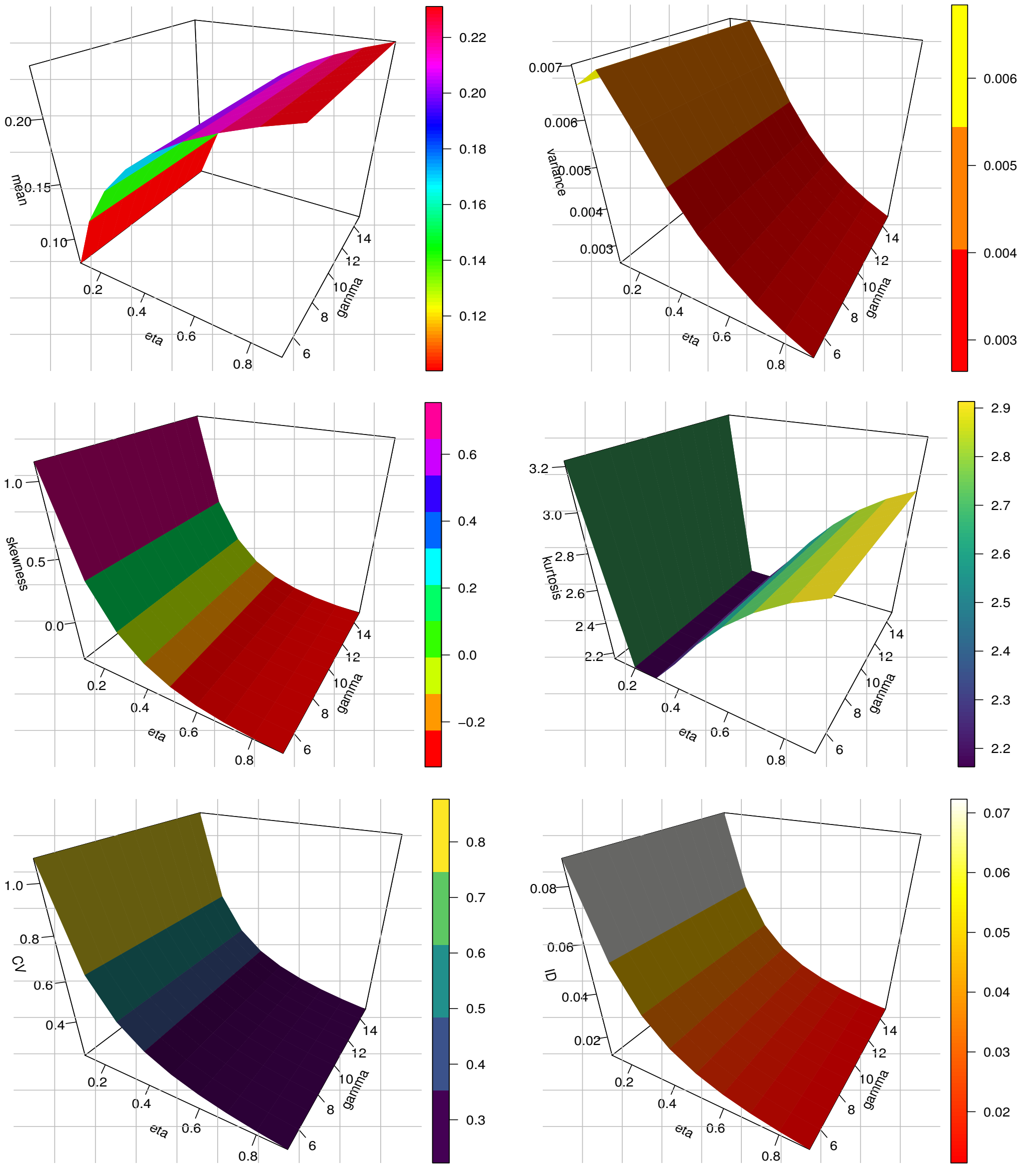

Figure 3. The 3D plots of mean, variance, SK, KU, CV and ID for the HLMKEx model at $\theta = 2.5$.

3.4. Entropy Measures

Entropy is the amount of variability or uncertainty present in a random variable. The degree of uncertainty in the data increases with the entropy value. The determination of the HLMKEx expression for various entropy measurements is the main topic here. The entropy of Z, according to Ref. [33], is defined mathematically as

$$\Phi(\rho) = (1-\rho)^{-1} \log\left(\int_0^\infty (f(z;\omega))^\rho dz \right). \tag{15}$$

To obtain $\Phi(\rho)$, we obtain $(f(z;\omega))^\rho$ using similar expansions (8) and (9) as given below:

$$[f(z;\omega)]^\rho = \sum_{u_1,u_2=0}^{\infty} \psi^\bullet_{u_1,u_2} e^{\theta\rho\gamma z} \left[1 - e^{-\theta z}\right]^{\rho(\gamma-1)} \exp\{-(u_2+\rho)\, K(z,\theta,\gamma)\},$$

where $\psi^\bullet_{u_1,u_2} = (-1)^{u_1+u_2}(2\eta\theta\gamma)^\rho \binom{2\rho+u_1-1}{u_1}\binom{\eta u_1+\rho(\eta-1)}{u_2}$. Using exponential and binomial expansions in (15), $[f(z;\omega)]^\rho$ can then be expressed by

$$[f(z;\omega)]^{\rho}=\sum_{u_m,j_m=0}^{\infty}\Xi^{\bullet}_{u_m,j_m}\exp\{-\theta z(j_2-(\rho+j_1)\gamma)\},\tag{16}$$

where $\Xi^{\bullet}_{u_m,j_m}=\frac{(-1)^{j_1+j_2}\psi^{\bullet}_{u_1,u_2}(u_2+\rho)^{j_1}}{j_1!}\binom{\gamma j_1+\rho(\gamma-1)}{j_2}, m=1,2.$

Substituting the PDF (16) in (15), then the $\Phi(\rho)$ of the HLMKEx distribution is

$$\Phi(\rho)=(1-\rho)^{-1}\log\left(\sum_{u_m,j_m=0}^{\infty}\frac{\Xi^{\bullet}_{u_m,j_m}}{\theta(j_2-(\rho+j_1)\gamma)}\right).$$

The Havrda and Charvat entropy measure (see Ref. [34]) of the HLMKEx distribution is given by

$$\begin{aligned}\lambda(\rho)&=\frac{1}{2^{1-\rho}-1}\left[\left(\int_0^{\infty}(f(z;\omega))^{\rho}dz\right)^{\frac{1}{\rho}}-1\right],\quad \rho\neq 1,\quad \rho>0\\&=\frac{1}{2^{1-\rho}-1}\left[\left(\sum_{u_m,j_m=0}^{\infty}\frac{\Xi^{\bullet}_{u_m,j_m}}{\theta(j_2-(\rho+j_1)\gamma)}\right)^{\frac{1}{\rho}}-1\right].\end{aligned}$$

Furthermore, the Tsallis entropy (Ref. [35]) of the HLMKEx distribution is obtained as follows:

$$\begin{aligned}\Delta(\rho)&=(\rho-1)^{-1}\left[1-\int_0^{\infty}(f(z;\omega))^{\rho}dz\right],\quad \rho\neq 1,\quad \rho>0\\&=(\rho-1)^{-1}\left[1-\left(\sum_{u_m,j_m=0}^{\infty}\frac{\Xi^{\bullet}_{u_m,j_m}}{\theta(j_2-(\rho+j_1)\gamma)}\right)\right].\end{aligned}$$

3.5. Stochastic Ordering

A crucial tool for comparing the behavior of system components is the stochastic ordering of a non-negative continuous random variable. A random variable Z_1 is said to be smaller than another random variable Z_2 in the following ways:

1. Stochastic order $(Z_1 \leqslant_{st} Z_2)$ $if\ H_{Z_1}(z_1)\geqslant H_{Z_2}(z_2)\ \forall z.$
2. Hazard rate order $(Z_1 \leqslant_{hr} Z_2)$ $if\ H_{Z_1}(z_1)\geqslant H_{Z_2}(z_2)\ \forall z.$
3. Mean residual life order $(Z_1 \leqslant_{mrl} Z_2)$ $if\ H_{Z_1}(z_1)\geqslant H_{Z_2}(z_2)\ \forall z.$
4. Likelihood ratio order $(Z_1 \leqslant_{lr} Z_2)$ if $h_{Z_1}(z)/h_{Z_2}(z)$ decreases in z. This implies that $Z_1 \leqslant_{lr} Z_2 \Rightarrow\ Z_1 \leqslant_{hr} Z_2 \Rightarrow\ Z_1 \leqslant_{st} Z_2 \Rightarrow\ Z_1 \leqslant_{mrl} Z_2.$

In this sub-section, we demonstrate that the HLMKEx distribution is ranked in accordance with the strongest "likelihood ratio", as indicated in the theorem below.

Theorem 1. *Let $Z_1\sim$ HLMKEx (ω_1) and $Z_2\sim$ HLMKEx (ω_2), where $\omega_i\equiv(\theta_i,\gamma_i,\eta_i)$, $i=1,2$. If $\theta_1>\theta_2,\gamma_1>\gamma_2,\eta_1>\eta_2$, then $Z_1\leqslant_{lr}Z_2$, hence $Z_1\leqslant_{hr}Z_2$, $Z_1\leqslant_{st}Z_2$, and $Z_1\leqslant_{mrl}Z_2$.*

Proof.

$$\frac{h_{Z_1}(z;\omega_1)}{h_{Z_2}(z;\omega_2)}=\frac{\eta_1\theta_1\gamma_1 e^{\theta_1\gamma_1 z}\left[1-e^{-\theta_1 z}\right]^{\gamma_1-1}e^{-K(z,\theta_1,\gamma_1)}\left[1-e^{-K(z,\theta_1,\gamma_1)}\right]^{\eta_1-1}\left(1+\left[1-e^{-K(z,\theta_1,\gamma_1)}\right]^{\eta_2}\right)^2}{\eta_2\theta_2\gamma_2 e^{\theta_2\gamma_2 z}\left[1-e^{-\theta_2 z}\right]^{\gamma_2-1}e^{-K(z,\theta_2,\gamma_2)}\left[1-e^{-K(z,\theta_2,\gamma_2)}\right]^{\eta_2-1}\left(1+\left[1-e^{-K(z,\theta_2,\gamma_2)}\right]^{\eta_1}\right)^2}.$$

Hence,

$$\begin{aligned}\frac{d}{dz}\log\frac{h_{Z_1}(z;\omega_1)}{h_{Z_2}(z;\omega_2)}&=\frac{\theta_1(\gamma_1-1)}{e^{\theta_1 z}-1}-\frac{\theta_2(\gamma_2-1)}{e^{\theta_2 z}-1}-K'(z,\theta_1,\gamma_1)-\frac{(\eta_2-1)K'(z,\theta_2,\gamma_2)}{e^{K(z,\theta_2,\gamma_2)}-1}+\frac{(\eta_1-1)K'(z,\theta_1,\gamma_1)}{e^{K(z,\theta_1,\gamma_1)}-1}\\&+K'(z,\theta_2,\gamma_2)+2\eta_2\left(1+\left[1-e^{-K(z,\theta_2,\gamma_2)}\right]^{\eta_2}\right)^{-1}\left[1-e^{-K(z,\theta_2,\gamma_2)}\right]^{\eta_2-1}e^{-K(z,\theta_2,\gamma_2)}K'(z,\theta_2,\gamma_2)\\&-2\eta_1\left(1+\left[1-e^{-K(z,\theta_1,\gamma_1)}\right]^{\eta_1}\right)^{-1}\left[1-e^{-K(z,\theta_1,\gamma_1)}\right]^{\eta_1-1}e^{-K(z,\theta_1,\gamma_1)}K'(z,\theta_1,\gamma_1),\end{aligned}$$

where $K'(z, \theta_i, \gamma_i) = \gamma_i \theta_i [\exp(\theta_i z) - 1]^{\gamma - 1} \exp(\theta_i z),\ i = 1, 2$. If $\theta_1 > \theta_2, \gamma_1 > \gamma_2, \eta_1 > \eta_2$, then $\frac{d}{dz} \log \frac{h_{Z_1}(z)}{h_{Z_2}(z)} < 0$ and $\frac{h_{Z_1}(z;\omega_1)}{h_{Z_2}(z;\omega_2)}$ is decreasing function in z, that is $Z_1 \leqslant_{lr} Z_2$, hence $Z_1 \leqslant_{hr} Z_2$, $Z_1 \leqslant_{st} Z_2$, and $Z_1 \leqslant_{mrl} Z_2$. □

4. Estimation Methods

In this part, we seek to identify estimators for the model parameters ($\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$) that we propose using various estimate techniques that are determined by maximizing or minimizing an objective function.

Our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$ are derived using the ML estimation technique (EM_1) by maximizing the following equation:

$$Logl = n \log(2\theta\gamma\eta) + \theta\gamma \sum_{i=1}^{n} z_i + (\gamma - 1) \sum_{i=1}^{n} \log\left(1 - e^{-\theta z_i}\right) - \sum_{i=1}^{n} \left(e^{\theta z_i} - 1\right)^{\gamma}$$
$$+ (\eta - 1) \sum_{i=1}^{n} \log\left(1 - e^{-\left(e^{\theta z_i} - 1\right)^{\gamma}}\right) - 2 \sum_{i=1}^{n} \log\left(1 + \left(1 - e^{-\left(e^{\theta z_i} - 1\right)^{\gamma}}\right)^{\eta}\right).$$

Using the Anderson–Darling estimation technique (EM_2), the following equation is minimized to produce our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$A = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1)[\log F(z_{(i)}) + \log \bar{F}(z_{(n-i+1)})] = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1)$$
$$\left[\log\left(\frac{2\left(1 - e^{-\left(e^{\theta z_{(i)}} - 1\right)^{\gamma}}\right)^{\eta}}{\left(1 - e^{-\left(e^{\theta z_{(i)}} - 1\right)^{\gamma}}\right)^{\eta} + 1} \right) + \log\left(1 - \frac{2\left(1 - e^{-\left(e^{\theta z_{(n-i+1)}} - 1\right)^{\gamma}}\right)^{\eta}}{\left(1 - e^{-\left(e^{\theta z_{(n-i+1)}} - 1\right)^{\gamma}}\right)^{\eta} + 1} \right) \right].$$

Using the Cramér–von Mises estimation technique (EM_3), the following equation is minimized to produce our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$C = -\frac{1}{12n} + \sum_{i=1}^{n} \left[F(z_{(i)}) - \frac{2i - 1}{2n} \right]^2 = -\frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{2\left(1 - e^{-\left(e^{\theta z_{(i)}} - 1\right)^{\gamma}}\right)^{\eta}}{\left(1 - e^{-\left(e^{\theta z_{(i)}} - 1\right)^{\gamma}}\right)^{\eta} + 1} - \frac{2i - 1}{2n} \right]^2.$$

Using the maximum product of the spacings estimation technique (EM_4), the following equation is maximized to produce our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$J = \frac{1}{n + 1} \sum_{i=1}^{n+1} \log H_i,\ H_i = F\left(z_{(i)}\right) - F\left(z_{(i-1)}\right).$$

Using the least-squares estimation technique (EM_5), the following equation is minimized to produce our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$V = \sum_{i=1}^{n} \left[F(z_{(i)}) - \frac{i}{n + 1} \right]^2 = \sum_{i=1}^{n} \left[\frac{2\left(1 - e^{-\left(e^{\theta z_{(i)}} - 1\right)^{\gamma}}\right)^{\eta}}{\left(1 - e^{-\left(e^{\theta z_{(i)}} - 1\right)^{\gamma}}\right)^{\eta} + 1} - \frac{i}{n + 1} \right]^2.$$

Using the percentile estimation technique (EM_6), the following equation is minimized to produce our suggested model estimators, $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$PC = \sum_{i=1}^{n}\left[z_{(i)} - Q(p_i)\right]^2 = \sum_{i=1}^{n}\left[z_{(i)} - \frac{1}{\theta}\log\left(\sqrt[\gamma]{\log\left(\frac{1}{1-\left(-\frac{p_i}{p_i-2}\right)^{1/\eta}}\right)}+1\right)\right]^2.$$

Using the right-tail Anderson–Darling estimation technique (EM_7), the following equation is minimized to produce our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$\begin{aligned} R &= \frac{n}{2} - 2\sum_{i=1}^{n} F\left(z_{(i)}\right) - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bar{F}\left(z_{(n+1-i)}\right) \\ &= \frac{n}{2} - 2\sum_{i=1}^{n}\left(\frac{2\left(1-e^{-\left(e^{\theta z_{(i)}}-1\right)^{\gamma}}\right)^{\eta}}{\left(1-e^{-\left(e^{\theta z_{(i)}}-1\right)^{\gamma}}\right)^{\eta}+1}\right) \\ &\quad - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(\frac{2\left(1-e^{-\left(e^{\theta z_{(n+1-i)}}-1\right)^{\gamma}}\right)^{\eta}}{\left(1-e^{-\left(e^{\theta z_{(n+1-i)}}-1\right)^{\gamma}}\right)^{\eta}+1}\right). \end{aligned}$$

Using the weighted least-squares estimation technique (EM_8), the following equation is minimized to produce our suggested model estimators $\hat{\theta}$, $\hat{\eta}$, and $\hat{\gamma}$:

$$\begin{aligned} W &= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(z_{(i)}) - \frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{2\left(1-e^{-\left(e^{\theta z_{(i)}}-1\right)^{\gamma}}\right)^{\eta}}{\left(1-e^{-\left(e^{\theta z_{(i)}}-1\right)^{\gamma}}\right)^{\eta}+1} - \frac{i}{n+1}\right]^2. \end{aligned}$$

5. Numerical Simulation

This section examines the behavior of all estimation techniques identified in Section 4, utilizing our suggested model to produce data sets at random and identify suggested model estimators using these estimation techniques. For this study, we employ several measures such as the average of absolute bias (BIAS), $|BIAS(\widehat{\omega})| = \frac{1}{M}\sum_{i=1}^{M}|\widehat{\omega}-\omega|$, mean squared error (MSE), MSE = $\frac{1}{M}\sum_{i=1}^{M}(\widehat{\omega}-\omega)^2$, and mean absolute relative error (MRE), MRE = $\frac{1}{M}\sum_{i=1}^{M}|\widehat{\omega}-\omega|/\omega$, $\omega = (\theta,\eta,\gamma)$. The second goal of this simulation is to determine the optimal estimating technique to use when calculating the model estimators we provide. In order to run this simulation, we create random samples from our model of various sizes, calculate the measures we will be using, and then repeat those processes many times.

Our simulation results are shown in Tables 3–8. Any value's ranking is determined by where it ranks among all estimation techniques. The partial and total ranks of our estimators are shown in Table 9.

As a result of the simulation and ranking tables, we draw the following conclusions:

- All of the estimates exhibit consistency properties.
- The BIAS of all estimates decreases as n grows for all estimation techniques.
- The MSE of all estimates decreases as n grows for all estimation techniques.
- The MRE of all estimates decreases as n grows for all estimation techniques.

- The best estimation technique is the maximum product of the spacings estimation method. Therefore, if researchers have data sets from our suggested model, we urge them to apply this method.

Table 3. Simulation values of BIAS, MSE and MRE for ($\theta = 0.25,\ \eta = 0.5,\ \gamma = 0.75$).

n	Measures	$\hat{\omega}$	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
75	BIAS	$\hat{\theta}$	0.13351 $^{\{5\}}$	0.12278 $^{\{4\}}$	0.15203 $^{\{8\}}$	0.08819 $^{\{1\}}$	0.14381 $^{\{7\}}$	0.11367 $^{\{3\}}$	0.10524 $^{\{2\}}$	0.13439 $^{\{6\}}$
		$\hat{\eta}$	0.29314 $^{\{7\}}$	0.2538 $^{\{4\}}$	0.30184 $^{\{8\}}$	0.20497 $^{\{1\}}$	0.29272 $^{\{6\}}$	0.22699 $^{\{2\}}$	0.2298 $^{\{3\}}$	0.27496 $^{\{5\}}$
		$\hat{\gamma}$	0.39892 $^{\{2\}}$	0.42407 $^{\{3\}}$	0.573 $^{\{7\}}$	0.35947 $^{\{1\}}$	0.56393 $^{\{6\}}$	0.89818 $^{\{8\}}$	0.52501 $^{\{5\}}$	0.46353 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.03584 $^{\{5\}}$	0.0254 $^{\{3\}}$	0.0412 $^{\{7\}}$	0.0143 $^{\{1\}}$	0.03741 $^{\{6\}}$	0.04725 $^{\{8\}}$	0.01808 $^{\{2\}}$	0.03126 $^{\{4\}}$
		$\hat{\eta}$	0.13595 $^{\{8\}}$	0.09216 $^{\{3\}}$	0.12186 $^{\{7\}}$	0.06844 $^{\{1\}}$	0.11576 $^{\{6\}}$	0.09595 $^{\{4\}}$	0.07444 $^{\{2\}}$	0.10477 $^{\{5\}}$
		$\hat{\gamma}$	0.25423 $^{\{2\}}$	0.31258 $^{\{3\}}$	0.57959 $^{\{6\}}$	0.25149 $^{\{1\}}$	0.56492 $^{\{5\}}$	3.2629 $^{\{8\}}$	0.6364 $^{\{7\}}$	0.38062 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.53403 $^{\{5\}}$	0.49112 $^{\{4\}}$	0.60812 $^{\{8\}}$	0.35278 $^{\{1\}}$	0.57525 $^{\{7\}}$	0.45467 $^{\{3\}}$	0.42097 $^{\{2\}}$	0.53757 $^{\{6\}}$
		$\hat{\eta}$	0.58628 $^{\{7\}}$	0.5076 $^{\{4\}}$	0.60368 $^{\{8\}}$	0.40994 $^{\{1\}}$	0.58544 $^{\{6\}}$	0.45397 $^{\{2\}}$	0.4596 $^{\{3\}}$	0.54993 $^{\{5\}}$
		$\hat{\gamma}$	0.5319 $^{\{2\}}$	0.56542 $^{\{3\}}$	0.76399 $^{\{7\}}$	0.47929 $^{\{1\}}$	0.7519 $^{\{6\}}$	1.19757 $^{\{8\}}$	0.70001 $^{\{5\}}$	0.61805 $^{\{4\}}$
	$\sum Ranks$		57 $^{\{4\}}$	42 $^{\{3\}}$	89 $^{\{8\}}$	12 $^{\{1\}}$	74 $^{\{7\}}$	59 $^{\{6\}}$	41 $^{\{2\}}$	58 $^{\{5\}}$
120	BIAS	$\hat{\theta}$	0.11805 $^{\{5\}}$	0.11621 $^{\{4\}}$	0.13793 $^{\{8\}}$	0.07712 $^{\{1\}}$	0.12666 $^{\{7\}}$	0.08872 $^{\{2\}}$	0.09517 $^{\{3\}}$	0.12108 $^{\{6\}}$
		$\hat{\eta}$	0.25889 $^{\{6\}}$	0.24411 $^{\{4\}}$	0.28081 $^{\{8\}}$	0.17582 $^{\{1\}}$	0.2692 $^{\{7\}}$	0.1981 $^{\{2\}}$	0.21105 $^{\{3\}}$	0.25599 $^{\{5\}}$
		$\hat{\gamma}$	0.33177 $^{\{2\}}$	0.36848 $^{\{3\}}$	0.50228 $^{\{7\}}$	0.27835 $^{\{1\}}$	0.49111 $^{\{6\}}$	0.70043 $^{\{8\}}$	0.42922 $^{\{5\}}$	0.41289 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.02774 $^{\{7\}}$	0.02273 $^{\{4\}}$	0.03358 $^{\{8\}}$	0.01174 $^{\{1\}}$	0.02727 $^{\{6\}}$	0.02155 $^{\{3\}}$	0.0142 $^{\{2\}}$	0.02501 $^{\{5\}}$
		$\hat{\eta}$	0.10864 $^{\{8\}}$	0.0873 $^{\{4\}}$	0.10776 $^{\{7\}}$	0.05394 $^{\{1\}}$	0.09875 $^{\{6\}}$	0.0766 $^{\{3\}}$	0.06373 $^{\{2\}}$	0.09302 $^{\{5\}}$
		$\hat{\gamma}$	0.1712 $^{\{2\}}$	0.23113 $^{\{3\}}$	0.46157 $^{\{7\}}$	0.1544 $^{\{1\}}$	0.43885 $^{\{6\}}$	2.11149 $^{\{8\}}$	0.42585 $^{\{5\}}$	0.30103 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.4722 $^{\{5\}}$	0.46483 $^{\{4\}}$	0.5517 $^{\{8\}}$	0.30849 $^{\{1\}}$	0.50663 $^{\{7\}}$	0.35488 $^{\{2\}}$	0.3807 $^{\{3\}}$	0.48431 $^{\{6\}}$
		$\hat{\eta}$	0.51778 $^{\{6\}}$	0.48822 $^{\{4\}}$	0.56161 $^{\{8\}}$	0.35164 $^{\{1\}}$	0.53839 $^{\{7\}}$	0.3962 $^{\{2\}}$	0.4221 $^{\{3\}}$	0.51197 $^{\{5\}}$
		$\hat{\gamma}$	0.44236 $^{\{2\}}$	0.4913 $^{\{3\}}$	0.66971 $^{\{7\}}$	0.37114 $^{\{1\}}$	0.65481 $^{\{6\}}$	0.9339 $^{\{8\}}$	0.57229 $^{\{5\}}$	0.55051 $^{\{4\}}$
	$\sum Ranks$		43 $^{\{5\}}$	33 $^{\{3\}}$	68 $^{\{8\}}$	9 $^{\{1\}}$	58 $^{\{7\}}$	38 $^{\{4\}}$	31 $^{\{2\}}$	44 $^{\{6\}}$
150	BIAS	$\hat{\theta}$	0.10886 $^{\{5\}}$	0.10848 $^{\{4\}}$	0.13506 $^{\{8\}}$	0.07128 $^{\{1\}}$	0.12362 $^{\{7\}}$	0.07463 $^{\{2\}}$	0.09093 $^{\{3\}}$	0.11527 $^{\{6\}}$
		$\hat{\eta}$	0.24267 $^{\{5\}}$	0.23426 $^{\{4\}}$	0.28289 $^{\{8\}}$	0.16734 $^{\{1\}}$	0.26751 $^{\{7\}}$	0.17733 $^{\{2\}}$	0.20677 $^{\{3\}}$	0.24842 $^{\{6\}}$
		$\hat{\gamma}$	0.30975 $^{\{2\}}$	0.35585 $^{\{3\}}$	0.47072 $^{\{7\}}$	0.26073 $^{\{1\}}$	0.44735 $^{\{6\}}$	0.59445 $^{\{8\}}$	0.39526 $^{\{5\}}$	0.38116 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.0235 $^{\{6\}}$	0.01982 $^{\{4\}}$	0.03168 $^{\{8\}}$	0.00939 $^{\{1\}}$	0.02618 $^{\{7\}}$	0.01366 $^{\{3\}}$	0.01286 $^{\{2\}}$	0.02265 $^{\{5\}}$
		$\hat{\eta}$	0.09703 $^{\{6\}}$	0.07891 $^{\{4\}}$	0.11098 $^{\{8\}}$	0.04879 $^{\{1\}}$	0.10148 $^{\{7\}}$	0.06193 $^{\{3\}}$	0.06039 $^{\{2\}}$	0.08961 $^{\{5\}}$
		$\hat{\gamma}$	0.15035 $^{\{2\}}$	0.2121 $^{\{3\}}$	0.398 $^{\{7\}}$	0.13528 $^{\{1\}}$	0.36312 $^{\{6\}}$	1.53994 $^{\{8\}}$	0.33486 $^{\{5\}}$	0.25389 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.43545 $^{\{5\}}$	0.43392 $^{\{4\}}$	0.54023 $^{\{8\}}$	0.28512 $^{\{1\}}$	0.49448 $^{\{7\}}$	0.29853 $^{\{2\}}$	0.36371 $^{\{3\}}$	0.46107 $^{\{6\}}$
		$\hat{\eta}$	0.48534 $^{\{5\}}$	0.46852 $^{\{4\}}$	0.56579 $^{\{8\}}$	0.33468 $^{\{1\}}$	0.53502 $^{\{7\}}$	0.35466 $^{\{2\}}$	0.41354 $^{\{3\}}$	0.49685 $^{\{6\}}$
		$\hat{\gamma}$	0.413 $^{\{2\}}$	0.47447 $^{\{3\}}$	0.62763 $^{\{7\}}$	0.34763 $^{\{1\}}$	0.59646 $^{\{6\}}$	0.7926 $^{\{8\}}$	0.52701 $^{\{5\}}$	0.50821 $^{\{4\}}$
	$\sum Ranks$		38 $^{\{4.5\}}$	33 $^{\{3\}}$	69 $^{\{8\}}$	9 $^{\{1\}}$	60 $^{\{7\}}$	38 $^{\{4.5\}}$	31 $^{\{2\}}$	46 $^{\{6\}}$
200	BIAS	$\hat{\theta}$	0.1026 $^{\{4\}}$	0.10335 $^{\{5\}}$	0.12079 $^{\{8\}}$	0.06391 $^{\{1\}}$	0.1175 $^{\{7\}}$	0.06735 $^{\{2\}}$	0.08416 $^{\{3\}}$	0.10614 $^{\{6\}}$
		$\hat{\eta}$	0.22493 $^{\{5\}}$	0.22472 $^{\{4\}}$	0.25888 $^{\{8\}}$	0.14629 $^{\{1\}}$	0.25854 $^{\{7\}}$	0.16416 $^{\{2\}}$	0.19303 $^{\{3\}}$	0.23148 $^{\{6\}}$
		$\hat{\gamma}$	0.27629 $^{\{2\}}$	0.31973 $^{\{3\}}$	0.42027 $^{\{6\}}$	0.22443 $^{\{1\}}$	0.42723 $^{\{7\}}$	0.50072 $^{\{8\}}$	0.34637 $^{\{4\}}$	0.34713 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.02144 $^{\{6\}}$	0.01807 $^{\{4\}}$	0.02493 $^{\{8\}}$	0.00789 $^{\{1\}}$	0.02352 $^{\{7\}}$	0.00999 $^{\{2\}}$	0.01064 $^{\{3\}}$	0.01949 $^{\{5\}}$
		$\hat{\eta}$	0.08689 $^{\{6\}}$	0.07502 $^{\{4\}}$	0.09551 $^{\{8\}}$	0.03952 $^{\{1\}}$	0.09544 $^{\{7\}}$	0.05201 $^{\{2\}}$	0.05287 $^{\{3\}}$	0.07871 $^{\{5\}}$
		$\hat{\gamma}$	0.11813 $^{\{2\}}$	0.16762 $^{\{3\}}$	0.32652 $^{\{6\}}$	0.10672 $^{\{1\}}$	0.33314 $^{\{7\}}$	1.02657 $^{\{8\}}$	0.25409 $^{\{5\}}$	0.207 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.41041 $^{\{4\}}$	0.41341 $^{\{5\}}$	0.48318 $^{\{8\}}$	0.25564 $^{\{1\}}$	0.47001 $^{\{7\}}$	0.26939 $^{\{2\}}$	0.33665 $^{\{3\}}$	0.42457 $^{\{6\}}$
		$\hat{\eta}$	0.44986 $^{\{5\}}$	0.44945 $^{\{4\}}$	0.51776 $^{\{8\}}$	0.29258 $^{\{1\}}$	0.51708 $^{\{7\}}$	0.32832 $^{\{2\}}$	0.38606 $^{\{3\}}$	0.46297 $^{\{6\}}$
		$\hat{\gamma}$	0.36838 $^{\{2\}}$	0.4263 $^{\{3\}}$	0.56036 $^{\{6\}}$	0.29924 $^{\{1\}}$	0.56964 $^{\{7\}}$	0.66762 $^{\{8\}}$	0.46183 $^{\{4\}}$	0.46284 $^{\{5\}}$
	$\sum Ranks$		36 $^{\{4.5\}}$	35 $^{\{3\}}$	66 $^{\{8\}}$	9 $^{\{1\}}$	63 $^{\{7\}}$	36 $^{\{4.5\}}$	31 $^{\{2\}}$	48 $^{\{6\}}$
300	BIAS	$\hat{\theta}$	0.08661 $^{\{4\}}$	0.09118 $^{\{5\}}$	0.10973 $^{\{7\}}$	0.0565 $^{\{1\}}$	0.11071 $^{\{8\}}$	0.0602 $^{\{2\}}$	0.07502 $^{\{3\}}$	0.09941 $^{\{6\}}$
		$\hat{\eta}$	0.19388 $^{\{4\}}$	0.2047 $^{\{5\}}$	0.23825 $^{\{7\}}$	0.12706 $^{\{1\}}$	0.24281 $^{\{8\}}$	0.14787 $^{\{2\}}$	0.1726 $^{\{3\}}$	0.21777 $^{\{6\}}$
		$\hat{\gamma}$	0.22972 $^{\{2\}}$	0.27718 $^{\{3\}}$	0.35862 $^{\{6\}}$	0.18091 $^{\{1\}}$	0.35926 $^{\{7\}}$	0.38353 $^{\{8\}}$	0.2845 $^{\{4\}}$	0.29812 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.01545 $^{\{5\}}$	0.01419 $^{\{4\}}$	0.02117 $^{\{8\}}$	0.00661 $^{\{1\}}$	0.02114 $^{\{7\}}$	0.00717 $^{\{2\}}$	0.00846 $^{\{3\}}$	0.01692 $^{\{6\}}$
		$\hat{\eta}$	0.06818 $^{\{5\}}$	0.06422 $^{\{4\}}$	0.08376 $^{\{7\}}$	0.0334 $^{\{1\}}$	0.0874 $^{\{8\}}$	0.0407 $^{\{2\}}$	0.04237 $^{\{3\}}$	0.07098 $^{\{6\}}$
		$\hat{\gamma}$	0.08394 $^{\{2\}}$	0.12123 $^{\{3\}}$	0.23625 $^{\{6\}}$	0.07198 $^{\{1\}}$	0.23697 $^{\{7\}}$	0.52017 $^{\{8\}}$	0.16581 $^{\{5\}}$	0.1447 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.34643 $^{\{4\}}$	0.36474 $^{\{5\}}$	0.43893 $^{\{7\}}$	0.22601 $^{\{1\}}$	0.44284 $^{\{8\}}$	0.24079 $^{\{2\}}$	0.30007 $^{\{3\}}$	0.39766 $^{\{6\}}$
		$\hat{\eta}$	0.38776 $^{\{4\}}$	0.4094 $^{\{5\}}$	0.47651 $^{\{7\}}$	0.25412 $^{\{1\}}$	0.48562 $^{\{8\}}$	0.29575 $^{\{2\}}$	0.34521 $^{\{3\}}$	0.43554 $^{\{6\}}$
		$\hat{\gamma}$	0.30629 $^{\{2\}}$	0.36958 $^{\{3\}}$	0.47816 $^{\{6\}}$	0.24121 $^{\{1\}}$	0.47901 $^{\{7\}}$	0.51137 $^{\{8\}}$	0.37933 $^{\{4\}}$	0.39749 $^{\{5\}}$
	$\sum Ranks$		32 $^{\{3\}}$	37 $^{\{5\}}$	61 $^{\{7\}}$	9 $^{\{1\}}$	68 $^{\{8\}}$	36 $^{\{4\}}$	31 $^{\{2\}}$	50 $^{\{6\}}$

Table 4. Simulation values of BIAS, MSE and MRE for ($\theta = 0.75,\ \eta = 1.50,\ \gamma = 0.25$).

n	Measures	$\hat{\omega}$	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
75	BIAS	$\hat{\theta}$	0.26151 $^{\{1\}}$	0.28071 $^{\{4\}}$	0.34671 $^{\{7\}}$	0.28024 $^{\{3\}}$	0.34364 $^{\{6\}}$	0.53901 $^{\{8\}}$	0.27735 $^{\{2\}}$	0.30061 $^{\{5\}}$
		$\hat{\eta}$	0.37667 $^{\{1\}}$	0.40831 $^{\{2\}}$	0.44505 $^{\{6\}}$	0.43981 $^{\{5\}}$	0.44748 $^{\{7\}}$	0.521 $^{\{8\}}$	0.41504 $^{\{3\}}$	0.42241 $^{\{4\}}$
		$\hat{\gamma}$	0.08762 $^{\{1\}}$	0.09492 $^{\{2\}}$	0.10336 $^{\{5\}}$	0.13388 $^{\{7\}}$	0.10409 $^{\{6\}}$	0.19049 $^{\{8\}}$	0.09978 $^{\{4\}}$	0.0995 $^{\{3\}}$
	MSE	$\hat{\theta}$	0.1179 $^{\{1\}}$	0.12471 $^{\{2\}}$	0.2136 $^{\{7\}}$	0.12594 $^{\{4\}}$	0.20641 $^{\{6\}}$	0.69507 $^{\{8\}}$	0.12489 $^{\{3\}}$	0.15256 $^{\{5\}}$
		$\hat{\eta}$	0.23299 $^{\{1\}}$	0.25637 $^{\{2\}}$	0.29756 $^{\{6\}}$	0.31676 $^{\{7\}}$	0.29703 $^{\{5\}}$	0.40317 $^{\{8\}}$	0.26785 $^{\{3\}}$	0.27266 $^{\{4\}}$
		$\hat{\gamma}$	0.02854 $^{\{3\}}$	0.02712 $^{\{1\}}$	0.03143 $^{\{6\}}$	0.06463 $^{\{7\}}$	0.03001 $^{\{5\}}$	0.08734 $^{\{8\}}$	0.02836 $^{\{2\}}$	0.02895 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.34868 $^{\{1\}}$	0.37428 $^{\{4\}}$	0.46228 $^{\{7\}}$	0.37365 $^{\{3\}}$	0.45818 $^{\{6\}}$	0.71868 $^{\{8\}}$	0.3698 $^{\{2\}}$	0.40082 $^{\{5\}}$
		$\hat{\eta}$	0.25112 $^{\{1\}}$	0.27221 $^{\{2\}}$	0.2967 $^{\{6\}}$	0.29321 $^{\{5\}}$	0.29832 $^{\{7\}}$	0.34733 $^{\{8\}}$	0.2767 $^{\{3\}}$	0.28161 $^{\{4\}}$
		$\hat{\gamma}$	0.35049 $^{\{1\}}$	0.3797 $^{\{2\}}$	0.41342 $^{\{5\}}$	0.53552 $^{\{7\}}$	0.41637 $^{\{6\}}$	0.76196 $^{\{8\}}$	0.3991 $^{\{4\}}$	0.398 $^{\{3\}}$
	$\sum Ranks$		14 $^{\{1\}}$	29 $^{\{2\}}$	73 $^{\{6.5\}}$	63 $^{\{5\}}$	73 $^{\{6.5\}}$	96 $^{\{8\}}$	35 $^{\{3\}}$	49 $^{\{4\}}$
120	BIAS	$\hat{\theta}$	0.20386 $^{\{1\}}$	0.22321 $^{\{3\}}$	0.28155 $^{\{6\}}$	0.22807 $^{\{5\}}$	0.28603 $^{\{7\}}$	0.41497 $^{\{8\}}$	0.2231 $^{\{2\}}$	0.22786 $^{\{4\}}$
		$\hat{\eta}$	0.30739 $^{\{1\}}$	0.32931 $^{\{2\}}$	0.37209 $^{\{6\}}$	0.35623 $^{\{5\}}$	0.37582 $^{\{7\}}$	0.45354 $^{\{8\}}$	0.33563 $^{\{4\}}$	0.33102 $^{\{3\}}$
		$\hat{\gamma}$	0.06909 $^{\{1\}}$	0.07301 $^{\{2\}}$	0.08429 $^{\{6\}}$	0.10142 $^{\{7\}}$	0.08406 $^{\{5\}}$	0.16589 $^{\{8\}}$	0.07913 $^{\{4\}}$	0.07384 $^{\{3\}}$
	MSE	$\hat{\theta}$	0.07031 $^{\{1\}}$	0.08128 $^{\{3\}}$	0.1331 $^{\{6\}}$	0.08586 $^{\{5\}}$	0.13367 $^{\{7\}}$	0.33391 $^{\{8\}}$	0.08082 $^{\{2\}}$	0.0837 $^{\{4\}}$
		$\hat{\eta}$	0.16051 $^{\{1\}}$	0.17839 $^{\{2\}}$	0.21571 $^{\{5\}}$	0.22225 $^{\{7\}}$	0.22044 $^{\{6\}}$	0.3249 $^{\{8\}}$	0.1847 $^{\{4\}}$	0.1786 $^{\{3\}}$
		$\hat{\gamma}$	0.02051 $^{\{4\}}$	0.01682 $^{\{1\}}$	0.02161 $^{\{6\}}$	0.04219 $^{\{7\}}$	0.02085 $^{\{5\}}$	0.06083 $^{\{8\}}$	0.01991 $^{\{3\}}$	0.01785 $^{\{2\}}$
	MRE	$\hat{\theta}$	0.27181 $^{\{1\}}$	0.29761 $^{\{3\}}$	0.37541 $^{\{6\}}$	0.30409 $^{\{5\}}$	0.38138 $^{\{7\}}$	0.5533 $^{\{8\}}$	0.29746 $^{\{2\}}$	0.30381 $^{\{4\}}$
		$\hat{\eta}$	0.20492 $^{\{1\}}$	0.21954 $^{\{2\}}$	0.24806 $^{\{6\}}$	0.23749 $^{\{5\}}$	0.25055 $^{\{7\}}$	0.30236 $^{\{8\}}$	0.22375 $^{\{4\}}$	0.22068 $^{\{3\}}$
		$\hat{\gamma}$	0.27635 $^{\{1\}}$	0.29205 $^{\{2\}}$	0.33716 $^{\{6\}}$	0.40569 $^{\{7\}}$	0.33626 $^{\{5\}}$	0.66355 $^{\{8\}}$	0.31652 $^{\{4\}}$	0.29535 $^{\{3\}}$
	$\sum Ranks$		12 $^{\{1\}}$	20 $^{\{2\}}$	53 $^{\{5.5\}}$	53 $^{\{5.5\}}$	56 $^{\{7\}}$	72 $^{\{8\}}$	29 $^{\{3.5\}}$	29 $^{\{3.5\}}$
150	BIAS	$\hat{\theta}$	0.17878 $^{\{1\}}$	0.2035 $^{\{4\}}$	0.25038 $^{\{7\}}$	0.20313 $^{\{3\}}$	0.24461 $^{\{6\}}$	0.36818 $^{\{8\}}$	0.19861 $^{\{2\}}$	0.21102 $^{\{5\}}$
		$\hat{\eta}$	0.26958 $^{\{1\}}$	0.30119 $^{\{4\}}$	0.33851 $^{\{7\}}$	0.31034 $^{\{5\}}$	0.336 $^{\{6\}}$	0.42947 $^{\{8\}}$	0.29686 $^{\{2\}}$	0.29911 $^{\{3\}}$
		$\hat{\gamma}$	0.05646 $^{\{1\}}$	0.06651 $^{\{4\}}$	0.07607 $^{\{6\}}$	0.08669 $^{\{7\}}$	0.07361 $^{\{5\}}$	0.15786 $^{\{8\}}$	0.06627 $^{\{3\}}$	0.06408 $^{\{2\}}$
	MSE	$\hat{\theta}$	0.05368 $^{\{1\}}$	0.06788 $^{\{3\}}$	0.10155 $^{\{7\}}$	0.06879 $^{\{4\}}$	0.09731 $^{\{6\}}$	0.23782 $^{\{8\}}$	0.06336 $^{\{2\}}$	0.07346 $^{\{5\}}$
		$\hat{\eta}$	0.12315 $^{\{1\}}$	0.15305 $^{\{4\}}$	0.18283 $^{\{7\}}$	0.17686 $^{\{5\}}$	0.18045 $^{\{6\}}$	0.29827 $^{\{8\}}$	0.14774 $^{\{3\}}$	0.14665 $^{\{2\}}$
		$\hat{\gamma}$	0.01152 $^{\{1\}}$	0.01525 $^{\{4\}}$	0.01809 $^{\{6\}}$	0.0353 $^{\{7\}}$	0.01731 $^{\{5\}}$	0.05792 $^{\{8\}}$	0.01464 $^{\{3\}}$	0.01403 $^{\{2\}}$
	MRE	$\hat{\theta}$	0.23838 $^{\{1\}}$	0.27134 $^{\{4\}}$	0.33384 $^{\{7\}}$	0.27085 $^{\{3\}}$	0.32615 $^{\{6\}}$	0.49091 $^{\{8\}}$	0.26481 $^{\{2\}}$	0.28136 $^{\{5\}}$
		$\hat{\eta}$	0.17972 $^{\{1\}}$	0.20079 $^{\{4\}}$	0.22567 $^{\{7\}}$	0.20689 $^{\{5\}}$	0.224 $^{\{6\}}$	0.28631 $^{\{8\}}$	0.1979 $^{\{2\}}$	0.19941 $^{\{3\}}$
		$\hat{\gamma}$	0.22584 $^{\{1\}}$	0.26604 $^{\{4\}}$	0.30428 $^{\{6\}}$	0.34675 $^{\{7\}}$	0.29444 $^{\{5\}}$	0.63142 $^{\{8\}}$	0.2651 $^{\{3\}}$	0.25633 $^{\{2\}}$
	$\sum Ranks$		9 $^{\{1\}}$	35 $^{\{4\}}$	60 $^{\{7\}}$	46 $^{\{5\}}$	51 $^{\{6\}}$	72 $^{\{8\}}$	22 $^{\{2\}}$	29 $^{\{3\}}$
200	BIAS	$\hat{\theta}$	0.16218 $^{\{1\}}$	0.17429 $^{\{4\}}$	0.20797 $^{\{6\}}$	0.17166 $^{\{3\}}$	0.21796 $^{\{7\}}$	0.32153 $^{\{8\}}$	0.16932 $^{\{2\}}$	0.18027 $^{\{5\}}$
		$\hat{\eta}$	0.24297 $^{\{1\}}$	0.25924 $^{\{3\}}$	0.28391 $^{\{6\}}$	0.27052 $^{\{5\}}$	0.29175 $^{\{7\}}$	0.36748 $^{\{8\}}$	0.24633 $^{\{2\}}$	0.26393 $^{\{4\}}$
		$\hat{\gamma}$	0.04804 $^{\{1\}}$	0.05569 $^{\{4\}}$	0.06095 $^{\{5\}}$	0.06672 $^{\{7\}}$	0.06264 $^{\{6\}}$	0.13818 $^{\{8\}}$	0.05234 $^{\{2\}}$	0.05533 $^{\{3\}}$
	MSE	$\hat{\theta}$	0.04211 $^{\{1\}}$	0.0483 $^{\{3\}}$	0.07009 $^{\{6\}}$	0.04854 $^{\{4\}}$	0.07496 $^{\{7\}}$	0.17722 $^{\{8\}}$	0.04699 $^{\{2\}}$	0.05293 $^{\{5\}}$
		$\hat{\eta}$	0.09748 $^{\{1\}}$	0.11346 $^{\{3\}}$	0.13433 $^{\{6\}}$	0.13239 $^{\{5\}}$	0.14057 $^{\{7\}}$	0.23818 $^{\{8\}}$	0.1034 $^{\{2\}}$	0.11628 $^{\{4\}}$
		$\hat{\gamma}$	0.00782 $^{\{1\}}$	0.01052 $^{\{3\}}$	0.01209 $^{\{5\}}$	0.0206 $^{\{7\}}$	0.01293 $^{\{6\}}$	0.04997 $^{\{8\}}$	0.00888 $^{\{2\}}$	0.0109 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.21624 $^{\{1\}}$	0.23239 $^{\{4\}}$	0.2773 $^{\{6\}}$	0.22889 $^{\{3\}}$	0.29061 $^{\{7\}}$	0.42871 $^{\{8\}}$	0.22577 $^{\{2\}}$	0.24036 $^{\{5\}}$
		$\hat{\eta}$	0.16198 $^{\{1\}}$	0.17282 $^{\{3\}}$	0.18927 $^{\{6\}}$	0.18035 $^{\{5\}}$	0.1945 $^{\{7\}}$	0.24499 $^{\{8\}}$	0.16422 $^{\{2\}}$	0.17595 $^{\{4\}}$
		$\hat{\gamma}$	0.19214 $^{\{1\}}$	0.22278 $^{\{4\}}$	0.24382 $^{\{5\}}$	0.26688 $^{\{7\}}$	0.25055 $^{\{6\}}$	0.55271 $^{\{8\}}$	0.20937 $^{\{2\}}$	0.22134 $^{\{3\}}$
	$\sum Ranks$		9 $^{\{1\}}$	31 $^{\{3\}}$	51 $^{\{6\}}$	46 $^{\{5\}}$	60 $^{\{7\}}$	72 $^{\{8\}}$	18 $^{\{2\}}$	37 $^{\{4\}}$
300	BIAS	$\hat{\theta}$	0.12772 $^{\{1\}}$	0.14162 $^{\{4\}}$	0.17372 $^{\{7\}}$	0.13688 $^{\{2\}}$	0.17201 $^{\{6\}}$	0.26014 $^{\{8\}}$	0.14019 $^{\{3\}}$	0.14354 $^{\{5\}}$
		$\hat{\eta}$	0.19302 $^{\{1\}}$	0.20502 $^{\{2\}}$	0.23269 $^{\{7\}}$	0.21015 $^{\{5\}}$	0.23105 $^{\{6\}}$	0.30755 $^{\{8\}}$	0.20864 $^{\{4\}}$	0.2061 $^{\{3\}}$
		$\hat{\gamma}$	0.03783 $^{\{1\}}$	0.03986 $^{\{3\}}$	0.04656 $^{\{6\}}$	0.04788 $^{\{7\}}$	0.04592 $^{\{5\}}$	0.10864 $^{\{8\}}$	0.04181 $^{\{4\}}$	0.03882 $^{\{2\}}$
	MSE	$\hat{\theta}$	0.02713 $^{\{1\}}$	0.0325 $^{\{2\}}$	0.04801 $^{\{7\}}$	0.03256 $^{\{3.5\}}$	0.04754 $^{\{6\}}$	0.1152 $^{\{8\}}$	0.03263 $^{\{5\}}$	0.03256 $^{\{3.5\}}$
		$\hat{\eta}$	0.0648 $^{\{1\}}$	0.07176 $^{\{3\}}$	0.09158 $^{\{7\}}$	0.08605 $^{\{5\}}$	0.08962 $^{\{6\}}$	0.17231 $^{\{8\}}$	0.07483 $^{\{4\}}$	0.06933 $^{\{2\}}$
		$\hat{\gamma}$	0.00525 $^{\{3\}}$	0.00468 $^{\{2\}}$	0.00706 $^{\{6\}}$	0.01145 $^{\{7\}}$	0.0065 $^{\{5\}}$	0.03316 $^{\{8\}}$	0.00563 $^{\{4\}}$	0.004 $^{\{1\}}$
	MRE	$\hat{\theta}$	0.17029 $^{\{1\}}$	0.18883 $^{\{4\}}$	0.23163 $^{\{7\}}$	0.1825 $^{\{2\}}$	0.22935 $^{\{6\}}$	0.34686 $^{\{8\}}$	0.18692 $^{\{3\}}$	0.19139 $^{\{5\}}$
		$\hat{\eta}$	0.12868 $^{\{1\}}$	0.13668 $^{\{2\}}$	0.15513 $^{\{7\}}$	0.1401 $^{\{5\}}$	0.15403 $^{\{6\}}$	0.20503 $^{\{8\}}$	0.13909 $^{\{4\}}$	0.1374 $^{\{3\}}$
		$\hat{\gamma}$	0.15132 $^{\{1\}}$	0.15946 $^{\{3\}}$	0.18626 $^{\{6\}}$	0.19153 $^{\{7\}}$	0.1837 $^{\{5\}}$	0.43457 $^{\{8\}}$	0.16724 $^{\{4\}}$	0.15528 $^{\{2\}}$
	$\sum Ranks$		11 $^{\{1\}}$	25 $^{\{2\}}$	60 $^{\{7\}}$	43.5 $^{\{5\}}$	51 $^{\{6\}}$	72 $^{\{8\}}$	35 $^{\{4\}}$	26.5 $^{\{3\}}$

Table 5. Simulation values of BIAS, MSE and MRE for ($\theta = 1.50,\ \eta = 0.25,\ \gamma = 0.50$).

n	Measures	$\hat{\omega}$	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
75	BIAS	$\hat{\theta}$	0.08268 $^{\{2\}}$	0.329 $^{\{3\}}$	2.36894 $^{\{7\}}$	0.01409 $^{\{1\}}$	2.06945 $^{\{6\}}$	3.18532 $^{\{8\}}$	1.22495 $^{\{4\}}$	1.81121 $^{\{5\}}$
		$\hat{\eta}$	0.01952 $^{\{2\}}$	0.02972 $^{\{3\}}$	0.14829 $^{\{7\}}$	0.0156 $^{\{1\}}$	0.14297 $^{\{6\}}$	0.20789 $^{\{8\}}$	0.14213 $^{\{5\}}$	0.13552 $^{\{4\}}$
		$\hat{\gamma}$	0.02922 $^{\{2\}}$	0.03589 $^{\{3\}}$	0.25056 $^{\{7\}}$	0.0244 $^{\{1\}}$	0.23045 $^{\{5\}}$	0.43911 $^{\{8\}}$	0.23988 $^{\{6\}}$	0.19734 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.14549 $^{\{2\}}$	1.04433 $^{\{3\}}$	18.45666 $^{\{7\}}$	0.00543 $^{\{1\}}$	13.53289 $^{\{6\}}$	54.38416 $^{\{8\}}$	3.56615 $^{\{4\}}$	11.97545 $^{\{5\}}$
		$\hat{\eta}$	0.00371 $^{\{2\}}$	0.00729 $^{\{3\}}$	0.03879 $^{\{6\}}$	0.00179 $^{\{1\}}$	0.03649 $^{\{4\}}$	0.07199 $^{\{8\}}$	0.043 $^{\{7\}}$	0.03854 $^{\{5\}}$
		$\hat{\gamma}$	0.00568 $^{\{2\}}$	0.00754 $^{\{3\}}$	0.08207 $^{\{6\}}$	0.00446 $^{\{1\}}$	0.07067 $^{\{5\}}$	1.00815 $^{\{8\}}$	0.09376 $^{\{7\}}$	0.05375 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.05512 $^{\{2\}}$	0.21934 $^{\{3\}}$	1.57929 $^{\{7\}}$	0.0094 $^{\{1\}}$	1.37963 $^{\{6\}}$	2.12354 $^{\{8\}}$	0.81663 $^{\{4\}}$	1.20747 $^{\{5\}}$
		$\hat{\eta}$	0.07807 $^{\{2\}}$	0.11889 $^{\{3\}}$	0.59316 $^{\{7\}}$	0.0624 $^{\{1\}}$	0.57186 $^{\{6\}}$	0.83158 $^{\{8\}}$	0.56852 $^{\{5\}}$	0.54208 $^{\{4\}}$
		$\hat{\gamma}$	0.05844 $^{\{2\}}$	0.07179 $^{\{3\}}$	0.50113 $^{\{7\}}$	0.04879 $^{\{1\}}$	0.4609 $^{\{5\}}$	0.87823 $^{\{8\}}$	0.47977 $^{\{6\}}$	0.39467 $^{\{4\}}$
	$\sum Ranks$		24 $^{\{2\}}$	36 $^{\{3\}}$	81 $^{\{7\}}$	12 $^{\{1\}}$	65 $^{\{5\}}$	96 $^{\{8\}}$	66 $^{\{6\}}$	52 $^{\{4\}}$
120	BIAS	$\hat{\theta}$	0.0102 $^{\{2\}}$	0.12028 $^{\{3\}}$	1.93339 $^{\{8\}}$	0.00359 $^{\{1\}}$	1.75501 $^{\{6\}}$	1.90063 $^{\{7\}}$	1.04603 $^{\{4\}}$	1.33951 $^{\{5\}}$
		$\hat{\eta}$	0.00401 $^{\{1\}}$	0.01221 $^{\{3\}}$	0.13203 $^{\{7\}}$	0.00407 $^{\{2\}}$	0.12908 $^{\{6\}}$	0.168 $^{\{8\}}$	0.12225 $^{\{5\}}$	0.11689 $^{\{4\}}$
		$\hat{\gamma}$	0.00677 $^{\{2\}}$	0.01501 $^{\{3\}}$	0.21273 $^{\{6\}}$	0.00555 $^{\{1\}}$	0.20644 $^{\{5\}}$	0.29434 $^{\{8\}}$	0.21735 $^{\{7\}}$	0.17808 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.00465 $^{\{2\}}$	0.33535 $^{\{3\}}$	11.10209 $^{\{7\}}$	0.00333 $^{\{1\}}$	9.59388 $^{\{6\}}$	13.9206 $^{\{8\}}$	2.28066 $^{\{4\}}$	5.54841 $^{\{5\}}$
		$\hat{\eta}$	0.00031 $^{\{1\}}$	0.00284 $^{\{3\}}$	0.02953 $^{\{5\}}$	0.00038 $^{\{2\}}$	0.03054 $^{\{7\}}$	0.04478 $^{\{8\}}$	0.02974 $^{\{6\}}$	0.02594 $^{\{4\}}$
		$\hat{\gamma}$	0.00092 $^{\{2\}}$	0.00311 $^{\{3\}}$	0.06003 $^{\{6\}}$	0.00073 $^{\{1\}}$	0.05623 $^{\{5\}}$	0.35748 $^{\{8\}}$	0.07828 $^{\{7\}}$	0.04416 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.0068 $^{\{2\}}$	0.08019 $^{\{3\}}$	1.28893 $^{\{8\}}$	0.00239 $^{\{1\}}$	1.17001 $^{\{6\}}$	1.26709 $^{\{7\}}$	0.69735 $^{\{4\}}$	0.893 $^{\{5\}}$
		$\hat{\eta}$	0.01606 $^{\{1\}}$	0.04883 $^{\{3\}}$	0.52811 $^{\{7\}}$	0.01626 $^{\{2\}}$	0.5163 $^{\{6\}}$	0.67198 $^{\{8\}}$	0.48899 $^{\{5\}}$	0.46758 $^{\{4\}}$
		$\hat{\gamma}$	0.01354 $^{\{2\}}$	0.03001 $^{\{3\}}$	0.42547 $^{\{6\}}$	0.0111 $^{\{1\}}$	0.41289 $^{\{5\}}$	0.58869 $^{\{8\}}$	0.4347 $^{\{7\}}$	0.35615 $^{\{4\}}$
	$\sum Ranks$		15 $^{\{2\}}$	27 $^{\{3\}}$	57 $^{\{7\}}$	12 $^{\{1\}}$	54 $^{\{6\}}$	67 $^{\{8\}}$	51 $^{\{5\}}$	41 $^{\{4\}}$
150	BIAS	$\hat{\theta}$	0.00537 $^{\{2\}}$	0.06317 $^{\{3\}}$	1.80033 $^{\{8\}}$	0.00154 $^{\{1\}}$	1.60955 $^{\{6\}}$	1.69141 $^{\{7\}}$	0.92445 $^{\{4\}}$	1.13627 $^{\{5\}}$
		$\hat{\eta}$	0.00196 $^{\{1\}}$	0.00603 $^{\{3\}}$	0.12632 $^{\{7\}}$	0.00245 $^{\{2\}}$	0.12448 $^{\{6\}}$	0.16382 $^{\{8\}}$	0.11362 $^{\{5\}}$	0.10692 $^{\{4\}}$
		$\hat{\gamma}$	0.00353 $^{\{1\}}$	0.00696 $^{\{3\}}$	0.20204 $^{\{6\}}$	0.00376 $^{\{2\}}$	0.1961 $^{\{5\}}$	0.29891 $^{\{8\}}$	0.21778 $^{\{7\}}$	0.16747 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.00126 $^{\{2\}}$	0.13958 $^{\{3\}}$	9.91974 $^{\{8\}}$	0.00023 $^{\{1\}}$	7.8568 $^{\{6\}}$	9.32181 $^{\{7\}}$	1.73081 $^{\{4\}}$	3.59389 $^{\{5\}}$
		$\hat{\eta}$	0.00016 $^{\{1\}}$	0.00132 $^{\{3\}}$	0.02826 $^{\{7\}}$	0.00019 $^{\{2\}}$	0.02725 $^{\{6\}}$	0.04215 $^{\{8\}}$	0.02344 $^{\{5\}}$	0.02149 $^{\{4\}}$
		$\hat{\gamma}$	0.00043 $^{\{1\}}$	0.00134 $^{\{3\}}$	0.05364 $^{\{6\}}$	0.00044 $^{\{2\}}$	0.05097 $^{\{5\}}$	0.39842 $^{\{8\}}$	0.07807 $^{\{7\}}$	0.04034 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.00358 $^{\{2\}}$	0.04211 $^{\{3\}}$	1.20022 $^{\{8\}}$	0.00103 $^{\{1\}}$	1.07304 $^{\{6\}}$	1.1276 $^{\{7\}}$	0.6163 $^{\{4\}}$	0.75751 $^{\{5\}}$
		$\hat{\eta}$	0.00786 $^{\{1\}}$	0.02411 $^{\{3\}}$	0.50527 $^{\{7\}}$	0.00981 $^{\{2\}}$	0.49794 $^{\{6\}}$	0.65529 $^{\{8\}}$	0.45447 $^{\{5\}}$	0.4277 $^{\{4\}}$
		$\hat{\gamma}$	0.00707 $^{\{1\}}$	0.01392 $^{\{3\}}$	0.40408 $^{\{6\}}$	0.00752 $^{\{2\}}$	0.3922 $^{\{5\}}$	0.59781 $^{\{8\}}$	0.43556 $^{\{7\}}$	0.33494 $^{\{4\}}$
	$\sum Ranks$		12 $^{\{1\}}$	27 $^{\{3\}}$	63 $^{\{7\}}$	15 $^{\{2\}}$	51 $^{\{6\}}$	69 $^{\{8\}}$	48 $^{\{5\}}$	39 $^{\{4\}}$
200	BIAS	$\hat{\theta}$	0.00164 $^{\{2\}}$	0.01227 $^{\{3\}}$	1.42107 $^{\{7\}}$	0.00048 $^{\{1\}}$	1.44659 $^{\{8\}}$	1.33121 $^{\{6\}}$	0.83444 $^{\{4\}}$	0.97353 $^{\{5\}}$
		$\hat{\eta}$	0.00083 $^{\{2\}}$	0.0014 $^{\{3\}}$	0.11693 $^{\{6\}}$	0.00082 $^{\{1\}}$	0.12106 $^{\{7\}}$	0.14905 $^{\{8\}}$	0.10487 $^{\{5\}}$	0.10239 $^{\{4\}}$
		$\hat{\gamma}$	0.00117 $^{\{1\}}$	0.0019 $^{\{3\}}$	0.18742 $^{\{6\}}$	0.00124 $^{\{2\}}$	0.18719 $^{\{5\}}$	0.23751 $^{\{8\}}$	0.20852 $^{\{7\}}$	0.16283 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.00026 $^{\{2\}}$	0.02085 $^{\{3\}}$	5.58379 $^{\{7\}}$	3×10^{-5} $^{\{1\}}$	5.85021 $^{\{8\}}$	4.65919 $^{\{6\}}$	1.35292 $^{\{4\}}$	2.41824 $^{\{5\}}$
		$\hat{\eta}$	6×10^{-5} $^{\{2\}}$	0.00024 $^{\{3\}}$	0.0243 $^{\{6\}}$	5×10^{-5} $^{\{1\}}$	0.02712 $^{\{7\}}$	0.03412 $^{\{8\}}$	0.01976 $^{\{5\}}$	0.01957 $^{\{4\}}$
		$\hat{\gamma}$	0.00014 $^{\{2\}}$	0.00033 $^{\{3\}}$	0.0472 $^{\{5\}}$	0.00012 $^{\{1\}}$	0.04736 $^{\{6\}}$	0.15881 $^{\{8\}}$	0.07264 $^{\{7\}}$	0.03731 $^{\{4\}}$
	MRE	$\hat{\theta}$	0.0011 $^{\{2\}}$	0.00818 $^{\{3\}}$	0.94738 $^{\{7\}}$	0.00032 $^{\{1\}}$	0.96439 $^{\{8\}}$	0.88748 $^{\{6\}}$	0.55629 $^{\{4\}}$	0.64902 $^{\{5\}}$
		$\hat{\eta}$	0.00332 $^{\{2\}}$	0.00558 $^{\{3\}}$	0.46771 $^{\{6\}}$	0.00329 $^{\{1\}}$	0.48423 $^{\{7\}}$	0.59619 $^{\{8\}}$	0.41946 $^{\{5\}}$	0.40958 $^{\{4\}}$
		$\hat{\gamma}$	0.00233 $^{\{1\}}$	0.0038 $^{\{3\}}$	0.37483 $^{\{6\}}$	0.00248 $^{\{2\}}$	0.37438 $^{\{5\}}$	0.47502 $^{\{8\}}$	0.41703 $^{\{7\}}$	0.32565 $^{\{4\}}$
	$\sum Ranks$		21 $^{\{1.5\}}$	24 $^{\{3\}}$	54 $^{\{6\}}$	21 $^{\{1.5\}}$	59 $^{\{7\}}$	64 $^{\{8\}}$	45 $^{\{5\}}$	36 $^{\{4\}}$
300	BIAS	$\hat{\theta}$	3×10^{-5} $^{\{2\}}$	0.00398 $^{\{3\}}$	1.16933 $^{\{8\}}$	2×10^{-5} $^{\{1\}}$	1.13756 $^{\{7\}}$	1.06187 $^{\{6\}}$	0.73301 $^{\{4\}}$	0.76496 $^{\{5\}}$
		$\hat{\eta}$	4×10^{-5} $^{\{1\}}$	0.00033 $^{\{3\}}$	0.10464 $^{\{6\}}$	6×10^{-5} $^{\{2\}}$	0.10506 $^{\{7\}}$	0.13138 $^{\{8\}}$	0.09624 $^{\{5\}}$	0.08528 $^{\{4\}}$
		$\hat{\gamma}$	3×10^{-5} $^{\{1\}}$	0.00038 $^{\{3\}}$	0.17109 $^{\{5\}}$	8×10^{-5} $^{\{2\}}$	0.17237 $^{\{6\}}$	0.20785 $^{\{8\}}$	0.19615 $^{\{7\}}$	0.14608 $^{\{4\}}$
	MSE	$\hat{\theta}$	0 $^{\{1.5\}}$	0.00599 $^{\{3\}}$	3.58255 $^{\{8\}}$	0 $^{\{1.5\}}$	3.33986 $^{\{7\}}$	2.57736 $^{\{6\}}$	0.98121 $^{\{4\}}$	1.3758 $^{\{5\}}$
		$\hat{\eta}$	0 $^{\{1.5\}}$	4×10^{-5} $^{\{3\}}$	0.0194 $^{\{6\}}$	0 $^{\{1.5\}}$	0.01964 $^{\{7\}}$	0.02464 $^{\{8\}}$	0.01576 $^{\{5\}}$	0.01291 $^{\{4\}}$
		$\hat{\gamma}$	0 $^{\{1.5\}}$	5×10^{-5} $^{\{3\}}$	0.04073 $^{\{6\}}$	0 $^{\{1.5\}}$	0.04055 $^{\{5\}}$	0.06341 $^{\{7\}}$	0.06402 $^{\{8\}}$	0.03154 $^{\{4\}}$
	MRE	$\hat{\theta}$	2×10^{-5} $^{\{2\}}$	0.00265 $^{\{3\}}$	0.77956 $^{\{8\}}$	1×10^{-5} $^{\{1\}}$	0.75838 $^{\{7\}}$	0.70791 $^{\{6\}}$	0.48868 $^{\{4\}}$	0.50997 $^{\{5\}}$
		$\hat{\eta}$	0.00015 $^{\{1\}}$	0.00134 $^{\{3\}}$	0.41856 $^{\{6\}}$	0.00025 $^{\{2\}}$	0.42023 $^{\{7\}}$	0.52552 $^{\{8\}}$	0.38494 $^{\{5\}}$	0.34112 $^{\{4\}}$
		$\hat{\gamma}$	7×10^{-5} $^{\{1\}}$	0.00075 $^{\{3\}}$	0.34218 $^{\{5\}}$	0.00015 $^{\{2\}}$	0.34474 $^{\{6\}}$	0.4157 $^{\{8\}}$	0.3923 $^{\{7\}}$	0.29216 $^{\{4\}}$
	$\sum Ranks$		43.5 $^{\{5\}}$	28 $^{\{1.5\}}$	47 $^{\{6\}}$	37.5 $^{\{3\}}$	48 $^{\{7\}}$	54 $^{\{8\}}$	38 $^{\{4\}}$	28 $^{\{1.5\}}$

Table 6. Simulation values of BIAS, MSE and MRE for ($\theta = 0.50,\ \eta = 0.75,\ \gamma = 1.50$).

n	Measures	$\hat{\omega}$	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
75	BIAS	$\hat{\theta}$	0.1876 $^{\{7\}}$	0.17184 $^{\{2\}}$	0.19473 $^{\{8\}}$	0.14535 $^{\{1\}}$	0.18111 $^{\{5\}}$	0.1755 $^{\{4\}}$	0.1821 $^{\{6\}}$	0.17192 $^{\{3\}}$
		$\hat{\eta}$	0.59564 $^{\{5\}}$	0.55058 $^{\{2\}}$	0.63891 $^{\{8\}}$	0.46693 $^{\{1\}}$	0.6037 $^{\{7\}}$	0.56828 $^{\{4\}}$	0.59697 $^{\{6\}}$	0.56052 $^{\{3\}}$
		$\hat{\gamma}$	0.91366 $^{\{4\}}$	0.81803 $^{\{2\}}$	1.17092 $^{\{8\}}$	0.73404 $^{\{1\}}$	1.13404 $^{\{7\}}$	0.85028 $^{\{3\}}$	0.93252 $^{\{6\}}$	0.93062 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.08345 $^{\{8\}}$	0.0592 $^{\{5\}}$	0.071 $^{\{7\}}$	0.04562 $^{\{1\}}$	0.05929 $^{\{6\}}$	0.05136 $^{\{2\}}$	0.05827 $^{\{4\}}$	0.05814 $^{\{3\}}$
		$\hat{\eta}$	0.73476 $^{\{8\}}$	0.55026 $^{\{3\}}$	0.66641 $^{\{7\}}$	0.43469 $^{\{1\}}$	0.58523 $^{\{5\}}$	0.5123 $^{\{2\}}$	0.5866 $^{\{6\}}$	0.55421 $^{\{4\}}$
		$\hat{\gamma}$	1.62787 $^{\{4\}}$	1.28986 $^{\{2\}}$	2.91034 $^{\{8\}}$	1.1446 $^{\{1\}}$	2.71443 $^{\{7\}}$	1.55071 $^{\{3\}}$	2.0562 $^{\{6\}}$	1.73932 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.37521 $^{\{7\}}$	0.34369 $^{\{2\}}$	0.38947 $^{\{8\}}$	0.29069 $^{\{1\}}$	0.36223 $^{\{5\}}$	0.351 $^{\{4\}}$	0.3642 $^{\{6\}}$	0.34383 $^{\{3\}}$
		$\hat{\eta}$	0.79418 $^{\{5\}}$	0.73411 $^{\{2\}}$	0.85188 $^{\{8\}}$	0.62257 $^{\{1\}}$	0.80494 $^{\{7\}}$	0.7577 $^{\{4\}}$	0.79596 $^{\{6\}}$	0.74736 $^{\{3\}}$
		$\hat{\gamma}$	0.60911 $^{\{4\}}$	0.54535 $^{\{2\}}$	0.78062 $^{\{8\}}$	0.48936 $^{\{1\}}$	0.75603 $^{\{7\}}$	0.56685 $^{\{3\}}$	0.62168 $^{\{6\}}$	0.62041 $^{\{5\}}$
	$\sum$ *Ranks*		68 $^{\{5\}}$	28 $^{\{2\}}$	94 $^{\{8\}}$	12 $^{\{1\}}$	75 $^{\{7\}}$	40 $^{\{3\}}$	70 $^{\{6\}}$	45 $^{\{4\}}$
120	BIAS	$\hat{\theta}$	0.16126 $^{\{6\}}$	0.1484 $^{\{2\}}$	0.16796 $^{\{8\}}$	0.12516 $^{\{1\}}$	0.16593 $^{\{7\}}$	0.15187 $^{\{5\}}$	0.1507 $^{\{4\}}$	0.14978 $^{\{3\}}$
		$\hat{\eta}$	0.50791 $^{\{6\}}$	0.47203 $^{\{2\}}$	0.5542 $^{\{8\}}$	0.39552 $^{\{1\}}$	0.5517 $^{\{7\}}$	0.50224 $^{\{5\}}$	0.49516 $^{\{4\}}$	0.48598 $^{\{3\}}$
		$\hat{\gamma}$	0.70968 $^{\{5\}}$	0.61374 $^{\{2\}}$	0.90233 $^{\{7\}}$	0.54471 $^{\{1\}}$	0.90927 $^{\{8\}}$	0.69117 $^{\{4\}}$	0.68977 $^{\{3\}}$	0.71558 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.06697 $^{\{8\}}$	0.04863 $^{\{5\}}$	0.0564 $^{\{7\}}$	0.03849 $^{\{1\}}$	0.05337 $^{\{6\}}$	0.04095 $^{\{2\}}$	0.04466 $^{\{3\}}$	0.04784 $^{\{4\}}$
		$\hat{\eta}$	0.57825 $^{\{8\}}$	0.44708 $^{\{4\}}$	0.53752 $^{\{7\}}$	0.35825 $^{\{1\}}$	0.52086 $^{\{6\}}$	0.42782 $^{\{2\}}$	0.4453 $^{\{3\}}$	0.45724 $^{\{5\}}$
		$\hat{\gamma}$	0.93426 $^{\{3\}}$	0.67257 $^{\{2\}}$	1.68062 $^{\{7\}}$	0.58982 $^{\{1\}}$	1.68888 $^{\{8\}}$	0.95966 $^{\{5\}}$	0.95726 $^{\{4\}}$	1.00152 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.32251 $^{\{6\}}$	0.2968 $^{\{2\}}$	0.33592 $^{\{8\}}$	0.25032 $^{\{1\}}$	0.33186 $^{\{7\}}$	0.30374 $^{\{5\}}$	0.30141 $^{\{4\}}$	0.29955 $^{\{3\}}$
		$\hat{\eta}$	0.67721 $^{\{6\}}$	0.62938 $^{\{2\}}$	0.73894 $^{\{8\}}$	0.52735 $^{\{1\}}$	0.73561 $^{\{7\}}$	0.66965 $^{\{5\}}$	0.66021 $^{\{4\}}$	0.64798 $^{\{3\}}$
		$\hat{\gamma}$	0.47312 $^{\{5\}}$	0.40916 $^{\{2\}}$	0.60155 $^{\{7\}}$	0.36314 $^{\{1\}}$	0.60618 $^{\{8\}}$	0.46078 $^{\{4\}}$	0.45984 $^{\{3\}}$	0.47705 $^{\{6\}}$
	$\sum$ *Ranks*		53 $^{\{6\}}$	23 $^{\{2\}}$	67 $^{\{8\}}$	9 $^{\{1\}}$	64 $^{\{7\}}$	37 $^{\{4\}}$	32 $^{\{3\}}$	39 $^{\{5\}}$
150	BIAS	$\hat{\theta}$	0.13682 $^{\{3\}}$	0.13303 $^{\{2\}}$	0.15536 $^{\{8\}}$	0.1195 $^{\{1\}}$	0.14969 $^{\{7\}}$	0.14469 $^{\{6\}}$	0.13912 $^{\{5\}}$	0.137 $^{\{4\}}$
		$\hat{\eta}$	0.4306 $^{\{2\}}$	0.43337 $^{\{3\}}$	0.52171 $^{\{8\}}$	0.37602 $^{\{1\}}$	0.50289 $^{\{7\}}$	0.47743 $^{\{6\}}$	0.45864 $^{\{5\}}$	0.4484 $^{\{4\}}$
		$\hat{\gamma}$	0.57947 $^{\{4\}}$	0.56478 $^{\{2\}}$	0.83261 $^{\{8\}}$	0.46664 $^{\{1\}}$	0.76979 $^{\{7\}}$	0.57561 $^{\{3\}}$	0.61207 $^{\{5\}}$	0.63783 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.05352 $^{\{8\}}$	0.04112 $^{\{4\}}$	0.05001 $^{\{7\}}$	0.03671 $^{\{1\}}$	0.04659 $^{\{6\}}$	0.03951 $^{\{3\}}$	0.03843 $^{\{2\}}$	0.04364 $^{\{5\}}$
		$\hat{\eta}$	0.46379 $^{\{6\}}$	0.39222 $^{\{3\}}$	0.49531 $^{\{8\}}$	0.33661 $^{\{1\}}$	0.46501 $^{\{7\}}$	0.40797 $^{\{4\}}$	0.3912 $^{\{2\}}$	0.40975 $^{\{5\}}$
		$\hat{\gamma}$	0.60896 $^{\{4\}}$	0.56743 $^{\{2\}}$	1.42914 $^{\{8\}}$	0.42459 $^{\{1\}}$	1.16637 $^{\{7\}}$	0.57616 $^{\{3\}}$	0.66245 $^{\{5\}}$	0.7407 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.27365 $^{\{3\}}$	0.26606 $^{\{2\}}$	0.31072 $^{\{8\}}$	0.23899 $^{\{1\}}$	0.29937 $^{\{7\}}$	0.28938 $^{\{6\}}$	0.27823 $^{\{5\}}$	0.27399 $^{\{4\}}$
		$\hat{\eta}$	0.57413 $^{\{2\}}$	0.57782 $^{\{3\}}$	0.69561 $^{\{8\}}$	0.50136 $^{\{1\}}$	0.67052 $^{\{7\}}$	0.63658 $^{\{6\}}$	0.61152 $^{\{5\}}$	0.59787 $^{\{4\}}$
		$\hat{\gamma}$	0.38631 $^{\{4\}}$	0.37652 $^{\{2\}}$	0.55507 $^{\{8\}}$	0.31109 $^{\{1\}}$	0.51319 $^{\{7\}}$	0.38374 $^{\{3\}}$	0.40805 $^{\{5\}}$	0.42522 $^{\{6\}}$
	$\sum$ *Ranks*		36 $^{\{3\}}$	23 $^{\{2\}}$	71 $^{\{8\}}$	9 $^{\{1\}}$	62 $^{\{7\}}$	40 $^{\{5\}}$	39 $^{\{4\}}$	44 $^{\{6\}}$
200	BIAS	$\hat{\theta}$	0.1131 $^{\{2\}}$	0.1201 $^{\{4\}}$	0.13891 $^{\{8\}}$	0.09771 $^{\{1\}}$	0.13883 $^{\{7\}}$	0.12646 $^{\{6\}}$	0.12526 $^{\{5\}}$	0.11812 $^{\{3\}}$
		$\hat{\eta}$	0.36045 $^{\{2\}}$	0.38527 $^{\{3\}}$	0.47001 $^{\{7\}}$	0.30838 $^{\{1\}}$	0.47157 $^{\{8\}}$	0.41712 $^{\{6\}}$	0.40974 $^{\{5\}}$	0.38692 $^{\{4\}}$
		$\hat{\gamma}$	0.4772 $^{\{2\}}$	0.48977 $^{\{3\}}$	0.70107 $^{\{7\}}$	0.39294 $^{\{1\}}$	0.71042 $^{\{8\}}$	0.51361 $^{\{4\}}$	0.51617 $^{\{5\}}$	0.52688 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.03981 $^{\{6\}}$	0.03476 $^{\{5\}}$	0.04266 $^{\{8\}}$	0.02725 $^{\{1\}}$	0.04076 $^{\{7\}}$	0.03168 $^{\{2\}}$	0.03344 $^{\{3\}}$	0.03367 $^{\{4\}}$
		$\hat{\eta}$	0.34746 $^{\{6\}}$	0.32213 $^{\{2\}}$	0.42893 $^{\{8\}}$	0.25103 $^{\{1\}}$	0.41969 $^{\{7\}}$	0.32415 $^{\{4\}}$	0.33147 $^{\{5\}}$	0.32261 $^{\{3\}}$
		$\hat{\gamma}$	0.42009 $^{\{3\}}$	0.40225 $^{\{2\}}$	0.9689 $^{\{8\}}$	0.29716 $^{\{1\}}$	0.96728 $^{\{7\}}$	0.45639 $^{\{4\}}$	0.4641 $^{\{5\}}$	0.52209 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.22619 $^{\{2\}}$	0.24021 $^{\{4\}}$	0.27782 $^{\{8\}}$	0.19542 $^{\{1\}}$	0.27767 $^{\{7\}}$	0.25292 $^{\{6\}}$	0.25051 $^{\{5\}}$	0.23623 $^{\{3\}}$
		$\hat{\eta}$	0.48061 $^{\{2\}}$	0.51369 $^{\{3\}}$	0.62668 $^{\{7\}}$	0.41117 $^{\{1\}}$	0.62876 $^{\{8\}}$	0.55616 $^{\{6\}}$	0.54632 $^{\{5\}}$	0.51589 $^{\{4\}}$
		$\hat{\gamma}$	0.31814 $^{\{2\}}$	0.32652 $^{\{3\}}$	0.46738 $^{\{7\}}$	0.26196 $^{\{1\}}$	0.47361 $^{\{8\}}$	0.34241 $^{\{4\}}$	0.34411 $^{\{5\}}$	0.35125 $^{\{6\}}$
	$\sum$ *Ranks*		27 $^{\{2\}}$	29 $^{\{3\}}$	68 $^{\{8\}}$	9 $^{\{1\}}$	67 $^{\{7\}}$	42 $^{\{5\}}$	43 $^{\{6\}}$	39 $^{\{4\}}$
300	BIAS	$\hat{\theta}$	0.08412 $^{\{2\}}$	0.10072 $^{\{4\}}$	0.11392 $^{\{7\}}$	0.07426 $^{\{1\}}$	0.11577 $^{\{8\}}$	0.10853 $^{\{6\}}$	0.106 $^{\{5\}}$	0.09818 $^{\{3\}}$
		$\hat{\eta}$	0.27159 $^{\{2\}}$	0.32548 $^{\{4\}}$	0.38722 $^{\{7\}}$	0.2376 $^{\{1\}}$	0.39628 $^{\{8\}}$	0.35798 $^{\{6\}}$	0.34882 $^{\{5\}}$	0.32438 $^{\{3\}}$
		$\hat{\gamma}$	0.36833 $^{\{2\}}$	0.41114 $^{\{3\}}$	0.53798 $^{\{7\}}$	0.29851 $^{\{1\}}$	0.57014 $^{\{8\}}$	0.42118 $^{\{4\}}$	0.43907 $^{\{6\}}$	0.42842 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.02447 $^{\{2\}}$	0.02712 $^{\{6\}}$	0.03143 $^{\{8\}}$	0.01613 $^{\{1\}}$	0.03074 $^{\{7\}}$	0.025 $^{\{4\}}$	0.02476 $^{\{3\}}$	0.02576 $^{\{5\}}$
		$\hat{\eta}$	0.21515 $^{\{2\}}$	0.2536 $^{\{5\}}$	0.31615 $^{\{7\}}$	0.15725 $^{\{1\}}$	0.31873 $^{\{8\}}$	0.2577 $^{\{6\}}$	0.24992 $^{\{3\}}$	0.25109 $^{\{4\}}$
		$\hat{\gamma}$	0.24406 $^{\{2\}}$	0.27821 $^{\{4\}}$	0.52259 $^{\{7\}}$	0.17526 $^{\{1\}}$	0.63325 $^{\{8\}}$	0.27699 $^{\{3\}}$	0.31451 $^{\{6\}}$	0.31003 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.16824 $^{\{2\}}$	0.20145 $^{\{4\}}$	0.22783 $^{\{7\}}$	0.14851 $^{\{1\}}$	0.23155 $^{\{8\}}$	0.21707 $^{\{6\}}$	0.212 $^{\{5\}}$	0.19637 $^{\{3\}}$
		$\hat{\eta}$	0.36212 $^{\{2\}}$	0.43397 $^{\{4\}}$	0.5163 $^{\{7\}}$	0.31681 $^{\{1\}}$	0.52837 $^{\{8\}}$	0.4773 $^{\{6\}}$	0.4651 $^{\{5\}}$	0.4325 $^{\{3\}}$
		$\hat{\gamma}$	0.24556 $^{\{2\}}$	0.27409 $^{\{3\}}$	0.35866 $^{\{7\}}$	0.19901 $^{\{1\}}$	0.38009 $^{\{8\}}$	0.28079 $^{\{4\}}$	0.29271 $^{\{6\}}$	0.28562 $^{\{5\}}$
	$\sum$ *Ranks*		18 $^{\{2\}}$	37 $^{\{4\}}$	64 $^{\{7\}}$	9 $^{\{1\}}$	71 $^{\{8\}}$	45 $^{\{6\}}$	44 $^{\{5\}}$	36 $^{\{3\}}$

Table 7. Simulation values of BIAS, MSE and MRE for ($\theta = 1.5,\ \eta = 1.5,\ \gamma = 1.5$).

n	Measures	$\hat{\omega}$	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
75	BIAS	$\hat{\theta}$	0.52344 $^{\{5\}}$	0.50413 $^{\{2\}}$	0.57443 $^{\{7\}}$	0.43809 $^{\{1\}}$	0.56046 $^{\{6\}}$	0.51378 $^{\{4\}}$	0.60559 $^{\{8\}}$	0.50681 $^{\{3\}}$
		$\hat{\eta}$	1.19949 $^{\{5\}}$	1.1569 $^{\{3\}}$	1.3495 $^{\{7\}}$	0.98458 $^{\{1\}}$	1.31242 $^{\{6\}}$	1.1478 $^{\{2\}}$	1.38605 $^{\{8\}}$	1.18648 $^{\{4\}}$
		$\hat{\gamma}$	0.86572 $^{\{4\}}$	0.75236 $^{\{3\}}$	1.19252 $^{\{8\}}$	0.68199 $^{\{1\}}$	1.1904 $^{\{7\}}$	0.71935 $^{\{2\}}$	0.87581 $^{\{5\}}$	0.90677 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.54348 $^{\{6\}}$	0.47419 $^{\{3\}}$	0.57107 $^{\{7\}}$	0.36151 $^{\{1\}}$	0.52646 $^{\{5\}}$	0.49254 $^{\{4\}}$	0.72973 $^{\{8\}}$	0.45629 $^{\{2\}}$
		$\hat{\eta}$	2.71089 $^{\{5\}}$	2.43735 $^{\{3\}}$	3.05596 $^{\{7\}}$	1.84988 $^{\{1\}}$	2.75992 $^{\{6\}}$	2.34889 $^{\{2\}}$	3.44067 $^{\{8\}}$	2.4521 $^{\{4\}}$
		$\hat{\gamma}$	1.86514 $^{\{4\}}$	1.09922 $^{\{2\}}$	3.77617 $^{\{8\}}$	1.13491 $^{\{3\}}$	3.67113 $^{\{7\}}$	0.98641 $^{\{1\}}$	2.02331 $^{\{6\}}$	1.96434 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.34896 $^{\{5\}}$	0.33608 $^{\{2\}}$	0.38295 $^{\{7\}}$	0.29206 $^{\{1\}}$	0.37364 $^{\{6\}}$	0.34252 $^{\{4\}}$	0.40373 $^{\{8\}}$	0.33788 $^{\{3\}}$
		$\hat{\eta}$	0.79966 $^{\{5\}}$	0.77126 $^{\{3\}}$	0.89967 $^{\{7\}}$	0.65638 $^{\{1\}}$	0.87495 $^{\{6\}}$	0.7652 $^{\{2\}}$	0.92403 $^{\{8\}}$	0.79099 $^{\{4\}}$
		$\hat{\gamma}$	0.57715 $^{\{4\}}$	0.50157 $^{\{3\}}$	0.79501 $^{\{8\}}$	0.45466 $^{\{1\}}$	0.7936 $^{\{7\}}$	0.47957 $^{\{2\}}$	0.58387 $^{\{5\}}$	0.60451 $^{\{6\}}$
	$\sum$ *Ranks*		57 $^{\{5\}}$	32 $^{\{3\}}$	88 $^{\{8\}}$	14 $^{\{1\}}$	75 $^{\{6\}}$	31 $^{\{2\}}$	85 $^{\{7\}}$	50 $^{\{4\}}$
120	BIAS	$\hat{\theta}$	0.41138 $^{\{2\}}$	0.44912 $^{\{3\}}$	0.49768 $^{\{6\}}$	0.35184 $^{\{1\}}$	0.50193 $^{\{8\}}$	0.47049 $^{\{5\}}$	0.5013 $^{\{7\}}$	0.46802 $^{\{4\}}$
		$\hat{\eta}$	0.95093 $^{\{2\}}$	1.04631 $^{\{3\}}$	1.17674 $^{\{8\}}$	0.78849 $^{\{1\}}$	1.16771 $^{\{7\}}$	1.05908 $^{\{4\}}$	1.15865 $^{\{6\}}$	1.09232 $^{\{5\}}$
		$\hat{\gamma}$	0.60425 $^{\{3\}}$	0.61489 $^{\{4\}}$	0.87361 $^{\{8\}}$	0.50429 $^{\{1\}}$	0.82599 $^{\{7\}}$	0.56927 $^{\{2\}}$	0.66135 $^{\{5\}}$	0.69284 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.3708 $^{\{2\}}$	0.38216 $^{\{3\}}$	0.45099 $^{\{6\}}$	0.26152 $^{\{1\}}$	0.45408 $^{\{7\}}$	0.43384 $^{\{5\}}$	0.49196 $^{\{8\}}$	0.41113 $^{\{4\}}$
		$\hat{\eta}$	1.88113 $^{\{2\}}$	2.02037 $^{\{3\}}$	2.40797 $^{\{7\}}$	1.32297 $^{\{1\}}$	2.35318 $^{\{6\}}$	2.13221 $^{\{4\}}$	2.52293 $^{\{8\}}$	2.16658 $^{\{5\}}$
		$\hat{\gamma}$	0.73633 $^{\{4\}}$	0.63213 $^{\{3\}}$	1.78269 $^{\{8\}}$	0.55788 $^{\{2\}}$	1.50534 $^{\{7\}}$	0.54078 $^{\{1\}}$	0.80612 $^{\{5\}}$	0.9179 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.27425 $^{\{2\}}$	0.29941 $^{\{3\}}$	0.33179 $^{\{6\}}$	0.23456 $^{\{1\}}$	0.33462 $^{\{8\}}$	0.31366 $^{\{5\}}$	0.3342 $^{\{7\}}$	0.31202 $^{\{4\}}$
		$\hat{\eta}$	0.63395 $^{\{2\}}$	0.69754 $^{\{3\}}$	0.78449 $^{\{8\}}$	0.52566 $^{\{1\}}$	0.77848 $^{\{7\}}$	0.70605 $^{\{4\}}$	0.77243 $^{\{6\}}$	0.72821 $^{\{5\}}$
		$\hat{\gamma}$	0.40283 $^{\{3\}}$	0.40993 $^{\{4\}}$	0.58241 $^{\{8\}}$	0.33619 $^{\{1\}}$	0.55066 $^{\{7\}}$	0.37951 $^{\{2\}}$	0.4409 $^{\{5\}}$	0.46189 $^{\{6\}}$
	$\sum$ *Ranks*		22 $^{\{2\}}$	29 $^{\{3\}}$	65 $^{\{8\}}$	10 $^{\{1\}}$	64 $^{\{7\}}$	32 $^{\{4\}}$	57 $^{\{6\}}$	45 $^{\{5\}}$
150	BIAS	$\hat{\theta}$	0.39249 $^{\{2\}}$	0.3999 $^{\{3\}}$	0.48215 $^{\{8\}}$	0.31648 $^{\{1\}}$	0.47087 $^{\{6\}}$	0.44146 $^{\{5\}}$	0.47093 $^{\{7\}}$	0.40872 $^{\{4\}}$
		$\hat{\eta}$	0.89899 $^{\{2\}}$	0.92744 $^{\{3\}}$	1.13109 $^{\{8\}}$	0.71474 $^{\{1\}}$	1.10751 $^{\{7\}}$	0.98774 $^{\{5\}}$	1.0946 $^{\{6\}}$	0.94955 $^{\{4\}}$
		$\hat{\gamma}$	0.53462 $^{\{3\}}$	0.54146 $^{\{4\}}$	0.76096 $^{\{7\}}$	0.43903 $^{\{1\}}$	0.77132 $^{\{8\}}$	0.51988 $^{\{2\}}$	0.60049 $^{\{6\}}$	0.57672 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.35679 $^{\{4\}}$	0.32792 $^{\{2\}}$	0.42572 $^{\{7\}}$	0.22621 $^{\{1\}}$	0.40387 $^{\{6\}}$	0.4011 $^{\{5\}}$	0.45215 $^{\{8\}}$	0.34046 $^{\{3\}}$
		$\hat{\eta}$	1.75954 $^{\{4\}}$	1.69694 $^{\{2\}}$	2.23597 $^{\{7\}}$	1.15192 $^{\{1\}}$	2.12274 $^{\{6\}}$	1.93753 $^{\{5\}}$	2.35066 $^{\{8\}}$	1.75931 $^{\{3\}}$
		$\hat{\gamma}$	0.51709 $^{\{4\}}$	0.49554 $^{\{3\}}$	1.21309 $^{\{7\}}$	0.35439 $^{\{1\}}$	1.24986 $^{\{8\}}$	0.43394 $^{\{2\}}$	0.61503 $^{\{6\}}$	0.58571 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.26166 $^{\{2\}}$	0.2666 $^{\{3\}}$	0.32143 $^{\{8\}}$	0.21098 $^{\{1\}}$	0.31391 $^{\{6\}}$	0.29431 $^{\{5\}}$	0.31395 $^{\{7\}}$	0.27248 $^{\{4\}}$
		$\hat{\eta}$	0.59933 $^{\{2\}}$	0.61829 $^{\{3\}}$	0.75406 $^{\{8\}}$	0.47649 $^{\{1\}}$	0.73834 $^{\{7\}}$	0.65849 $^{\{5\}}$	0.72973 $^{\{6\}}$	0.63303 $^{\{4\}}$
		$\hat{\gamma}$	0.35641 $^{\{3\}}$	0.36097 $^{\{4\}}$	0.50731 $^{\{7\}}$	0.29269 $^{\{1\}}$	0.51421 $^{\{8\}}$	0.34659 $^{\{2\}}$	0.40033 $^{\{6\}}$	0.38448 $^{\{5\}}$
	$\sum$ *Ranks*		26 $^{\{2\}}$	27 $^{\{3\}}$	67 $^{\{8\}}$	9 $^{\{1\}}$	62 $^{\{7\}}$	36 $^{\{4\}}$	60 $^{\{6\}}$	37 $^{\{5\}}$
200	BIAS	$\hat{\theta}$	0.33155 $^{\{2\}}$	0.362 $^{\{3\}}$	0.43402 $^{\{7\}}$	0.26647 $^{\{1\}}$	0.43071 $^{\{6\}}$	0.37868 $^{\{5\}}$	0.44223 $^{\{8\}}$	0.36523 $^{\{4\}}$
		$\hat{\eta}$	0.76643 $^{\{2\}}$	0.8408 $^{\{3\}}$	1.02171 $^{\{7\}}$	0.5971 $^{\{1\}}$	1.01276 $^{\{6\}}$	0.86562 $^{\{5\}}$	1.02796 $^{\{8\}}$	0.85179 $^{\{4\}}$
		$\hat{\gamma}$	0.44768 $^{\{2\}}$	0.47591 $^{\{4\}}$	0.64501 $^{\{8\}}$	0.35049 $^{\{1\}}$	0.63905 $^{\{7\}}$	0.45384 $^{\{3\}}$	0.54606 $^{\{6\}}$	0.50876 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.27545 $^{\{2\}}$	0.27796 $^{\{3\}}$	0.36311 $^{\{7\}}$	0.1739 $^{\{1\}}$	0.35137 $^{\{6\}}$	0.31978 $^{\{5\}}$	0.41658 $^{\{8\}}$	0.28336 $^{\{4\}}$
		$\hat{\eta}$	1.37612 $^{\{2\}}$	1.42507 $^{\{3\}}$	1.91777 $^{\{7\}}$	0.89526 $^{\{1\}}$	1.85332 $^{\{6\}}$	1.5858 $^{\{5\}}$	2.15692 $^{\{8\}}$	1.46238 $^{\{4\}}$
		$\hat{\gamma}$	0.33568 $^{\{3\}}$	0.35611 $^{\{4\}}$	0.73538 $^{\{8\}}$	0.2317 $^{\{1\}}$	0.71439 $^{\{7\}}$	0.33033 $^{\{2\}}$	0.48167 $^{\{6\}}$	0.44443 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.22103 $^{\{2\}}$	0.24133 $^{\{3\}}$	0.28935 $^{\{7\}}$	0.17765 $^{\{1\}}$	0.28714 $^{\{6\}}$	0.25245 $^{\{5\}}$	0.29482 $^{\{8\}}$	0.24349 $^{\{4\}}$
		$\hat{\eta}$	0.51095 $^{\{2\}}$	0.56053 $^{\{3\}}$	0.68114 $^{\{7\}}$	0.39807 $^{\{1\}}$	0.67517 $^{\{6\}}$	0.57708 $^{\{5\}}$	0.68531 $^{\{8\}}$	0.56786 $^{\{4\}}$
		$\hat{\gamma}$	0.29845 $^{\{2\}}$	0.31727 $^{\{4\}}$	0.43001 $^{\{8\}}$	0.23366 $^{\{1\}}$	0.42603 $^{\{7\}}$	0.30256 $^{\{3\}}$	0.36404 $^{\{6\}}$	0.33917 $^{\{5\}}$
	$\sum$ *Ranks*		19 $^{\{2\}}$	30 $^{\{3\}}$	66 $^{\{7.5\}}$	9 $^{\{1\}}$	57 $^{\{6\}}$	38 $^{\{4\}}$	66 $^{\{7.5\}}$	39 $^{\{5\}}$
300	BIAS	$\hat{\theta}$	0.26014 $^{\{2\}}$	0.28551 $^{\{3\}}$	0.37512 $^{\{8\}}$	0.21111 $^{\{1\}}$	0.36051 $^{\{6\}}$	0.32959 $^{\{5\}}$	0.36642 $^{\{7\}}$	0.29749 $^{\{4\}}$
		$\hat{\eta}$	0.60356 $^{\{2\}}$	0.66839 $^{\{3\}}$	0.88016 $^{\{8\}}$	0.47431 $^{\{1\}}$	0.85173 $^{\{6\}}$	0.7558 $^{\{5\}}$	0.85535 $^{\{7\}}$	0.69178 $^{\{4\}}$
		$\hat{\gamma}$	0.3666 $^{\{2\}}$	0.38224 $^{\{4\}}$	0.51553 $^{\{7\}}$	0.28418 $^{\{1\}}$	0.52394 $^{\{8\}}$	0.37307 $^{\{3\}}$	0.44431 $^{\{6\}}$	0.40724 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.17583 $^{\{2\}}$	0.18894 $^{\{3\}}$	0.29005 $^{\{7\}}$	0.12198 $^{\{1\}}$	0.26689 $^{\{6\}}$	0.26145 $^{\{5\}}$	0.29873 $^{\{8\}}$	0.20425 $^{\{4\}}$
		$\hat{\eta}$	0.88678 $^{\{2\}}$	0.98051 $^{\{3\}}$	1.52198 $^{\{7\}}$	0.63683 $^{\{1\}}$	1.41641 $^{\{6\}}$	1.30922 $^{\{5\}}$	1.56296 $^{\{8\}}$	1.04787 $^{\{4\}}$
		$\hat{\gamma}$	0.22148 $^{\{3\}}$	0.2359 $^{\{4\}}$	0.44793 $^{\{7\}}$	0.16208 $^{\{1\}}$	0.46138 $^{\{8\}}$	0.2149 $^{\{2\}}$	0.30915 $^{\{6\}}$	0.26617 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.17343 $^{\{2\}}$	0.19034 $^{\{3\}}$	0.25008 $^{\{8\}}$	0.14074 $^{\{1\}}$	0.24034 $^{\{6\}}$	0.21973 $^{\{5\}}$	0.24428 $^{\{7\}}$	0.19832 $^{\{4\}}$
		$\hat{\eta}$	0.40237 $^{\{2\}}$	0.4456 $^{\{3\}}$	0.58677 $^{\{8\}}$	0.31621 $^{\{1\}}$	0.56782 $^{\{6\}}$	0.50387 $^{\{5\}}$	0.57023 $^{\{7\}}$	0.46118 $^{\{4\}}$
		$\hat{\gamma}$	0.2444 $^{\{2\}}$	0.25483 $^{\{4\}}$	0.34369 $^{\{7\}}$	0.18945 $^{\{1\}}$	0.34929 $^{\{8\}}$	0.24871 $^{\{3\}}$	0.29621 $^{\{6\}}$	0.27149 $^{\{5\}}$
	$\sum$ *Ranks*		19 $^{\{2\}}$	30 $^{\{3\}}$	67 $^{\{8\}}$	9 $^{\{1\}}$	60 $^{\{6\}}$	38 $^{\{4\}}$	62 $^{\{7\}}$	39 $^{\{5\}}$

Table 8. Simulation values of BIAS, MSE and MRE for ($\theta = 1.25,\ \eta = 2.0,\ \gamma = 0.8$).

n	Measures	$\hat{\omega}$	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
75	BIAS	$\hat{\theta}$	0.397 $^{\{8\}}$	0.33902 $^{\{4\}}$	0.3697 $^{\{7\}}$	0.13538 $^{\{1\}}$	0.36302 $^{\{6\}}$	0.30106 $^{\{2\}}$	0.31825 $^{\{3\}}$	0.36009 $^{\{5\}}$
		$\hat{\eta}$	0.8647 $^{\{8\}}$	0.71849 $^{\{4\}}$	0.77579 $^{\{7\}}$	0.24054 $^{\{1\}}$	0.73586 $^{\{5\}}$	0.63298 $^{\{2\}}$	0.66797 $^{\{3\}}$	0.7741 $^{\{6\}}$
		$\hat{\gamma}$	0.54435 $^{\{6\}}$	0.45159 $^{\{2\}}$	0.69017 $^{\{8\}}$	0.2154 $^{\{1\}}$	0.64371 $^{\{7\}}$	0.48199 $^{\{4\}}$	0.47265 $^{\{3\}}$	0.53403 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.25248 $^{\{6\}}$	0.21559 $^{\{4\}}$	0.29204 $^{\{7\}}$	0.04885 $^{\{1\}}$	0.31114 $^{\{8\}}$	0.18868 $^{\{3\}}$	0.18763 $^{\{2\}}$	0.22997 $^{\{5\}}$
		$\hat{\eta}$	1.0276 $^{\{8\}}$	0.8024 $^{\{4\}}$	0.98956 $^{\{7\}}$	0.25857 $^{\{1\}}$	0.97329 $^{\{6\}}$	0.72326 $^{\{2\}}$	0.75837 $^{\{3\}}$	0.90307 $^{\{5\}}$
		$\hat{\gamma}$	0.92017 $^{\{6\}}$	0.60948 $^{\{2\}}$	1.82426 $^{\{8\}}$	0.30254 $^{\{1\}}$	1.64154 $^{\{7\}}$	0.85474 $^{\{4\}}$	0.78332 $^{\{3\}}$	0.86311 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.3176 $^{\{8\}}$	0.27121 $^{\{4\}}$	0.29576 $^{\{7\}}$	0.10831 $^{\{1\}}$	0.29042 $^{\{6\}}$	0.24085 $^{\{2\}}$	0.2546 $^{\{3\}}$	0.28807 $^{\{5\}}$
		$\hat{\eta}$	0.43235 $^{\{8\}}$	0.35924 $^{\{4\}}$	0.3879 $^{\{7\}}$	0.12027 $^{\{1\}}$	0.36793 $^{\{5\}}$	0.31649 $^{\{2\}}$	0.33399 $^{\{3\}}$	0.38705 $^{\{6\}}$
		$\hat{\gamma}$	0.68044 $^{\{6\}}$	0.56449 $^{\{2\}}$	0.86271 $^{\{8\}}$	0.26925 $^{\{1\}}$	0.80464 $^{\{7\}}$	0.60249 $^{\{4\}}$	0.59081 $^{\{3\}}$	0.66754 $^{\{5\}}$
	$\sum$ *Ranks*		86 $^{\{7\}}$	40 $^{\{4\}}$	88 $^{\{8\}}$	12 $^{\{1\}}$	75 $^{\{6\}}$	33 $^{\{2\}}$	35 $^{\{3\}}$	63 $^{\{5\}}$
120	BIAS	$\hat{\theta}$	0.36543 $^{\{8\}}$	0.30498 $^{\{4\}}$	0.32443 $^{\{6\}}$	0.10253 $^{\{1\}}$	0.32905 $^{\{7\}}$	0.25448 $^{\{2\}}$	0.28504 $^{\{3\}}$	0.31837 $^{\{5\}}$
		$\hat{\eta}$	0.80066 $^{\{8\}}$	0.66775 $^{\{4\}}$	0.70817 $^{\{7\}}$	0.17114 $^{\{1\}}$	0.70007 $^{\{6\}}$	0.56433 $^{\{2\}}$	0.61883 $^{\{3\}}$	0.69451 $^{\{5\}}$
		$\hat{\gamma}$	0.4252 $^{\{6\}}$	0.38095 $^{\{3\}}$	0.50452 $^{\{8\}}$	0.14084 $^{\{1\}}$	0.49849 $^{\{7\}}$	0.37616 $^{\{2\}}$	0.39316 $^{\{4\}}$	0.42058 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.19626 $^{\{7\}}$	0.13696 $^{\{3\}}$	0.18 $^{\{6\}}$	0.03151 $^{\{1\}}$	0.21863 $^{\{8\}}$	0.12048 $^{\{2\}}$	0.14907 $^{\{4\}}$	0.15568 $^{\{5\}}$
		$\hat{\eta}$	0.87657 $^{\{8\}}$	0.65658 $^{\{4\}}$	0.7984 $^{\{6\}}$	0.16958 $^{\{1\}}$	0.81827 $^{\{7\}}$	0.58574 $^{\{2\}}$	0.64986 $^{\{3\}}$	0.71572 $^{\{5\}}$
		$\hat{\gamma}$	0.48507 $^{\{4\}}$	0.36817 $^{\{2\}}$	0.82662 $^{\{8\}}$	0.13199 $^{\{1\}}$	0.79365 $^{\{7\}}$	0.45962 $^{\{3\}}$	0.503 $^{\{5\}}$	0.50844 $^{\{6\}}$
	MRE	$\hat{\theta}$	0.29235 $^{\{8\}}$	0.24399 $^{\{4\}}$	0.25955 $^{\{6\}}$	0.08203 $^{\{1\}}$	0.26324 $^{\{7\}}$	0.20359 $^{\{2\}}$	0.22803 $^{\{3\}}$	0.25469 $^{\{5\}}$
		$\hat{\eta}$	0.40033 $^{\{8\}}$	0.33387 $^{\{4\}}$	0.35408 $^{\{7\}}$	0.08557 $^{\{1\}}$	0.35004 $^{\{6\}}$	0.28217 $^{\{2\}}$	0.30942 $^{\{3\}}$	0.34725 $^{\{5\}}$
		$\hat{\gamma}$	0.5315 $^{\{6\}}$	0.47618 $^{\{3\}}$	0.63065 $^{\{8\}}$	0.17605 $^{\{1\}}$	0.62312 $^{\{7\}}$	0.4702 $^{\{2\}}$	0.49145 $^{\{4\}}$	0.52572 $^{\{5\}}$
	$\sum$ *Ranks*		63 $^{\{8\}}$	31 $^{\{3\}}$	62 $^{\{6.5\}}$	9 $^{\{1\}}$	62 $^{\{6.5\}}$	19 $^{\{2\}}$	32 $^{\{4\}}$	46 $^{\{5\}}$
150	BIAS	$\hat{\theta}$	0.34759 $^{\{8\}}$	0.29747 $^{\{5\}}$	0.30556 $^{\{6\}}$	0.08035 $^{\{1\}}$	0.3074 $^{\{7\}}$	0.23894 $^{\{2\}}$	0.28268 $^{\{3\}}$	0.29588 $^{\{4\}}$
		$\hat{\eta}$	0.7628 $^{\{8\}}$	0.64963 $^{\{4\}}$	0.67065 $^{\{7\}}$	0.11781 $^{\{1\}}$	0.66732 $^{\{6\}}$	0.52664 $^{\{2\}}$	0.6217 $^{\{3\}}$	0.65308 $^{\{5\}}$
		$\hat{\gamma}$	0.36704 $^{\{5\}}$	0.35227 $^{\{3\}}$	0.45519 $^{\{8\}}$	0.10041 $^{\{1\}}$	0.43315 $^{\{7\}}$	0.31472 $^{\{2\}}$	0.38786 $^{\{6\}}$	0.36196 $^{\{4\}}$
	MSE	$\hat{\theta}$	0.1735 $^{\{7\}}$	0.14171 $^{\{5\}}$	0.1579 $^{\{6\}}$	0.02167 $^{\{1\}}$	0.17885 $^{\{8\}}$	0.10172 $^{\{2\}}$	0.13341 $^{\{4\}}$	0.12895 $^{\{3\}}$
		$\hat{\eta}$	0.78893 $^{\{8\}}$	0.62741 $^{\{4\}}$	0.71482 $^{\{6\}}$	0.10999 $^{\{1\}}$	0.73616 $^{\{7\}}$	0.50351 $^{\{2\}}$	0.62946 $^{\{5\}}$	0.62387 $^{\{3\}}$
		$\hat{\gamma}$	0.3326 $^{\{4\}}$	0.29051 $^{\{3\}}$	0.64508 $^{\{8\}}$	0.07271 $^{\{1\}}$	0.56005 $^{\{7\}}$	0.28359 $^{\{2\}}$	0.43527 $^{\{6\}}$	0.33707 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.27807 $^{\{8\}}$	0.23798 $^{\{5\}}$	0.24445 $^{\{6\}}$	0.06428 $^{\{1\}}$	0.24592 $^{\{7\}}$	0.19115 $^{\{2\}}$	0.22615 $^{\{3\}}$	0.2367 $^{\{4\}}$
		$\hat{\eta}$	0.3814 $^{\{8\}}$	0.32482 $^{\{4\}}$	0.33532 $^{\{7\}}$	0.05891 $^{\{1\}}$	0.33366 $^{\{6\}}$	0.26332 $^{\{2\}}$	0.31085 $^{\{3\}}$	0.32654 $^{\{5\}}$
		$\hat{\gamma}$	0.4588 $^{\{5\}}$	0.44033 $^{\{3\}}$	0.56898 $^{\{8\}}$	0.12551 $^{\{1\}}$	0.54144 $^{\{7\}}$	0.3934 $^{\{2\}}$	0.48483 $^{\{6\}}$	0.45245 $^{\{4\}}$
	$\sum$ *Ranks*		61 $^{\{6\}}$	36 $^{\{3\}}$	62 $^{\{7.5\}}$	9 $^{\{1\}}$	62 $^{\{7.5\}}$	18 $^{\{2\}}$	39 $^{\{5\}}$	37 $^{\{4\}}$
200	BIAS	$\hat{\theta}$	0.32002 $^{\{8\}}$	0.28558 $^{\{4\}}$	0.29482 $^{\{7\}}$	0.06218 $^{\{1\}}$	0.29001 $^{\{6\}}$	0.22335 $^{\{2\}}$	0.26481 $^{\{3\}}$	0.28728 $^{\{5\}}$
		$\hat{\eta}$	0.70028 $^{\{8\}}$	0.62519 $^{\{4\}}$	0.66399 $^{\{7\}}$	0.08419 $^{\{1\}}$	0.63331 $^{\{5\}}$	0.49384 $^{\{2\}}$	0.58085 $^{\{3\}}$	0.64084 $^{\{6\}}$
		$\hat{\gamma}$	0.3067 $^{\{3\}}$	0.31299 $^{\{4\}}$	0.40859 $^{\{8\}}$	0.07437 $^{\{1\}}$	0.37746 $^{\{7\}}$	0.26875 $^{\{2\}}$	0.32223 $^{\{5\}}$	0.33293 $^{\{6\}}$
	MSE	$\hat{\theta}$	0.15029 $^{\{8\}}$	0.11857 $^{\{5\}}$	0.1297 $^{\{6\}}$	0.01464 $^{\{1\}}$	0.14539 $^{\{7\}}$	0.08438 $^{\{2\}}$	0.11385 $^{\{3\}}$	0.11412 $^{\{4\}}$
		$\hat{\eta}$	0.67868 $^{\{8\}}$	0.56199 $^{\{4\}}$	0.66555 $^{\{7\}}$	0.07373 $^{\{1\}}$	0.63331 $^{\{6\}}$	0.4357 $^{\{2\}}$	0.5483 $^{\{3\}}$	0.58493 $^{\{5\}}$
		$\hat{\gamma}$	0.19246 $^{\{3\}}$	0.2228 $^{\{4\}}$	0.43515 $^{\{8\}}$	0.04769 $^{\{1\}}$	0.39807 $^{\{7\}}$	0.18866 $^{\{2\}}$	0.25684 $^{\{6\}}$	0.25595 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.25601 $^{\{8\}}$	0.22847 $^{\{4\}}$	0.23585 $^{\{7\}}$	0.04975 $^{\{1\}}$	0.23201 $^{\{6\}}$	0.17868 $^{\{2\}}$	0.21185 $^{\{3\}}$	0.22982 $^{\{5\}}$
		$\hat{\eta}$	0.35014 $^{\{8\}}$	0.3126 $^{\{4\}}$	0.33199 $^{\{7\}}$	0.0421 $^{\{1\}}$	0.31666 $^{\{5\}}$	0.24692 $^{\{2\}}$	0.29042 $^{\{3\}}$	0.32042 $^{\{6\}}$
		$\hat{\gamma}$	0.38338 $^{\{3\}}$	0.39124 $^{\{4\}}$	0.51074 $^{\{8\}}$	0.09297 $^{\{1\}}$	0.47182 $^{\{7\}}$	0.33594 $^{\{2\}}$	0.40278 $^{\{5\}}$	0.41617 $^{\{6\}}$
	$\sum$ *Ranks*		57 $^{\{7\}}$	37 $^{\{4\}}$	65 $^{\{8\}}$	9 $^{\{1\}}$	56 $^{\{6\}}$	18 $^{\{2\}}$	34 $^{\{3\}}$	48 $^{\{5\}}$
300	BIAS	$\hat{\theta}$	0.29699 $^{\{8\}}$	0.26199 $^{\{4\}}$	0.2688 $^{\{6\}}$	0.04232 $^{\{1\}}$	0.26771 $^{\{5\}}$	0.21566 $^{\{2\}}$	0.24795 $^{\{3\}}$	0.26958 $^{\{7\}}$
		$\hat{\eta}$	0.65175 $^{\{8\}}$	0.58076 $^{\{4\}}$	0.60399 $^{\{7\}}$	0.04435 $^{\{1\}}$	0.59782 $^{\{5\}}$	0.47631 $^{\{2\}}$	0.5502 $^{\{3\}}$	0.59953 $^{\{6\}}$
		$\hat{\gamma}$	0.25804 $^{\{3\}}$	0.26344 $^{\{4\}}$	0.32622 $^{\{8\}}$	0.04274 $^{\{1\}}$	0.32011 $^{\{7\}}$	0.24095 $^{\{2\}}$	0.28362 $^{\{6\}}$	0.27945 $^{\{5\}}$
	MSE	$\hat{\theta}$	0.12406 $^{\{8\}}$	0.0943 $^{\{3\}}$	0.10398 $^{\{7\}}$	0.00649 $^{\{1\}}$	0.1024 $^{\{6\}}$	0.07301 $^{\{2\}}$	0.09493 $^{\{4\}}$	0.09784 $^{\{5\}}$
		$\hat{\eta}$	0.57224 $^{\{8\}}$	0.46866 $^{\{3\}}$	0.54182 $^{\{7\}}$	0.02913 $^{\{1\}}$	0.53205 $^{\{6\}}$	0.38615 $^{\{2\}}$	0.47911 $^{\{4\}}$	0.49628 $^{\{5\}}$
		$\hat{\gamma}$	0.12292 $^{\{2\}}$	0.14237 $^{\{4\}}$	0.26638 $^{\{8\}}$	0.01578 $^{\{1\}}$	0.24219 $^{\{7\}}$	0.13786 $^{\{3\}}$	0.1907 $^{\{6\}}$	0.16248 $^{\{5\}}$
	MRE	$\hat{\theta}$	0.23759 $^{\{8\}}$	0.20959 $^{\{4\}}$	0.21504 $^{\{6\}}$	0.03386 $^{\{1\}}$	0.21417 $^{\{5\}}$	0.17253 $^{\{2\}}$	0.19836 $^{\{3\}}$	0.21566 $^{\{7\}}$
		$\hat{\eta}$	0.32588 $^{\{8\}}$	0.29038 $^{\{4\}}$	0.302 $^{\{7\}}$	0.02218 $^{\{1\}}$	0.29891 $^{\{5\}}$	0.23816 $^{\{2\}}$	0.2751 $^{\{3\}}$	0.29976 $^{\{6\}}$
		$\hat{\gamma}$	0.32255 $^{\{3\}}$	0.32931 $^{\{4\}}$	0.40778 $^{\{8\}}$	0.05343 $^{\{1\}}$	0.40013 $^{\{7\}}$	0.30119 $^{\{2\}}$	0.35453 $^{\{6\}}$	0.34931 $^{\{5\}}$
	$\sum$ *Ranks*		56 $^{\{7\}}$	34 $^{\{3\}}$	64 $^{\{8\}}$	9 $^{\{1\}}$	53 $^{\{6\}}$	19 $^{\{2\}}$	38 $^{\{4\}}$	51 $^{\{5\}}$

Table 9. Partial and overall ranks of all the methods of estimation of proposed distribution by various values of model parameters.

Parameter	n	EM_1	EM_2	EM_3	EM_4	EM_5	EM_6	EM_7	EM_8
$\theta = 0.25,\ \eta = 0.5,\ \gamma = 0.75$	75	4.0	3.0	8.0	1.0	7.0	6.0	2.0	5.0
	120	5.0	3.0	8.0	1.0	7.0	4.0	2.0	6.0
	150	4.5	3.0	8.0	1.0	7.0	4.5	2.0	6.0
	200	4.5	3.0	8.0	1.0	7.0	4.5	2.0	6.0
	300	3.0	5.0	7.0	1.0	8.0	4.0	2.0	6.0
$\theta = 0.75,\ \eta = 1.5,\ \gamma = 0.25$	75	1.0	2.0	6.5	5.0	6.5	8.0	3.0	4.0
	120	1.0	2.0	5.5	5.5	7.0	8.0	3.5	3.5
	150	1.0	4.0	7.0	5.0	6.0	8.0	2.0	3.0
	200	1.0	3.0	6.0	5.0	7.0	8.0	2.0	4.0
	300	1.0	2.0	7.0	5.0	6.0	8.0	4.0	3.0
$\theta = 1.5,\ \eta = 0.25,\ \gamma = 0.50$	75	2.0	3.0	7.0	1.0	5.0	8.0	6.0	4.0
	120	2.0	3.0	7.0	1.0	6.0	8.0	5.0	4.0
	150	1.0	3.0	7.0	2.0	6.0	8.0	5.0	4.0
	200	1.5	3.0	6.0	1.5	7.0	8.0	5.0	4.0
	300	5.0	1.5	6.0	3.0	7.0	8.0	4.0	1.5
$\theta = 0.5,\ \eta = 0.75,\ \gamma = 1.5$	75	5.0	2.0	8.0	1.0	7.0	3.0	6.0	4.0
	120	6.0	2.0	8.0	1.0	7.0	4.0	3.0	5.0
	150	3.0	2.0	8.0	1.0	7.0	5.0	4.0	6.0
	200	2.0	3.0	8.0	1.0	7.0	5.0	6.0	4.0
	300	2.0	4.0	7.0	1.0	8.0	6.0	5.0	3.0
$\theta = 1.5,\ \eta = 1.5,\ \gamma = 1.5$	75	5.0	3.0	8.0	1.0	6.0	2.0	7.0	4.0
	120	2.0	3.0	8.0	1.0	7.0	4.0	6.0	5.0
	150	2.0	3.0	8.0	1.0	7.0	4.0	6.0	5.0
	200	2.0	3.0	7.5	1.0	6.0	4.0	7.5	5.0
	300	2.0	3.0	8.0	1.0	6.0	4.0	7.0	5.0
$\theta = 1.25,\ \eta = 2.0,\ \gamma = 0.8$	75	7.0	4.0	8.0	1.0	6.0	2.0	3.0	5.0
	120	8.0	3.0	6.5	1.0	6.5	2.0	4.0	5.0
	150	6.0	3.0	7.5	1.0	7.5	2.0	5.0	4.0
	200	7.0	4.0	8.0	1.0	6.0	2.0	3.0	5.0
	300	7.0	3.0	8.0	1.0	6.0	2.0	4.0	5.0
$\sum$ Ranks		103.5	88.5	220.5	54.0	199.5	154.0	126.0	134.0
Overall Rank		3	2	8	1	7	6	4	5

6. Modeling to Engineering Data

In this section, we explore two real data sets in the engineering field to demonstrate the significance and capability of the HLMKEx model. The first data set has 100 observations of carbon fiber breaking stress (in Gba) [36]:

3.7	3.11	4.42	3.28	3.75	2.96	3.39	3.31	3.15	2.81	1.41	2.76	3.19	1.59	2.17
3.51	1.84	1.61	1.57	1.89	2.74	3.27	2.41	3.09	2.43	2.53	2.81	3.31	2.35	2.77
2.68	4.91	1.57	0.65	2.12	1.8	0.85	4.38	2	1.18	1.71	1.17	2.17	0.39	2.79
1.08	2.73	0.98	1.73	1.59	1.92	2.38	3.56	2.55	3.22	3.39	4.9	1.69	3.11	3.6
2.05	1.61	2.03	2.48	1.25	2.48	1.12	2.88	2.87	3.19	1.87	2.95	2.67	4.2	2.85
2.55	2.17	2.97	3.68	0.81	1.22	5.08	1.69	3.68	4.70	2.03	2.82	2.50	3.22	3.15
1.47	5.56	1.84	1.36	2.59	2.83	2.56	3.33	2.93	2.97					

Ref. [37] investigated the second batch of data, which comprises gauge lengths of 20 mm from a sample of 74 observations. The data set is as follows:

3.585	3.585	3.433	3.233	3.128	3.096	3.09	3.084	3.067	3.012	2.954	2.88
2.848	2.821	2.818	2.809	2.88	2.848	2.821	2.818	2.809	2.8	2.773	2.77
2.726	2.697	2.684	2.648	2.642	2.633	2.629	2.586	2.57	2.566	2.554	2.535
2.514	2.511	2.49	2.478	2.435	2.434	2.426	2.382	2.382	2.359	2.301	2.301
2.274	2.272	2.27	2.253	2.24	2.224	2.179	2.14	2.098	2.063	2.055	2.027
2.021	2.006	1.997	1.966	1.958	1.944	1.865	1.861	1.803	1.7	1.552	1.479
1.314	1.312										

The descriptive analyses of all data sets are reported in Table 10.

Table 10. Some descriptive analyses of all data sets.

	n	**Mean**	**Median**	**Skewness**	**Kurtosis**	**Range**	**Minimum**	**Maximum**	**Sum**
data1	100	2.5814	2.63	0.3893	0.25467	5.17	0.39	5.56	258.14
data2	74	2.4773	2.5125	−0.1574	0.03345	2.273	1.312	3.585	183.318

Further, we shall compare the fits of the HLMKEx distribution with other models: MKEx, exponentiated exponential (EEx) [38], beta exponential (BEx) [39] and alpha power exponential (APEx) [40] models. The PDFs of these models are provided by the following:

$$\text{EEx:}\ \ f(z;\alpha,\lambda) = \alpha\lambda e^{-\lambda z}(1-e^{-\lambda z})^{(\alpha-1)},\ \ z,\alpha,\lambda > 0,$$

$$\text{BEx:}\ \ f(z;a,b,\lambda) = \frac{\lambda}{B(a,b)} e^{-b\lambda z}(1-e^{-\lambda z})^{(a-1)},\ \ z,a,b,\lambda > 0,$$

and

$$\text{APEx:}\ \ f(z;\alpha,\lambda) = \frac{log(\alpha)\lambda e^{-\lambda z}}{\alpha-1}\alpha^{1-e^{-\lambda z}},\ \ z,\alpha,\lambda > 0.$$

Cramér–von Mises (WST), Anderson–Darling (AST), and Kolmogorov–Smirnov (KOS) statistics with p-values (KOSP) are used to assess the fit of these statistical models.

Tables 11 and 12 provide the ML estimates (MLEs) of the model parameters and their standard errors (SEs), WST, AST, KOS, and KOSP statistics for the fitted HLMKEx, MKEx, BEx, EEx, and APEx models for the two data sets, respectively. The values in these tables indicate that the HLMKEx distribution has the lowest values of WST, AST, KOS, and largest KOSP, among all fitted models. The fitted PDF, CDF, and P-P plots for all models based on the two data sets are displayed in Figures 4–7, respectively.

Table 11. The WST, AST, KOS, KOSP, MLEs, and SEs for breaking stress of carbon fibers data.

Distribution	**WST**	**AST**	**KOS**	**KOSP**		**Estimates (SEs)**
HLMKEx	0.09202	0.51764	0.0673	0.7552	γ	1.637 (0.652)
					η	1.718 (1.0059)
					θ	0.2423 (0.0542)
MKEx	0.17398	0.61251	0.08574	0.4542	γ	1.966 (0.1548)
					θ	0.2334 (0.00887)
EEx	0.226	0.585	0.1077	0.1962	α	7.7882 (1.496)
					λ	1.0132 (0.087)
BEx	0.148	0.758	0.0935	0.3461	a	5.9605 (0.821)
					b	34.5462 (61.141)
					λ	0.0615 (0.102)
APEx	0.184	0.939	0.0954	0.3228	α	19,134.8 (8983)
					λ	1.0777 (0.0509)

Table 12. The WST, AST, KOS, KOSP, MLEs, and SEs for gauge lengths data.

Distribution	WST	AST	KOS	KOSP		Estimates (SEs)
HLMKEx	0.03002	0.22883	0.0602	0.9512	γ η θ	3.652 (1.343) 1.577 (0.908) 0.259 (0.0261)
MKEx	0.03215	0.30334	0.0753	0.7954	γ θ	4.152 (0.371) 0.258 (0.0055)
EEx	0.2172	1.4053	0.0953	0.5121	α λ	89.435 (32.476) 2.0192 (0.1716)
BEx	0.0874	0.5738	0.0682	0.8809	a b λ	24.317 (3.9884) 92.491 (154.90) 0.0947 (0.1426)
APEx	0.1153	0.7486	0.1924	0.0083	α λ	1,592,046 (16777) 1.2536 (0.0549)

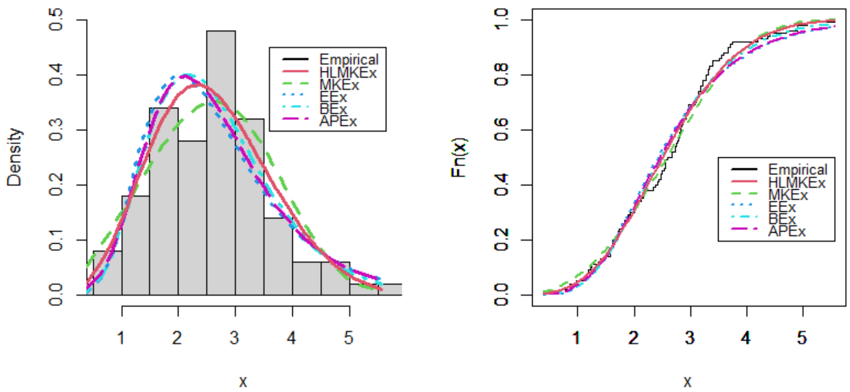

Figure 4. Estimated PDF and CDF plots of competitive models for breaking stress of carbon fibers data.

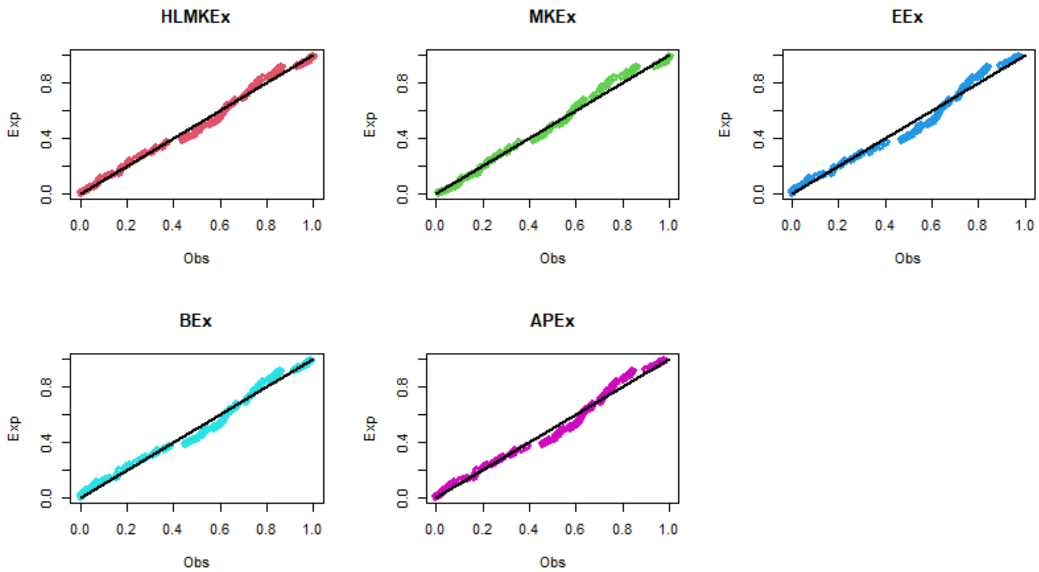

Figure 5. The P–P plots of the fitted models for breaking stress of carbon fibers data.

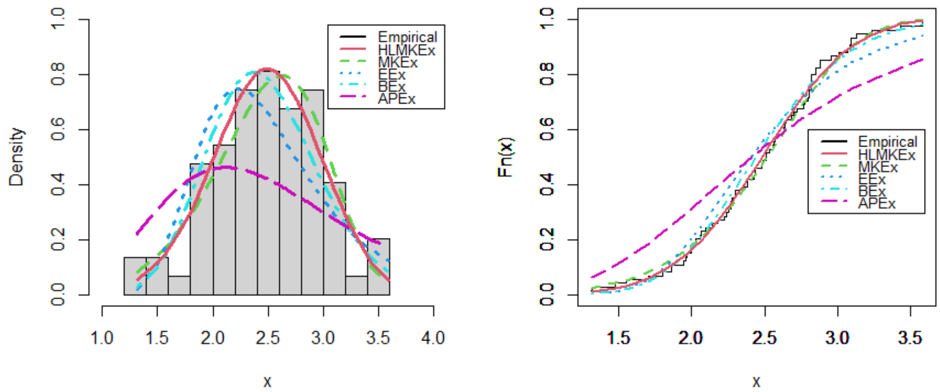

Figure 6. Estimated PDF and CDF plots of competitive models for gauge lengths data.

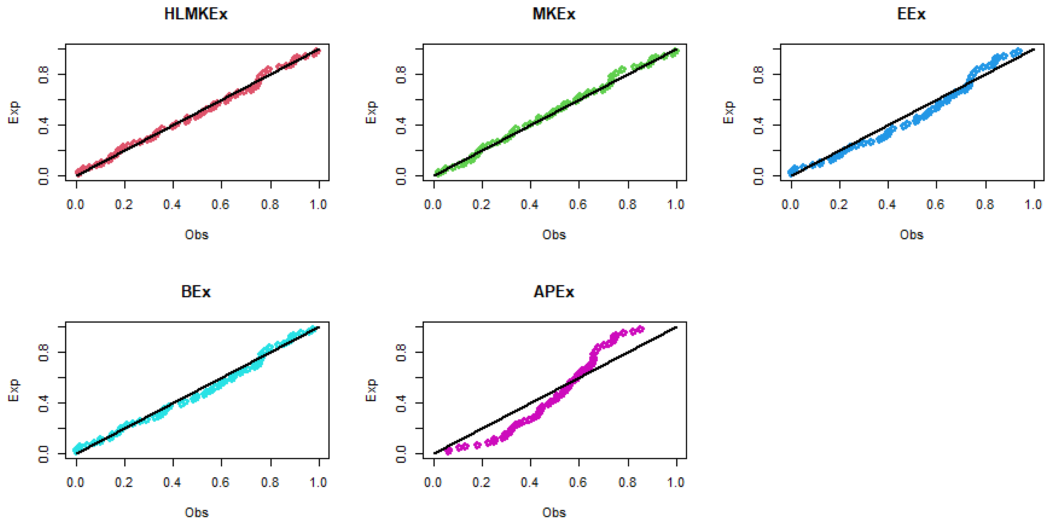

Figure 7. The P-P plots of the fitted models for gauge lengths data.

Different methods of estimation were used to determine estimated parameters WST, AST, KOS, and KOSP for the proposed model, which are presented in Tables 13 and 14 for the two real data sets, respectively. Figures 8 and 9 show P-P plots for the proposed model generated by different estimating approaches, as well as the estimated PDFs generated by these methods.

Table 13. Estimated parameters with goodness-of-fit measures by different estimation methods for the breaking stress of carbon fibers data set.

Method	$\hat{\theta}$	$\hat{\eta}$	$\hat{\gamma}$	AST	WST	KOS	KOSP
EM_1	0.2423	1.718	1.637	0.51764	0.09202	0.0673	0.7552
EM_2	1.99373	12.8273	0.182177	0.4415	0.0749703	0.0618446	0.838957
EM_3	2.39286	13.0523	0.152564	0.439598	0.0731711	0.0604884	0.857751
EM_4	1.93992	12.2451	0.182529	0.491826	0.0815856	0.0715943	0.684557
EM_5	1.56649	12.0819	0.226449	0.470536	0.080918	0.0678114	0.747293
EM_6	0.267448	2.1967	1.37191	0.541278	0.0993827	0.0733148	0.655578
EM_7	0.28814	2.67928	1.30599	0.64157	0.094863	0.0632017	0.81927
EM_8	0.199022	0.780023	3.04153	0.668256	0.0599927	0.0562461	0.909759

Table 14. Estimated parameters with goodness-of-fit measures by different estimation methods for the gauge lengths real data set.

Method	$\hat{\theta}$	$\hat{\eta}$	$\hat{\gamma}$	AST	WST	KOS	KOSP
EM_1	0.259	1.577	3.652	0.22883	0.03002	0.0602	0.9512
EM_2	2.57578	39.7856	0.20216	0.527451	0.067717	0.0743585	0.807842
EM_3	26.1644	45.1059	0.0203112	0.536762	0.0566676	0.0709002	0.8509
EM_4	0.688956	19.3123	0.721908	0.772891	0.117443	0.0740044	0.812428
EM_5	25.6194	43.215	0.0205635	0.524675	0.0575394	0.0748339	0.801631
EM_6	0.251169	1.33358	3.92256	0.258384	0.0337077	0.0694041	0.868182
EM_7	0.266217	1.87355	3.38647	0.249827	0.0303681	0.0555073	0.97655
EM_8	21.2889	44.3588	0.0250144	0.506376	0.0618432	0.0643198	0.919464

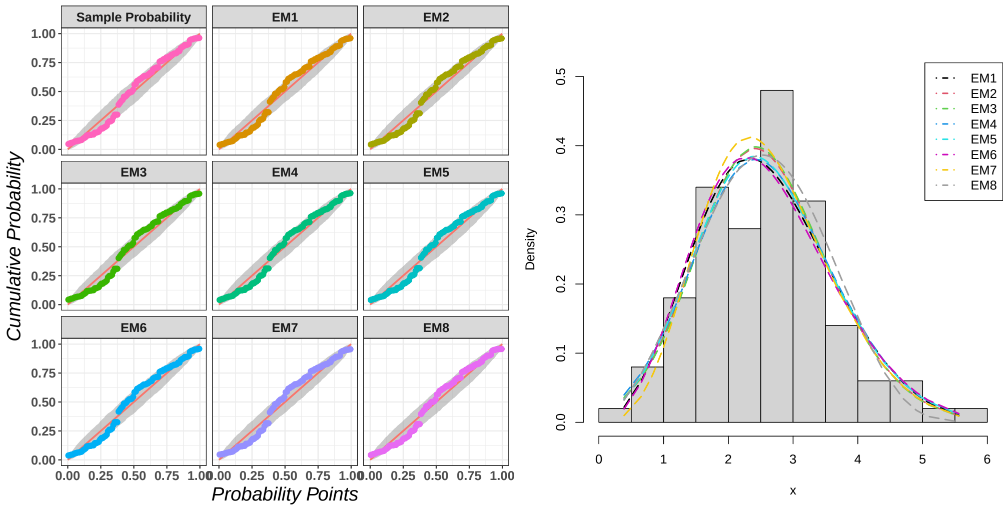

Figure 8. The P–P plots and the fitted PDFs of the proposed model for the first real data.

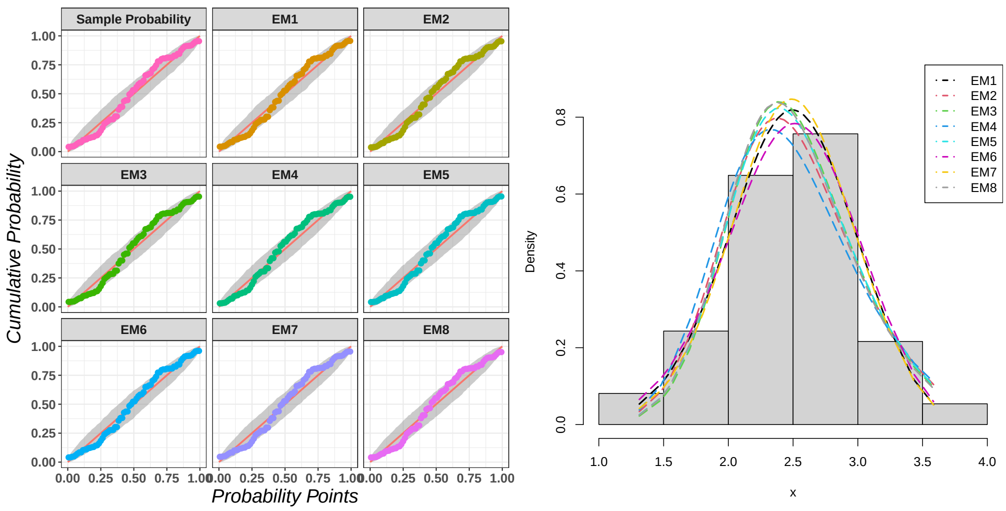

Figure 9. The P–P plots and the fitted PDFs of the proposed model for the second real data.

7. Conclusions

In the current work, a novel three-parameter model called the half-logistic modified Kies exponential distribution is introduced to enhance the adaptability of the modified Kies exponential distribution in modeling engineering data. The HLMKEx distribution's adaptability allows it to offer symmetrical, asymmetrical, unimodal, and reversed-J-shaped densities as well as increasing, reversed-J-shaped, and upside-down hazard rates forms. The quantile function, moments, incomplete moments, and residual moments are all obtained, which are the basic mathematical properties of the HLMKEx model. Furthermore, certain measures of uncertainty and stochastic ordering are provided. Eight estimating techniques are employed to estimate its parameters. We assess the performance of each estimating technique through a simulation study, and we derive partial and overall ranks for some accuracy measures. The simulation results show that the proposed distribution can indeed match the data more precisely than other competing distributions. To show how adaptable and practical the suggested distribution is, two real data sets from the engineering industry are studied. The results show that the proposed distribution can indeed fit the data more accurately than other competing distributions.

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