Thailand Statistician
April 2023; 21(2): 314-336
http://statassoc.or.th
Contributed paper

Power Quasi Lindley Power Series Class of Distributions: Theory and Applications

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Received: 27 May 2020
Revised: 15 April 2021
Accepted: 27 April 2021

Abstract

This article introduces a new class of lifetime distributions called the power quasi Lindley power series (PQLPS) which generalizes the Lindley power series class proposed by Warahena-Liyanage and Pararai (2015). This new class is obtained by compounding the power quasi Lindley and truncated power series distributions. The new class contains some new distributions such as power quasi Lindley geometric distribution, power quasi Lindley Poisson distribution, power quasi Lindley logarithmic distribution, power quasi Lindley binomial distribution and quasi-Lindley power series class of distributions. Some former works such as quasi Lindley geometric, Lindley geometric and Lindley Poisson distributions are special cases of the new compound class. Properties of the PQLPS class are studied, among them; quantile function, order statistics, moments and entropy. Some special models in the PQLPS class are provided. Maximum likelihood, least squares and weighted least squares methods are used to obtain parameter estimators of the PQLPS class. We assess and compare the performance of different parameter estimators of the power quasi Lindley Poisson model supported by a detailed simulation study. Additionally, the log-location-scale regression model based on a special member of the family is introduced. Two real data sets are employed to validate the distributions and the results demonstrate that the sub-models from the class can be considered as suitable models under several real situations.

Keywords: Hazard function, power quasi Lindley distribution, power series distribution, order statistics.

1. Introduction

In many practical situations, most of the classical distributions do not produce a good fit for real data. To overcome these difficulties, various distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions with discrete ones. Compounding lifetime distributions have been obtained by mixing up the distribution when the lifetime can be expressed as the minimum of a sequence of independent and identically distributed (i.i.d.) random variables with a discrete random variable. This idea was first pioneered by Adamidis and Loukas (1998). They introduced the exponential geometric distribution by compounding the exponential with geometric random variables. Several authors introduced new lifetime distributions (see for example; Kuş 2007, Barreto-Souza et al. 2011, Lu and Shi 2012, Hassan and Abdelghafar 2017, and Hassan and Nassr 2018).

In recent years, a great effort has been made to define new compounding classes by mixing some useful lifetime and power series (PS) distributions. The new families generalize some compound distributions and yield more flexibility in modeling several practical data. Some authors defined new compound families (see for example; exponential-PS (Chahkandi and Ganjali 2009) , Weibull-PS (Morais and Barreto-Souza 2011), generalized exponential PS (Mahmoudi and Jafari 2012), extended Weibull-PS (Silva et al. 2013), Burr III Poisson (Hassan et al. 2015a), Burr XII-PS (Silva and Corderio 2015), Lindley-PS (Warahena-Liyanage and Pararai 2015), complementary Poisson Lindley (Hassan et al. 2015b) complementary exponentiated inverted Weibull-PS (Hassan et al. 2016a) generalized inverse Weibull-PS (Hassan et al. 2016b), and power function-PS (Hassan and Assar 2019).

The Lindley distribution was proposed by Lindley (1958) in the context of Bayes' theorem as a counter example of fiducial statistics with the next probability density function (pdf):

$$g(x;\beta)=\frac{\beta^2}{\beta+1}(1+x)\;e^{-\beta x},\;\; x,\beta>0.$$

A detailed study about its important mathematical and statistical properties, estimation of parameter and application showing the superiority of Lindley distribution over exponential distribution for the waiting times before service of the bank customers has been done by Ghitany et al. (2008).

A two-parameter quasi Lindley (QL) distribution has been proposed by Shanker and Mishra (2013) with the following pdf:

$$g(x;\beta,\lambda)=\frac{\lambda}{\beta+1}(\beta+\lambda x)\;e^{-\lambda x},\;\; x,\lambda>0,\beta>-1.$$

Ghitany et al. (2013) proposed the power Lindley (PL) distribution as an extension of the Lindley (L) distribution. Using the transformation $X=Y^{1/\alpha}$, they derived and studied the PL distribution with pdf given by

$$g(x;\alpha,\beta)=\frac{\alpha\beta^2}{\beta+1}(1+x^{\alpha})\;x^{\alpha-1}e^{-\beta x^{\alpha}},\;\; x,\alpha,\beta>0.$$

Alkarni (2015) introduced a recent class of distributions called power quasi Lindley (PQL), which includes several distributions such as Lindley, PL, quasi Lindley, gamma, and generalized gamma. The pdf of the PQL distribution is given by

$$g(x;\lambda,\beta,\alpha)=\frac{\alpha\lambda}{\beta+1}(\beta+\lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}},\;\; x,\lambda,\alpha,\beta>0. \tag{1}$$

The cumulative distribution function (cdf) corresponding to (1) is given by

$$G(x;\lambda,\alpha,\beta)=1-(1+\frac{\lambda x^{\alpha}}{\beta+1})\;e^{-\lambda x^{\alpha}}\;\; x,\lambda,\alpha,\beta>0. \tag{2}$$

Also, a discrete random variable, Z is a member of PS distributions (truncated at zero) with probability mass function (pmf) given by

$$P(Z=z;\theta)=\frac{a_z\theta^z}{C(\theta)},\ z=1,2,3...,\tag{3}$$

where θ is the scale parameter. The coefficients a_z 's depend only on z, $C(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$, $C'(.)$ and $C''(.)$ denote the first and second derivatives of $C(\theta)$, respectively. The expression "PS distribution" is generally credited to Noack (1950). This class includes binomial, Poisson, geometric, negative binomial, and logarithmic distributions. Useful quantities, such as a_z, $C(\theta)$ is the first derivative of $C(\theta)$ and its inverse, for the mentioned distributions truncated for zero are presented in Table 1.

Table 1 Useful quantities of some PS distributions

Distributions	a_z	$C(\theta)$	$C'(.)$	$C''(.)$	$(C(\theta))^{-1}$	θ
Poisson	$Z!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\ln(\theta+1)$	$\theta\in(0,\infty)$
Logarithm	Z^{-1}	$-\ln(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(1+\theta)^{-1}$	$\theta\in(0,1)$
Binomial	$\binom{m}{z}$	$(1+\theta)^m-1$	$m(1+\theta)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta-1)^{\frac{1}{m}}-1$	$\theta\in(0,1)$

In this article, a new class of lifetime distributions is introduced by compounding PQL and PS distributions. The pdf as well as, distribution function, survival and hazard rate functions of the proposed class are obtained in Section 2. In the following section, some statistical properties are derived such as quantile, moments, entropy, and order statistics. In Section 4, some special sub-models and some of their statistical properties for two new sub-models; namely, PQL Poisson (PQLP) and PQL geometric (PQLG) are discussed. In Section 5, maximum likelihood (ML), least squares (LS) and weighted least squares (WLS) estimators of the population parameters on the basis of the class are obtained. In Section 6, the simulation study is carried out to examine the convergence of the parameters of the PQLP model and compare the performance of ML, LS and WLS estimates. A lifetime regression is introduced based on PQLP distribution in Section 7. Applications to real data sets are given to show the applicability and potentiality of the proposed class are provided in Section 8. Some final comments in Section 9 conclude the paper.

2. The Power Quasi Lindley Power Series Class

In this section, the power quasi Lindley power series (PQLPS) class is proposed. This new class is derived by compounding the PQL and PS distributions. The distribution function, survival and hazard rate function (hrf) are derived. Furthermore, two important propositions are provided. While in the second proposition we provide useful expansion for the pdf of the PQLPS distribution.

Following the same key idea of Adamidis and Loukas (1998), the pdf of the PQLPS is derived through the following steps:

Step 1: Let $X_1, X_2, ..., X_z$ be i.i.d. random variables having PQL distribution with pdf (1) and cdf (2). Define $X_{(1)} = \min\{X_1, X_2, ..., X_z\}$, where Z be a zero truncated PS distribution with the pmf (3) independent of X 's.

Step 2: Obtain the conditional pdf of $X_1 | Z$ as follows

$$f_{X_{(1)}|Z}(x| z; \lambda, \beta, \alpha) = \frac{z\alpha\lambda(\beta + \lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{\beta+1}\left((1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)^{z-1}.$$

Step 3: Obtain the joint pdf of $X_{(1)}$ and Z as follows

$$f_{X_{(1)}Z}(x, z; \psi) = \frac{z\alpha\lambda a_z\theta^z(\beta + \lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{(\beta+1)C(\theta)}\left((1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)^{z-1},$$

where $\psi \equiv (\alpha, \lambda, \beta, \theta)$ is a set of parameters.

Step 4: The pdf of the PQLPS family is defined as the marginal density of X, that is,

$$f(x;\psi) = \frac{\alpha\lambda\theta(\beta + \lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{(\beta+1)C(\theta)}\sum_{z=1}^{\infty} za_z\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)^{z-1}. \tag{4}$$

But,

$$C'\left(\theta\left((1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)\right) = \sum_{z=1}^{\infty} za_z\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)^{z-1}. \tag{5}$$

Therefore, from (4) and (5), the pdf of the PQLPS family takes the following form

$$f(x;\psi) = \frac{\alpha\lambda\theta(\beta + \lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{(\beta+1)C(\theta)}C'\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right), x, \lambda, \alpha, \theta, \beta > 0. \tag{6}$$

The cdf of PQLPS class corresponding to (6) is obtained as follows

$$F(x;\psi) = 1 - \frac{1}{C(\theta)}C\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right). \tag{7}$$

A random variable X with cdf (7) is denoted by X ~PQLPS $(\alpha, \lambda, \beta, \theta)$. In addition, the survival function and hrf for PQLPS class are given by

$$S(x;\psi) = \frac{1}{C(\theta)}C\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right),$$

and

$$h(x;\psi) = \frac{\alpha\lambda\theta(\beta + \lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}C'\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)}{(\beta+1)C\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)}.$$

Proposition 2.1 *The PQL distribution with parameters α, λ and β is a limiting special case of PQLPS class when $\theta \to 0^+$.*

Proof: Applying $C(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$, for $x>0$ in cdf (7), then we obtain

$$\lim_{\theta\to 0^+}F(x;\psi)=1-\lim_{\theta\to 0^+}\frac{\sum_{z=1}^{\infty}a_z\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)^z}{\sum_{z=1}^{\infty}a_z\theta^z}.$$

Using L'Hospital's rule in previous equation leads to

$$\lim_{\theta\to 0^+}F(x;\psi)=1-\frac{\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\left[1+a_1^{-1}\lim_{\theta\to 0^+}\sum_{z=2}^{\infty}za_z\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})e^{-\lambda x^{\alpha}}\right)^{z-1}\right]}{1+a_1^{-1}\lim_{\theta\to 0^+}\sum_{z=2}^{\infty}za_z\theta^{z-1}}.$$

Hence,

$$\lim_{\theta\to 0^+}F(x;\psi)=1-\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}},$$

which is the cdf of the PQL distribution.

Proposition 2.2 *The pdf of PQLPS class can be expressed as a linear combination of the density of* $X_{(1)}=\min\{X_1,X_2,...X_z\}$.

Proof: Since $C'(\theta)=\sum_{z=1}^{\infty}za_z\theta^{z-1}$, then the pdf (6) can be expressed as follows

$$f(x;\psi)=\sum_{z=1}^{\infty}P(Z=z;\theta)g_{x_{(1)}}(x;z), \tag{8}$$

where $g_{x_{(1)}}(x;z)$ is the pdf of $X_{(1)}=\min\{X_1,X_2,...,X_z\}$ given by

$$g_{X_{(1)}}(x;z)=\frac{z\alpha\lambda(\beta+\lambda x^{\alpha})x^{\alpha-1}(\beta+1+\lambda x^{\alpha})^{z-1}e^{-z\lambda x^{\alpha}}}{(\beta+1)^z}.$$

3. Statistical Properties

Here, some structure properties of the POLPS class including, expansion for pdf (6), quantile function, the r^{th} moment, Rényi entropy and distribution of order statistics are obtained.

3.1. Quantile Function

In current subsection, the quantile function of the PQLPS distribution is derived. The quantile function, denoted by, $Q(p)$, defined by $Q(p)=p$, is the root of the following equation

$$1-\frac{1}{C(\theta)}C\left(\theta(1+\frac{\lambda(Q(p))^{\alpha}}{\beta+1})e^{-\lambda(Q(p))^{\alpha}}\right)=p,\ 0<p<1.$$

Let $D(p)=-(\beta+1+\lambda(Q(p))^{\alpha})$. Then,

$$D(p)e^{D(p)} = -\frac{(\beta+1)C^{-1}\left((1-p)C(\theta)\right)}{\theta e^{\beta+1}}.$$

Then solving for $D(p)$,

$$D(p)e^{D(p)} = W\left[-\frac{(\beta+1)C^{-1}\left((1-p)C(\theta)\right)}{\theta e^{\beta+1}}\right],$$

where $W(.)$ is the negative branch of the Lambert W function (see Corless et al. 1996). Consequently, the quantile function of the PQLPS class is given by solving the following equation for $Q(p)$.

$$(Q(p))^{\alpha} = -\frac{\beta}{\lambda} - \frac{1}{\lambda} - W\left[-\frac{(\beta+1)C^{-1}\left((1-p)C(\theta)\right)}{\theta e^{\beta+1}}\right]. \tag{9}$$

3.2. Moments

The r^{th} moment of a random variable X from the PQLPS distribution, is derived by using pdf (8) as the following

$$\mu_r' = \sum_{z=1}^{\infty} P(Z=z;\theta) \int_0^{\infty} x^r g_{X_{(1)}}(x;z)\,dx.$$

Then,

$$\mu_r' = \sum_{z=1}^{\infty} P(Z=z;\theta) \int_0^{\infty} \frac{z\alpha\lambda x^{r+\alpha-1}(\beta+\lambda x^{\alpha})}{(\beta+1)^z} (\beta+1+\lambda x^{\alpha})^{z-1} e^{-z\lambda x^{\alpha}}\,dx.$$

Let $u = \lambda x^{\alpha} \to du = \alpha\lambda x^{\alpha-1}dx$, then $\mu_r' = \sum_{z=1}^{\infty} \frac{zP(Z=z;\theta)}{(\beta+1)^z} \int_0^{\infty} \left(\frac{u}{\lambda}\right)^{\frac{r}{\alpha}} (\beta+u)(1+\beta+u)^{z-1} e^{-uz}\,du.$

Using binomial series more than one times, then

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1} \binom{z-1}{j}\binom{j+1}{i} \frac{zP(Z=z;\theta)}{(\beta+1)^z} \int_0^{\infty} \left(\frac{u}{\lambda}\right)^{\frac{r}{\alpha}} (\beta)^{j+1} \left(\frac{u}{\beta}\right)^{i} e^{-zu}\,du.$$

After some simplifications, it takes the following form

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1} \binom{z-1}{j}\binom{j+1}{i} \frac{a_z\theta^z(\beta)^{j+1-i}}{(1+\beta)^z C(\theta) z^{\frac{r}{\alpha}+i} \lambda^{\frac{r}{\alpha}}} \Gamma\left(\frac{r}{\alpha}+i+1\right), \quad r = 1,2... \tag{10}$$

It is easy to show that, the moment generating $\Lambda_X(t)$ function is obtained as follows:

$$\Lambda_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1} \binom{z-1}{j}\binom{j+1}{i} \frac{a_z\theta^z t^r(\beta)^{j+1-i}}{r!(\beta+1)^z C(\theta) z^{\frac{r}{\alpha}+i} \lambda^{\frac{r}{\alpha}}} \Gamma\left(\frac{r}{\alpha}+i+1\right), \quad r = 1,2....,$$

where the r^{th} moment is μ_r'.

3.3. Order Statistics

In this subsection, an expression for the pdf of the i^{th} order statistics from the PQLPS distribution is derived. In addition, the distributions of the smallest and largest order statistics are obtained.

Let $X_1, X_2, ..., X_n$ be a simple random sample from a PQLPS class with pdf (6) and cdf (7).

Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denote the corresponding order statistics from the sample. The pdf of $X_{i:n}$, $i = 1, 2, ..., n$ is given by

$$f_{i:n}(x) = \frac{1}{\mathrm{B}(i, n-i+1)} f(x)[F(x)]^{i-1}[1-F(x)]^{n-i}, \tag{11}$$

where $\mathrm{B}(.,.)$ is the beta function. Using cdf (7) and applying the binomial expansion in (11), then we get

$$f_{i:n}(x;\psi) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \left(\frac{1}{C(\theta)} C\left(\theta \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) \right) e^{-\lambda x^\alpha} \right)^{n+j-i}.$$

Now, since an expansion for $\left(C\left(\theta \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) e^{-\lambda x^\alpha} \right) \right)^{n+j-i}$ can be written as follows

$$\begin{aligned} \left(C\left(\theta (1 + \frac{\lambda x^\alpha}{\beta+1}) e^{-\lambda x^\alpha} \right) \right)^{n+j-i} &= \left(\sum_{z=1}^{\infty} a_z \theta^z e^{-z\lambda x^\alpha} \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right)^z \right)^{n+j-i} \\ &= \left(a_1 \theta e^{-\lambda x^\alpha} \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) \right)^{n+j-i} \times \\ &\left[1 + \frac{a_2}{a_1} \theta e^{-\lambda x^\alpha} \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) + \frac{a_3}{a_2} \theta^2 e^{-2\lambda x^\alpha} \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right)^2 + ... \right]^{n+j-i}. \end{aligned}$$

Hence,

$$\begin{aligned} \left(C\left(\theta (1 + \frac{\lambda x^\alpha}{\beta+1}) \right) e^{-\lambda x^\alpha} \right)^{n+j-i} &= \left(a_1 \theta e^{-\lambda x^\alpha} \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) \right)^{n+j-i} \times \\ &\left(\sum_{m=0}^{\infty} \ell_m \theta e^{-\lambda x^\alpha} \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) \right)^{n+j-i}, \ell_m = \frac{a_{m+1}}{a_1}(m+1),\ m = 1, 2, ... \end{aligned} \tag{12}$$

According to Gradshteyn and Ryzhik (2000) for a positive integer, we have the following relation

$$\left(\sum_{m=0}^{\infty} \ell_m Y^m \right)^{n+j-i} = \sum_{m=0}^{\infty} d_{n+j-i,m} Y^m.$$

Then (12) can be written as follows

$$\left(C\left(\theta (1 + \frac{\lambda x^\alpha}{\beta+1}) \right) e^{-\lambda x^\alpha} \right)^{n+j-i} = (a_1)^{n+j-i} \sum_{m=0}^{\infty} d_{n+j-i,m} \left(\theta \left(1 + \frac{\lambda x^\alpha}{\beta+1} \right) e^{-\lambda x^\alpha} \right)^{n+j-i+m},$$

(13)

where $d_{n+j-i,0} = 1$ and the coefficients $d_{n+j-i,m}$ are easily determined from the following recurrence equation

$$d_{n+j-i,t} = t^{-1} \sum_{m=1}^{t} [m(n+j-i+1) - t] \ell_m d_{n+j-i,t-m}, t \geq 1.$$

In addition,

$$C'\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)=\sum_{z=1}^{\infty} z\,a_z\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{z-1}.$$

Let $k = z-1$, then the previous equation can be expressed as

$$C'\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)=\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{k},\ \ell_k=\frac{a_{k+1}}{a_1}. \quad (14)$$

Then, the pdf of the i^{th} order statistic from PQLPS class is obtained by substituting (13) and (14) in pdf (11) as follows

$$f_{i:n}(x;\psi)=\frac{\lambda\alpha\theta(\beta+\lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{\mathrm{B}(i,n-i+j)(\beta+1)(C(\theta))^{n+j-i+1}}\sum_{k=0}^{\infty}\ell_k(k+1)\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{k}$$

$$\times\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j a_1^{n+j-i+1}\sum_{m=0}^{\infty}d_{n+j-i,m}\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{n+j-i+m}.$$

Thus, the pdf of the i^{th} order statistics can be formed as follows

$$f_{i:n}(x;\psi)=\frac{\lambda\alpha(\beta+\lambda x^{\alpha})x^{\alpha-1}}{\mathrm{B}(i,n-i+j)(\beta+1)}\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}(-1)^j\binom{i-1}{j}\frac{\ell_k(k+1)}{(C(\theta))^{n+j-i+1}}$$

$$\times d_{n+j-i,m}a_1^{n+j-i+1}\theta^{n+j-i+m+k+1}e^{-(n+j-i+m+1+k)\lambda x^{\alpha}}\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)^{n+j-i+m+k},\quad x>0.$$

Or it can be written as follows

$$f_{i:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\tau_{j,k,m}(\beta+\lambda x^{\alpha})x^{\alpha-1}\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)^{n+j-i+m+k}e^{-(n+j-i+m+k+1)\lambda x^{\alpha}},$$

where

$$\tau_{j,k,m}=(-1)^j\binom{i-1}{j}\frac{\alpha\lambda\ell_k(k+1)\theta^{n+j-i+m+k+1}a_1^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(\beta+1)(C(\theta))^{n+j-i+1}}.$$

Another form can be written by using binomial expansion as follows:

$$f_{i:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{i-1}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\xi_{j,k,m,h}(\beta+\lambda x^{\alpha})x^{\alpha(h+1)-1}e^{-(n+j-i+m+k+1)\lambda x^{\alpha}}, \quad (15)$$

Where

$$\xi_{j,k,m,h}=(-1)^j\binom{i-1}{j}\binom{m+n+j-i+k}{h}\frac{\alpha\lambda^{h+1}\theta^{n+j-i+m+k+1}\ell_k(k+1)a_1^{n+j-i+1}d_{n+j-i,m}}{\mathrm{B}(i,n-i+j)(\beta+1)^{h+1}(C(\theta))^{n+j-i+1}}.$$

In particular, the pdf of the smallest and the largest order statistics of the PQLPS distribution is obtained by substituting $i=1$ and n, in (15), respectively, as follows

$$f_{1:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{h=0}^{n+j-i+m+k}\eta_{k,m,h}(\beta+\lambda x^{\alpha})x^{\alpha(h+1)-1}e^{-(n+m+k)\lambda x^{\alpha}},$$

$$\eta_{k,m,h}=\binom{m+n+-1+k}{h}\frac{n\alpha\lambda^{h+1}\ell_k(k+1)\theta^{n+m+k}a_1^{n}d_{n-1,m}}{(\beta+1)^{h+1}(C(\theta))^{n}},$$

and

$$f_{n:n}(x;\psi)=\sum_{k=0}^{\infty}\sum_{j=0}^{n-1}\sum_{m=0}^{\infty}\sum_{h=0}^{j+m+k}\varsigma_{j,k,m,h}(\beta+\lambda x^{\alpha})x^{\alpha(h+1)-1}e^{-(j+m+k+1)\lambda x^{\alpha}},$$

where

$$\varsigma_{k,m,h}=\binom{m+j+k}{h}\binom{n-1}{j}(-1)^{j}\frac{n\lambda^{h+1}\alpha\ell_{k}(k+1)\theta^{j+m+k+1}a_{1}^{j+1}d_{j,m}}{(\beta+1)^{h+1}(C(\theta))^{j+1}}.$$

3.4. Rényi Entropy

Entropy has been used in various situations in science and engineering. The entropy of a random variable X is a measure of variation of the uncertainty. If X is a random variable which distributed as PQLPS, then the Rényi entropy, for $\rho>0$, and $\rho\neq 1$, is defined as:

$$I_R(x)=(1-\rho)^{-1}\log_b\left(\int_0^{\infty}\left(f(x;\psi)\right)^{\rho}dx\right).$$

Let $\text{IP}=\int_0^{\infty}\left(f(x;\psi)\right)^{\rho}dx,$ then IP can be written as follows:

$$\text{IP}=\int_0^{\infty}\left(\frac{\alpha\lambda\theta(\beta+\lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{(\beta+1)}\right)^{\rho}\left\{\sum_{z=1}^{\infty}\frac{za_z}{C(\theta)}\left(\theta(1+\frac{\lambda x^{\alpha}}{\beta+1})\,e^{-\lambda x^{\alpha}}\right)^{z-1}\right\}^{\rho}dx.$$

But

$$\left(\sum_{z=1}^{\infty}za_z\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{z-1}\right)^{\rho}=a_1^{\rho}\left(\sum_{m=0}^{\infty}\delta_m\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{m}\right)^{\rho},$$

$$\delta_m=\frac{a_{m+1}}{a_1}(m+1),\ m=1,2,...$$

Using the same rule as provided in Gradshteyn and Ryzhik (2000), then we obtain

$$\left(\sum_{z=1}^{\infty}\delta_m\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{m}\right)^{\rho}=\sum_{m=0}^{\infty}d_{\rho,m}\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{m}.$$

Therefore,

$$\left(\sum_{z=1}^{\infty}za_z\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{z-1}\right)^{\rho}=a_1^{\rho}\sum_{z=1}^{\infty}d_{\rho,m}\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{m}. \quad (16)$$

The coefficients for $t>1$ are computed from the following recurrence equation:

$$d_{\rho,t}=t^{-1}\sum_{m=1}^{t}[m(\rho+1)-t]\delta_m d_{\rho,t-m},\ \text{and}\ d_{\rho,0}=1.$$

Using binomial expansion for $\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)^{m}$, then (16) can be written as follows:

$$\left(\sum_{z=1}^{\infty} z a_z \left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)^{z-1}\right)^{\rho} = a_1^{\rho}\sum_{z=1}^{\infty}\sum_{k=0}^{m}\binom{m}{k} d_{\rho,m}\theta^m e^{-m\lambda x^{\alpha}}\left(\frac{\lambda x^{\alpha}}{\beta+1}\right)^k .$$

Then the IP can be rewritten as follows:

$$\text{IP} = \int_0^{\infty}\left(\frac{\alpha\lambda\theta x^{\alpha-1}\beta a_1}{(\beta+1)C(\theta)}\right)^{\rho}(1+\frac{\lambda x^{\alpha}}{\beta})^{\rho}\sum_{m=0}^{\infty}\sum_{k=0}^{m} d_{\rho,m}\theta^m\binom{m}{k}\left(\frac{\lambda x^{\alpha}}{\beta+1}\right)^k e^{-(m+\rho)\lambda x^{\alpha}}dx$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h} d_{\rho,m}\theta^m\int_0^{\infty}\left(\frac{\alpha\lambda\theta x^{\alpha-1}\beta a_1}{(\beta+1)C(\theta)}\right)^{\rho}\frac{(\lambda x^{\alpha})^{k+h}}{(\beta+1)^k\beta^h}e^{-(m+\rho)\lambda x^{\alpha}} .$$

After some simplification, then the Rényi entropy takes the following form

$$I_R(x) = (1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^{m+\rho}\lambda^{\frac{\rho-1}{\alpha}}\alpha^{\rho-1}a_1^{\rho}\Gamma\left(\frac{\rho(\alpha-1)+k+h+1}{\alpha}\right)}{(\beta+1)^{k+\rho}\beta^{h-\rho}(C(\theta))^{\rho}(m+\rho)^{\frac{\rho(\alpha-1)+k+h+1}{\alpha}}}\right]. \tag{17}$$

4. Special Models of PQLPS

Some sub-models of the PQLPS class for selected values of the parameters are presented in this section. Two sub-models; namely, the power quasi Lindley Poisson and power quasi Lindley geometric distributions are discussed in more details in sub-sections 4.1 and 4.2, respectively.

4.1. Sub-classes and sub models

In this sub-section, we look at the sub-class of distributions of the PQLPS distribution and some sub-models based on cdf (7). The results are summarized in Table 2.

Table 2 Sub-classes and sub-models of the PQLPS

$C(\theta)$	Condition	Sub-Models	Sub-Classes	Authors
–	$\alpha=1$		Quasi Lindley power series	
–	$\alpha=1,\lambda=\beta$		Lindley power series	Warahena-Liyanage and Pararai (2015)
$e^{\theta}-1$	–	PQL Poisson		
$e^{\theta}-1$	$\alpha=1,\lambda=\beta$	Lindley Poisson		Gui et al. (2014)
$e^{\theta}-1$	$\lambda=\beta$	Power Lindley Poisson		
$e^{\theta}-1$	$\alpha=1$	Quasi Lindley Poisson		
$e^{\theta}-1$	$\alpha=1,\lambda=\beta$	Gamma Poisson (GP)		
$e^{\theta}-1$	–	Generalized GP		
$-\ln(1-\theta)$	$\alpha=1,\lambda=\beta$	PQL logarithmic		
$-\ln(1-\theta)$	$\lambda=\beta$	Lindley logarithmic		

$-\ln(1-\theta)$	$\alpha=1$	Power Lindley logarithmic		
$-\ln(1-\theta)$	$\alpha=1,\lambda=\beta$	Quasi Lindley logarithmic		
$-\ln(1-\theta)$	–	Gamma logarithmic (GL)		
$-\ln(1-\theta)$	$\alpha=1,\lambda=\beta$	Generalized GL		
$\theta(1-\theta)^{-1}$	$\lambda=\beta$	PQL geometric		
$\theta(1-\theta)^{-1}$	$\alpha=1$	Lindley geometric		Zakerzadeh and Mahmoudi (2012)
$\theta(1-\theta)^{-1}$	$\alpha=1,\lambda=\beta$	Power Lindley geometric		
$\theta(1-\theta)^{-1}$	$\alpha=1,\lambda=\beta$	Gamma geometric (GG)		
$\theta(1-\theta)^{-1}$	$\lambda=\beta$	Generalized GG		
$(1-\theta)^m-1$	$\alpha=1$	PQL binomial		
$(1-\theta)^m-1$	$\alpha=1,\lambda=\beta$	Lindley binomial		
$(1-\theta)^m-1$	–	Power Lindley binomial		

Table 2 (Continued)

$C(\theta)$	Condition	Sub-Models	Sub-Classes	Authors
$(1-\theta)^m-1$	$\alpha=1,\lambda=\beta$	Quasi Lindley binomial		
$(1-\theta)^m-1$	$\lambda=\beta$	Gamma binomial (GB)		
$(1-\theta)^m-1$	$\alpha=1$	Generalized GB		

4.2. The PQLP Distribution

The PQLP distribution is obtained from PQLPS class distribution as a special case. The cdf and pdf of the PQLP distribution are given by:

$$F(x;\psi)=\frac{1}{e^{\theta}-1}\left\{e^{\theta}-\exp\left[\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right]\right\},\ x,\alpha,\lambda,\beta,\theta>0, \tag{18}$$

and

$$f(x;\psi)=\frac{\alpha\lambda\theta(\beta+\lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}}{(\beta+1)(e^{\theta}-1)}\exp\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right),\ \ x,\lambda,\alpha,\theta,\beta>0. \tag{19}$$

In addition, the survival function and hrf take the following forms respectively:

$$S(x;\psi)=\frac{1}{e^{\theta}-1}\left\{\exp\left[\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right]-1\right\},$$

and
$$h(x;\psi)=\frac{\alpha\lambda\theta(\beta+\lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}\exp\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)^{-\lambda x^{\alpha}}\right)}{(\beta+1)\left[\exp\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)-1\right]}.$$

The pdf plots of the PQLP distribution are uni-modal as represented in Figure 1. The pdf plots are right skewed, reversed J- shaped, increasing and decreasing for various values of the parameters giving the shapes obtained in the below plots. The hrf plots of the PQLP distribution for some selected values of parameters are described in Figure 2. The hazard rate shapes for PQLP distribution are decreasing, increasing and reversed J-shaped for the selected values of parameters.

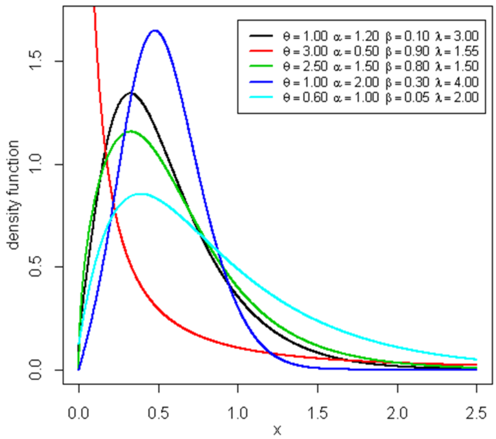

Figure 1 The pdf plots of the PQLP distribution

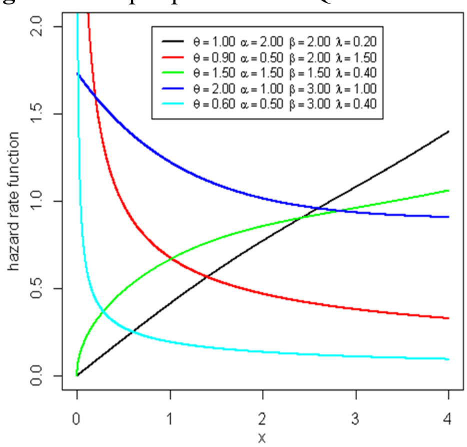

Figure 2 The hrf plots of the PQLP distribution

The quantile function for the PQLP distribution is obtained directly from expression (8) with $C(\theta)=e^{\theta}-1$, and $C^{-1}(\theta)=\ln(1+\theta)$ as follows

$$(Q(p))^{\alpha} = -\frac{\beta}{\lambda} - \frac{1}{\lambda} - W\left[-\frac{(\beta+1)\ln(p+(1-p)e^{\theta})}{\theta e^{\beta+1}}\right].$$

Solving this equation for $Q(p)$, the quantile function of PQLP is obtained. Furthermore, the r^{th} moment of the PQLP distribution about the origin is given by substituting the following pmf of truncated Poisson distribution

$$P(Z = z;\theta) = \frac{e^{-\theta}\theta^{z}}{z!(1-e^{-\theta})},\ z = 1,2,...$$

in (10) as follows

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z}(\beta)^{j+1-i}\Gamma\left(\frac{r}{\alpha}+i+1\right)}{z!(e^{\theta}-1)(\beta+1)^{z} z^{\frac{r}{\alpha}+i}\lambda^{\frac{r}{\alpha}}}, r = 1,2...$$

Additionally the Rényi entropy is obtained by substituting $C(\theta) = e^{\theta}-1,$ in (17) as follows

$$I_R(x) = (1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\lambda^{\frac{\rho-1}{\alpha}}\theta^{m+\rho}\alpha^{\rho-1}a_1^{\rho}\Gamma(\frac{\rho(\alpha-1)+k+h+1}{\alpha})}{(\beta+1)^{k+\rho}\beta^{h-\rho}(e^{\theta}-1)^{\rho}(m+\rho)^{\frac{\rho(\alpha-1)+k+h+1}{\alpha}}}\right].$$

4.3. The PQLG Distribution

The PQLG distribution is discussed as sub-model from PQLPS class. The pdf and cdf of the PQLG distribution are given by

$$f(x;\psi) = \frac{\alpha\lambda(\beta+\lambda x^{\alpha})x^{\alpha-1}e^{-\lambda x^{\alpha}}(1-\theta)}{(\beta+1)\left[1-\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)\right]^{2}},\quad x>0, 0<\theta<1, \alpha,\lambda,\beta>0.$$

and

$$F(x;\psi) = \frac{1-\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}}{1-\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}},$$

In addition, the hrf takes the following form

$$h(x;\psi) = \frac{\alpha\lambda(\beta+\lambda x^{\alpha})x^{\alpha-1}}{(\beta+1)\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)\left[1-\left(\theta\left(1+\frac{\lambda x^{\alpha}}{\beta+1}\right)e^{-\lambda x^{\alpha}}\right)\right]}.$$

Figure 3 represents the pdf plots for PQLG distribution for some selected values of parameters. It is observed from Figure 3 that the density function of PQLG distribution takes different shapes as symmetric, uni-modal, right skewed and reversed J-shaped. Figure 4 presents different shapes of the hazard function for the PQLG distribution. We observe that the hrf of the PQLG distribution is quite flexible. It exhibits decreasing, constant and reversed J-shaped.

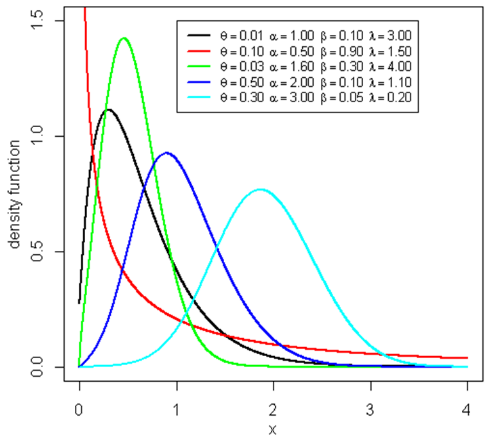

Figure 3 The pdf plots of the PQLG distribution

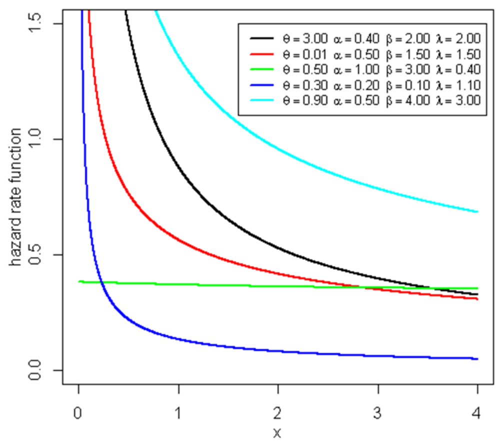

Figure 4 The hrf plots of the PQLG distribution

From this figure, it is observed that the shapes of the hrf are decreasing at some selected values. The quantile function for the PQLG distribution is obtained directly from expression (9) with $C(\theta)=\theta(1-\theta)^{-1}$, and $C^{-1}(\theta)=\theta(1+\theta)^{-1}$ as follows

$$(Q(p))^{\alpha}=-\frac{\beta}{\lambda}-\frac{1}{\lambda}-W\left[-\frac{(1+\beta)(1-p)}{(1-\theta p)e^{\beta+1}}\right].$$

Solving this equation for $Q(p)$, the quantile function PQLG is obtained. Additionally, the r^{th} moment of the PQLG distribution about the origin is given by substituting the following pmf of truncated geometric distribution

$$P(Z=z;\theta)=(1-\theta)\theta^{z-1},\ z=1,2,...$$

in (10) as follows

$$\mu_r' = \sum_{z=1}^{\infty}\sum_{j=0}^{z-1}\sum_{i=0}^{j+1}\binom{z-1}{j}\binom{j+1}{i}\frac{\theta^{z-1}(1-\theta)(\beta)^{j+1-i}\Gamma\left(\frac{r}{\alpha}+i+1\right)}{(1+\beta)^z z^{\frac{r}{\alpha}+i}\lambda^{\frac{r}{\alpha}}},\quad r=1,2,...$$

Further, the Rényi entropy is obtained by substituting $C(\theta)=\theta(1-\theta)^{-1}$ (17) as follows

$$I_R(x)=(1-\rho)^{-1}\log_b\left[\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{h=0}^{\rho}\binom{m}{k}\binom{\rho}{h}\frac{d_{\rho,m}\theta^m\lambda^{\frac{\rho-1}{\alpha}}\alpha^{\rho-1}a_1^{\rho}\Gamma\left(\frac{\rho(\alpha-1)+k+h+1}{\alpha}\right)}{(\beta+1)^{k+\rho}\beta^{h-\rho}(1-\theta)^{-\rho}(m+\rho)^{\frac{\rho(\alpha-1)+k+h+1}{\alpha}}}\right].$$

5. Parameter Estimation of the Class

In this section estimation of the model parameters of PQLPS class of distributions is obtained by using different methods of estimation, namely ML method, least squares method and weighted least squares method.

5.1 Maximum likelihood estimators

Let $X_1, X_2, ..., X_n$ be a simple random sample from the PQLPS class with set of parameters $\psi \equiv (\alpha,\lambda,\beta,\theta)$. The log-likelihood function, denoted by $\ln\ell$ based on the observed random sample of size n is given by

$$\ln\ell = n\ln\lambda + n\ln\alpha + n\ln\theta - n\ln(\beta+1) + \sum_{i=1}^{n}\ln(\beta+\lambda x_i^{\alpha})$$
$$-\lambda\sum_{i=1}^{n}x_i^{\alpha} + (\alpha-1)\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\ln\left(C'(\theta M_i)\right) - n\ln(C(\theta)),$$

where $M_i = \left(1+\frac{\lambda x_i^{\alpha}}{\beta+1}\right)e^{-\lambda x_i^{\alpha}}$. The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by

$$\frac{\partial\ln\ell}{\partial\alpha} = \frac{n}{\alpha} + \lambda\sum_{i=1}^{n}\ln x_i\left[\frac{x_i^{\alpha}}{\beta+\lambda x_i^{\alpha}} - x_i^{\alpha}\right] + \sum_{i=1}^{n}\ln x_i - \theta\sum_{i=1}^{n}\frac{C''(\theta M_i)}{C'(\theta M_i)}\frac{\partial M_i}{\partial\alpha},$$

$$\frac{\partial\ln\ell}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\left[\frac{x_i^{\alpha}}{\beta+\lambda x_i^{\alpha}} - x_i^{\alpha}\right] + \theta\sum_{i=1}^{n}\frac{C''(\theta M_i)}{C'(\theta M_i)}\frac{\partial M_i}{\partial\lambda},$$

$$\frac{\partial\ln\ell}{\partial\beta} = \frac{n}{\beta+1} + \sum_{i=1}^{n}\frac{1}{\beta+\lambda x_i^{\alpha}} + \theta\sum_{i=1}^{n}\frac{C''(\theta M_i)}{C'(\theta M_i)}\frac{\partial M_i}{\partial\beta},$$

$$\frac{\partial\ln\ell}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left[\frac{C''(\theta M_i)}{C'(\theta M_i)}\right]M_i - \frac{nC'(\theta)}{C(\theta)},$$

where $\frac{\partial M_i}{\partial\alpha} = -\left(\frac{\beta+\lambda x_i^{\alpha}}{\beta+1}\right)\lambda x_i^{\alpha}e^{-\lambda x_i^{\alpha}}\ln x_i$, $\frac{\partial M_i}{\partial\beta} = -\left(\frac{\lambda x_i^{\alpha}e^{-\lambda x_i^{\alpha}}}{(\beta+1)^2}\right)$, and $\frac{\partial M_i}{\partial\lambda} = -\left(\frac{\beta+\lambda x_i^{\alpha}}{\beta+1}\right)x_i^{\alpha}$.

The ML estimators of the model parameters are determined by solving the non-linear equations $\partial \ln \ell / \partial \alpha = 0, \partial \ln \ell / \partial \lambda = 0, \partial \ln \ell / \partial \theta = 0, \partial \ln \ell / \partial \beta = 0.$ These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative technique.

5.2. Least squares estimators

Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n from PQLPS class, and suppose $X_{1:n}, X_{2:n}, ..., X_{n:n}$ denotes the corresponding ordered sample. Hence, the least squares (LS) estimators for PQLPS parameters $\psi \equiv (\alpha, \lambda, \beta, \theta)$ can be obtained by minimizing the following quantity

$$LS = \sum_{i=1}^{n} \left(\left[1 - \frac{1}{C(\theta)} C\left(\theta \left(1 + \frac{\lambda x_{i:n}^{\alpha}}{\beta + 1} \right) e^{-\lambda x_{i:n}^{\alpha}} \right) \right] - \frac{i}{n+1} \right)^2,$$

with respect to $\alpha, \lambda, \beta,$ and $\theta,$ respectively.

5.3. Weighted least squares estimators

In this subsection the WLS estimators of the unknown parameters for PQLPS class are derived. Let $X_1, X_2, ..., X_n$ be a random sample of size n from PQLPS distribution and $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be the corresponding ordered sample. The PQLPS estimators can be obtained by minimizing the following sum of squares errors

$$\sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left(\left[1 - \frac{1}{C(\theta)} C\left(\theta \left(1 + \frac{\lambda x_{i:n}^{\alpha}}{\beta + 1} \right) e^{-\lambda x_{i:n}^{\alpha}} \right) \right] - \frac{i}{n+1} \right)^2,$$

with respect to the unknown parameters $\alpha, \lambda, \beta,$ and $\theta,$ respectively.

6. Simulation Study

In this section, an extensive simulation study is executed for PQLP distribution as a special distribution from PQLPS class. We compare the behavior of different estimates in the sense of their absolute biases (ABs) and mean square errors (MSEs) for different sample sizes and for different selected parameter values. The simulation study is designed as follows:

Step 1: Generate 1,000 random samples of size n =10, 20, 30 and 100 from the PQLP distribution.

Step 2: Three sets of parameter values are selected as;

Case 1 $\equiv (\alpha = 0.5,\ \theta = 0.1, \lambda = 1.5,\ \beta = 0.1)$;

Case 2 $\equiv (\alpha = 1.5,\ \theta = 0.1, \lambda = 1.5,\ \beta = 1.5)$ and

Case 3 $\equiv (\alpha = 0.1,\ \theta = 1, \lambda = 0.5,\ \beta = 0.1)$.

Step 3: The ML estimate (MLEs), LS estimates and WLS estimates of the population parameters are obtained.

Step 4: The ABs and MSEs of different estimates of population parameters are computed. All the results of the simulation are listed in Table 3.

Some conclusions can be deducted about the performance of different estimates:

1. For all different values of estimates and different methods of estimation we can realize that the MSEs decrease with increasing sample size.

2. The MSEs for MLEs, for all true values, are the smallest among the other estimates.

3. For the first and second set of parameter estimates, the estimate of λ has the largest MSEs values in most all cases for all sample sizes and different methods of estimation.

4. The third set of parameter estimates has the largest MSEs values among the three sets in most cases, for all sample sizes and different methods of estimation.

5. The MSEs for β estimates are the greatest for the third case of estimates for all different methods of estimation and sample sizes.

6. The ABs decrease for all sets of parameter estimates and for all methods of estimation with increasing sample size.

Table 3 Results of simulation study of ABs and MSEs of different estimates for the PQLP distribution

n	Method	Measure	Case 1				Case 2				Case 3			
			$\alpha=0.5$	$\theta=0.1$	$\lambda=1.5$	$\beta=0.1$	$\alpha=1.5$	$\theta=0.1$	$\lambda=1.5$	$\beta=1.5$	$\alpha=0.1$	$\theta=1$	$\lambda=0.5$	$\beta=0.1$
10	ML	AB	0.1321	0.7634	0.938	0.0243	0.7168	0.6999	1.1226	0.9973	0.7709	0.7245	1.5578	0.9062
		MSE	0.0563	0.2952	0.3727	0.0751	0.6293	0.3723	1.7931	0.9247	0.6912	0.5239	0.6491	0.8242
	LS	AB	0.4984	0.6798	0.8252	0.0157	0.8433	0.6172	0.741	0.9922	0.9021	0.5758	0.9159	0.9513
		MSE	0.2258	0.4117	1.835	0.1591	0.6701	0.4899	2.4001	0.9845	0.8434	1.3372	0.9061	1.4159
	WLS	AB	0.5621	0.6985	0.7042	0.0136	0.7064	0.6001	1.1096	0.9897	0.9094	0.5863	0.8427	0.9412
		MSE	0.3434	0.4889	2.465	0.1861	0.7672	0.3861	1.797	0.9597	0.8482	1.0494	0.8071	1.3865
20	ML	AB	0.0246	0.7064	0.681	0.0201	0.6313	0.4081	0.7186	0.8971	0.4969	0.2125	0.5456	0.1815
		MSE	0.0332	0.0597	0.0612	0.0534	0.3146	0.0916	0.6094	0.0976	0.6517	0.5019	0.6294	0.7644
	LS	AB	0.2134	0.6363	0.6741	0.0128	0.7651	0.2917	0.3203	0.5921	0.2096	0.1251	0.2005	0.1942
		MSE	0.1833	0.0646	0.2004	0.0907	0.3709	0.388	0.6695	0.2843	0.6669	0.8913	0.7328	0.892
	WLS	AB	0.5319	0.6313	0.5233	0.0129	0.5605	0.2369	0.6138	0.1911	0.1085	0.2249	0.1523	0.0943
		MSE	0.3167	0.3849	0.975	0.0704	0.5615	0.232	0.8786	0.1822	0.7462	0.791	0.7049	0.9024
30	ML	AB	0.0154	0.0099	0.4211	0.012	0.0565	0.0766	0.4324	0.0994	0.0565	0.0702	0.0727	0.0988
		MSE	0.0255	0.0194	0.0298	0.0041	0.0646	0.0559	0.2254	0.0129	0.0764	0.0803	0.0629	0.0977
	LS	AB	0.0016	0.0675	0.0972	0.0095	0.0773	0.0641	0.0807	0.0218	0.0887	0.0624	0.0647	0.0991
		MSE	0.0838	0.0526	0.1001	0.0083	0.0959	0.0403	0.0815	0.0942	0.0979	0.0881	0.0977	0.1868
	WLS	AB	0.0063	0.0657	0.0185	0.0091	0.0731	0.0626	0.0861	0.0099	0.0858	0.0623	0.0746	0.0626
		MSE	0.0127	0.0484	0.0886	0.0089	0.0836	0.0924	0.0921	0.0156	0.0952	0.0945	0.0832	0.0959
100	ML	AB	0.0014	0.0069	0.0631	0.0043	0.0157	0.0047	0.0217	0.0192	0.0151	0.0069	0.0098	0.0089
		MSE	0.0012	0.0048	0.0068	0.0002	0.0049	0.0040	0.0084	0.0078	0.0051	0.0049	0.0172	0.0898
	LS	AB	0.0539	0.0609	0.0724	0.0015	0.021	0.0063	0.0051	0.0097	0.0074	0.0063	0.0091	0.0051
		MSE	0.0034	0.0484	0.0943	0.0004	0.0514	0.0412	0.0626	0.0198	0.0064	0.0772	0.0896	0.0993
	WLS	AB	0.0055	0.0112	0.0067	0.0015	0.0076	0.0065	0.0064	0.0059	0.0074	0.0061	0.0159	0.0051
		MSE	0.008	0.0188	0.0532	0.0008	0.0058	0.0431	0.0167	0.0098	0.0095	0.0767	0.0441	0.0992

7. The PQLP Regression Model

In many practical applications, lifetimes are affected by variables, which are referred to covariates, such as the cholesterol level, blood pressure and many others. So, it is important to explore the relationship between the lifetime and explanatory variables and the approach based on a regression model can be used.

Let X is a random variable having the PQLP density function, then the vector of explanatory variables is denoted by $x=(x_1,...,x_n)^T$, which is related to response variable $Y=\log(X)$ through a regression model.

Let $Y=\log(X)$ have the log-PQLP distribution. The density of Y, parameterize in terms of $\lambda=e^{-\mu/\sigma}$ and $\alpha=1/\sigma$ where $-\infty \le Y \le \infty$, $-\infty \le \mu \le \infty$ is location parameter and $\sigma>0$ is the scale parameter, hence the density function of Y is given by:

$$f(y;\theta,\beta,\sigma,\mu)=\frac{1}{\sigma(\beta+1)(e^{\theta}-1)}e^{\frac{-\mu}{\sigma}}\theta\left(\beta+e^{\frac{-\mu}{\sigma}}e^{\frac{y}{\sigma}}\right)e^{y\left(\frac{1}{\sigma}-1\right)}e^{-\left(e^{\frac{-\mu}{\sigma}}e^{\frac{y}{\sigma}}\right)}\exp\left(\theta\left(1+\frac{e^{\frac{-\mu}{\sigma}}e^{\frac{y}{\sigma}}}{\beta+1}\right)e^{-\left(e^{\frac{-\mu}{\sigma}}e^{\frac{y}{\sigma}}\right)}\right)\left|e^{y}\right|. \tag{20}$$

After simplification, then (20) takes the following form

$$f(y;\theta,\beta,\sigma,\mu)=\frac{1}{\sigma(\beta+1)(e^{\theta}-1)}\theta e^{\frac{y-\mu}{\sigma}}\left(\beta+e^{\frac{y-\mu}{\sigma}}\right)e^{-\left(e^{\frac{y-\mu}{\sigma}}\right)}\exp\left(\theta\left(1+\frac{e^{\frac{y-\mu}{\sigma}}}{\beta+1}\right)e^{-\left(e^{\frac{y-\mu}{\sigma}}\right)}\right). \tag{21}$$

This new distribution will be referred to as the log-PQLP. The survival function corresponding to (21) is given by

$$S(y;\theta,\beta,\sigma,\mu)=\frac{1}{e^{\theta}-1}\left[\exp\left(\theta\left(1+\frac{e^{\frac{y-\mu}{\sigma}}}{\beta+1}\right)e^{-\left(e^{\frac{y-\mu}{\sigma}}\right)}\right)-1\right]. \tag{22}$$

If Y has the log-PQLP distribution then it is denoted by $Y \sim$ LPQLP $(\theta,\beta,\sigma,\mu)$. We now define the density function survival function in standardized form as follows

$$f(z)=\frac{\theta e^{z}\left(\beta+e^{z}\right)e^{-e^{z}}}{\sigma(\beta+1)(e^{\theta}-1)}\exp\left(\theta\left(1+\frac{e^{z}}{\beta+1}\right)e^{-\left(e^{z}\right)}\right),$$

and

$$S(z)=\frac{1}{e^{\theta}-1}\left[\exp\left(\theta\left(1+\frac{e^{z}}{\beta+1}\right)e^{-e^{z}}\right)-1\right].$$

Consider the following linear location-scale regression model, linking the response variable y_i and the explanatory variable vector $X_i^T=(x_1,x_2,...,x_n)^T$,

$$y_i=x_i^T\mathrm{B}+\sigma z_i, \tag{23}$$

where the random error z_i, the response variable y_i has the density given in (21), the parameter $\mu_i=x_i^T\mathrm{B}$ is the location of y_i. The location parameter vector $\mu=(\mu_1,\mu_2,...,\mu_n)^T$ is represented by a linear model $\mu=X\mathrm{B}$, $X=(x_1,x_2,...,x_n)^T$ is a model known matrix, and $\mathrm{B}=(\mathrm{B}_0,\mathrm{B}_1,...,\mathrm{B}_k)$ is the unknown regression coefficients.

Let the random sample $y_1, y_2, ..., y_n$ follow a LPQLP distribution and the response variable is defined as $y_i = \min\left(\log(x_i), \log(c_i)\right)$, where c_i is the censoring time and x_i is the observed lifetime. Assume that the censoring times and lifetimes are independent. Let F and C are the sets representing the observed lifetimes and censoring times. The log-likelihood function for the model given in (23) is given by:

$$l(\tau) = \sum_{i\in F} l_i(\tau) + \sum_{i\in C} l_i^{(c)}(\tau),$$

where $l_i(\tau) = \log(f(y_i))$, $l_i^{(c)}(\tau) = \log(S(y_i))$, $f(y_i)$ is the density (21) and $S(y_i)$ is the survival function (22) of Y_i. The total log-likelihood function for $\tau = (\alpha, \lambda, \beta, \theta, \mathrm{B}^T)$ is given by

$$l(\tau) = r\log\theta - r\log\sigma - r\log(\beta+1) - 2r\log(e^{\theta}-1) + \sum_{i\in F} z_i + \sum_{i\in F}\log\left[\beta + \left(e^{z_i}\right)\right] - \sum_{i\in F}\exp(z_i)$$
$$+ \sum_{i\in F}\left(\theta\left(1+\frac{e^{z_i}}{\beta+1}\right)e^{-\left(e^{z_i}\right)}\right) + \sum_{i\in C}\log\left(\theta\left(1+\frac{e^{z_i}}{\beta+1}\right)e^{-\left(e^{z_i}\right)} - 1\right). \quad (24)$$

where $z_i = \dfrac{y_i - x_i^T \mathrm{B}}{\sigma}$ and r is the number of uncensored observations (failures). The MLE $\hat{\tau}$ of the vector of unknown parameters can be evaluated by maximizing the log-likelihood (24).

8. Applications to Real Data

The flexibility of PQLP as a special distribution from PQLPS class is examined using two real data sets. The superiority of PQLP distribution is clarified as compared with some other sub-models. For both sets of data, the PQLP was compared to L, QL, PQL, LP and GP distributions.

The first data set has been obtained from Murthy et al. (2004) and is given by
0.04, 0.3, 0.31, 0.557, 0.943, 1.07, 1.124, 1.248, 1.281, 1.281, 1.303, 1.432, 1.48,1.51, 1.51, 1.568, 1.615, 1.619, 1.652, 1.652, 1.757, 1.795, 1.866, 1.876, 1.899, 1.911, 1.912, 1.9141, 0.981, 2.010, 2.038, 2.085, 2.089, 2.097, 2.135, 2.154, 2.190, 2.194, 2.223, 2.224, 2.23, 2.3, 2.324, 2.349, 2.385, 2.481, 2.610, 2.625, 2.632, 2.646, 2.661, 2.688, 2.823, 2.89, 2.9, 2.934, 2.962, 2.964, 3, 3.1, 3.114, 3.117, 3.166, 3.344, 3.376, 3.385, 3.443, 3.467, 3.478, 3.578, 3.595, 3.699, 3.779, 3.924, 4.035, 4.121, 4.167, 4.240, 4.255, 4.278, 4.305, 4.376, 4.449, 4.485, 4.570, 4.602, 4.663, 4.694.

Based on the ML method, the population parameters of each distribution are estimated. The MLEs and standard errors (SEs) of all models are obtained. Also, some selected measures as; Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct AIC (CAIC), Hannan-Quinn information criterion (HQIC), the Kolmogorov-Smirnov (KS) and p-value statistics are obtained to compare the fitted models. The best distribution is corresponding to the lower values of, AIC, CAIC, BIC, and K-S statistics. The results for mentioned estimates and measures for all models are reported in Table 4.

Table 4 Criteria for comparison for first data set

Model	MLEs (SEs)				Measurements					
	α	λ	θ	β	KS	p-value	AIC	CAIC	BIC	HQIC
PQLP	1.986 (0.151)	0.112 (0.028)	3.393 (1.591)	0.172 (0.069)	0.059	0.933	269.142	269.636	278.866	273.051
L	-	-	-	0.631 (0.051)	0.242	0.179	308.103	308.247	312.964	310.057
QL	-	0.78 (0.037)	-	0.004 (0.062)	0.18	0.808	290.035	290.179	294.897	291.989
LP	-	-	1.793 (0.408)	0.39 (0.034)	0.652	0.131	318.573	318.718	323.435	320.528
GP	-	0.5 (0.071)	1.792 (0.687)	-	0.172	0.145	298.29	298.435	303.152	300.244

The PQLP distribution is the best-fitted distribution in comparison with L, QL, LP and GP distributions as seen from Table 4. The histogram and the estimated densities of the fitted PQLP L, QL, LP and GP models for the first data are achieved in Figure 5.

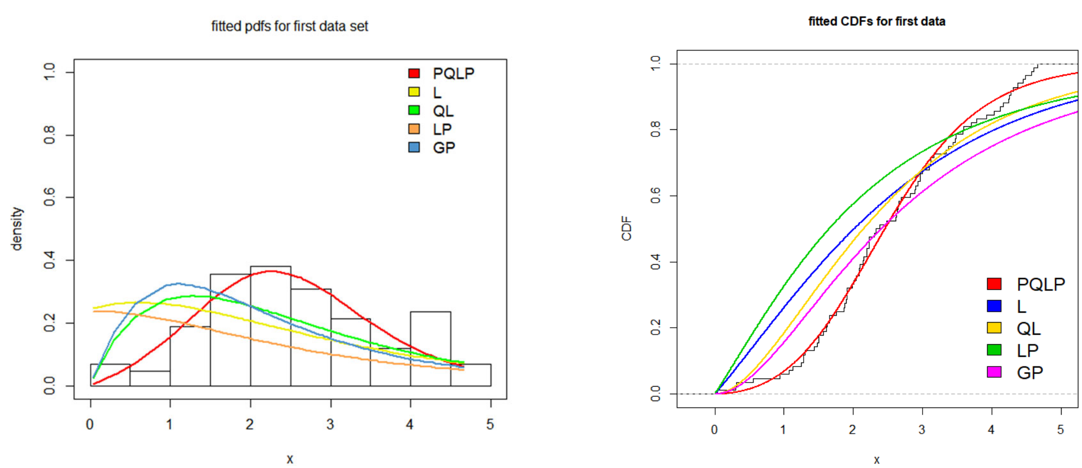

Figure 5 Estimated densities and estimated cdf of all models for the first data

From Figure 5, we observe that the PQLP distribution is the best-fitted distribution compared with L, QL, LP and GP distributions.

We consider the second data on failure and service times for a particular model windshield. The data represent the service times of 63 Aircraft Windshield (Murthy et al., 2004). The data are listed as follows

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

For the second real data set, the values of different measurements are recorded in Table 5.

Table 5 Criteria for comparison for second data set

Model	MLEs (SEs)				Measurements					
	α	λ	θ	β	KS	p-value	AIC	CAIC	BIC	HQIC
PQLP	1.295 (0.125)	0.084 (0.058)	20.068 (1.7368)	0.127 (0.035)	0.087	0.87	206.045	206.712	214.618	209.417
L	-	-	-	0.753 (0.07)	0.156	0.156	213.156	213.349	217.442	214.841
QL	-	0.909 (0.133)	-	0.116 (0.096)	0.138	0.138	208.201	208.395	216.488	209.887
LP	-	-	2.066 (0.483)	0.447 (0.046)	0.52	0.5	219.733	219.927	224.019	221.419
GP	-	0.552 (0.126)	2.434 (1.153)	-	0.136	0.194	214.876	215.07	219.162	216.562

We can see that the PQLP distribution is more suitable than the L, QL, LP and GP distributions as seen from Table 5. The plots of the estimated cumulative and estimated densities of the fitted models are achieved, respectively, in Figure 6.

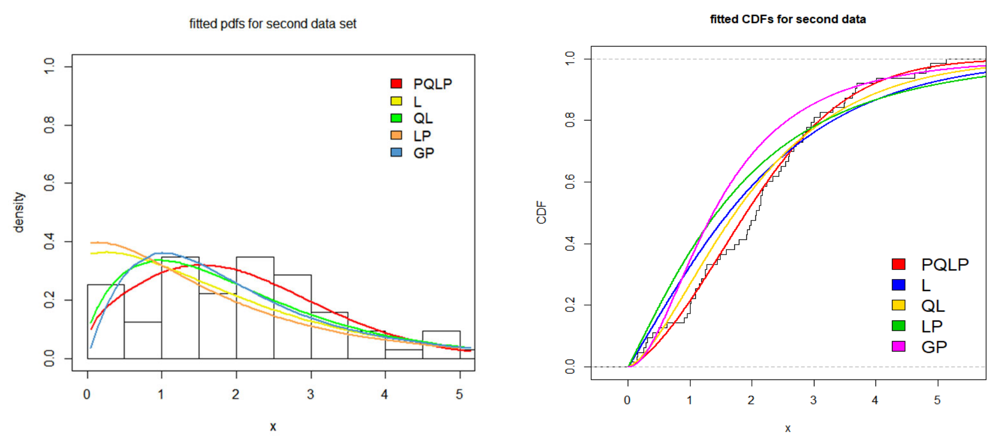

Figure 6 Estimated densities and estimated cdf densities of all models for the second data

It is clear from the above two figures that the PQLP model has the best fit than the L, QL, LP and GP models.

9. Conclusions

We introduce a new class of lifetime models called the power quasi Lindley power series. It includes the Lindley power series distributions (Warahena-Liyanage and Pararai 2015). The PQLPS class is obtained by mixing the PQL distribution together with the power series distribution. More specifically, the PQLPS covers several new distributions. Also, statistical properties of the new class, including expressions for density function, moments, moment generating function, quantile function, order statistics and entropy are provided. Maximum likelihood is implemented for estimating the model parameters. Two special models, namely; power quasi Lindley Poisson and power quasi Lindley geometric are considered. Further, the derived properties of the class are valid to the two selected models. We explore the estimation of the power quasi Lindley power series class parameters by the maximum likelihood, least squares and weighted least squares methods. The accuracy and comparison of different power quasi Lindley Poisson parameter estimates are checked via simulation study. We also introduced a new linear regression model based on the logarithm of the power quasi Lindley Poisson random variable. The power quasi Lindley Poisson model is fitted to two real data sets to demonstrate the potentiality of the introduced class. Eventually, we wish a broadly statistical application in some areas for this new compounding class.

Acknowledgments

We would like to thank the two anonymous reviewers for their suggestions and comments.

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