Partial Probability Weighted Moments Estimation for the Generalized Pareto Distribution

By

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Abstract

The method of partial probability-weighted moment (PPWM) with double bound censoring is used to derive estimators for the parameters of the generalized Pareto (GP) distribution. To study the properties of the new estimators a comprehensive numerical study will be carried out using Mathcad (2001) statistical package. The left and right PPWM may be obtained as special cases. Probability weighted moment (PWM) parameter estimation results for the GP derived by Hosking and Wallis (1987) may be considered as a special case from the present results.

Key Words: Double censoring partial probability weighted moments; Generalized Pareto distribution; Order statistic; Simulation.

1. Introduction

The Generalized Pareto (GP) distribution was introduced by Pickands (1975) and has been further studied by Davison (1984); Smith (1984, 1985) and Van Montfort and Witter (1985). The GP distribution is a two-parameter distribution that contains uniform, exponential, and Pareto distributions as special cases. Hosking and Wallis (1987) introduced the following cumulative distribution function and probability density function of a two-parameter GP distribution with scale parameter α and shape parameter k

$$F(x) = \begin{cases} 1-(1-\frac{kx}{\alpha})^{\frac{1}{k}}, & k \neq 0 \\ 1-\exp(-\frac{x}{\alpha}), & k = 0 \end{cases} \qquad (1.1)$$

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and

$$f(x) = \begin{cases} \alpha^{-1}\,(1-\dfrac{kx}{\alpha})^{\frac{1}{k}-1}, & k \neq 0 \\ \alpha^{-1}\exp(-\dfrac{x}{\alpha}) \quad , & k = 0 \end{cases} \tag{1.2}$$

where $0 \le x \le \infty$ for $k \le 0$ and $0 \le x \le \alpha/k$ for $k > 0$. Applications of GP distribution are numerous and include reliability studies, the analysis of environmental extreme events, and the modeling of large insurance claims. Moreover, the GP distribution can be used in any situation in which the exponential distribution might be used in which some robustness is required against heavier tailed or lighter tailed alternatives. Parameter estimates for the GP has been obtained using the following methods: maximum likelihood (ML), method of moments (MM), and probability weighted moments (PWM). The three methods have been compared by Hosking and Wallis (1987) where it is found that the MM and PWM have smaller bias and root mean square error than the ML for samples less than 500.

Observed data sets containing values above or below the analytical threshold of measuring equipments are referred to as censored. Such data are frequently encountered in quality and quantity monitoring applications of water, soil, and air samples. Censored data are categorized as either type I censoring, where the measurement threshold is fixed and the number of censored data points varies, or as type II censoring, where the number of censored data points is fixed and the implicit threshold varies (see David (1981)).

The aim of this article is to estimate the unknown parameters of the GP distribution by using the method of PPWM for double bound censoring. The estimates of the parameters from left and right censoring PPWM may be obtained as special cases. The results of the PWM parameter estimation for the GP obtained by Hosking and Wallis (1987) may be considered as a special case from the present results. We investigate the properties of these estimators numerically using Monte Carlo simulation and Mathcad (2001) statistical package.

2. Partial probability- weighted moments estimation

Wang (1990a, b, 1996a) introduced the concept of PPWM which are extensions of the usual PWM for the purpose of estimating the upper quantiles of flood flows when one's interested in the right tail of the distribution and there is some benefit to censoring some of the smaller observations in the left tail. For real p, r, and s the PPWM with double bound censoring observations of a continuous random variable X with the distribution function F are introduced by Wang (1990a, b) as follows:

$$M'''_{p,r,s} = \int_c^d [x(F)]^p F^r [1-F]^s \, dF. \tag{2.1}$$

where the lower bound c = F(x_{01}), the upper bound d = F(x_{02}) are the fraction of observations which are censored and x_{01}, x_{02} are the censoring threshold. For parameter estimation the special cases $M^{'''}_{1,r,0}$ (r = 0, 1, 2,…) or $M^{'''}_{1,0,s}$ (s = 0, 1, 2,…) are for sufficient generality. When p = 1 and s = 0, The PPWM become

$$\beta_r^{'''} = M^{'''}_{1,r,0} = \int_c^d x(\mathrm{F})(\mathrm{F})^{\mathrm{r}}\, d\mathrm{F}. \tag{2.2}$$

where $\beta_r^{'''}$ results when the observations below and above fixed threshold c and d are censored. The parameters can be estimated by using PPWM with double bound censoring by replacing the theoretical expression by their sampling estimators.

The main purpose of this **Section** is to obtain the PPWM estimators with double bound censoring observations for the GP distribution (1.2). To obtain the general expression of PPWM estimates of parameters for $k \neq 0$, it is convenient to use the quantities

$$\beta_s^{'''} = M^{''''}_{1,0,s} = \frac{\alpha}{\mathrm{k}\,(\mathrm{s}+1)}[\,(1\text{-}\mathrm{c})^{\mathrm{s}+1} - (1\text{-}\mathrm{d})^{\mathrm{s}+1}] + \frac{\alpha}{\mathrm{k}\,(\mathrm{k}+s+1)}[(1\text{-}\mathrm{d})^{\mathrm{k}+\mathrm{s}+1} - (1\text{-}\mathrm{c})^{\mathrm{k}+\mathrm{s}+1}] \tag{2.3}$$

which exist provided that k>-1. The Parameters may be expressed in terms $\beta_s^{'''}$'s as

$$\beta_0^{'''} = \frac{\alpha}{\mathrm{k}\,(\mathrm{k}+1)}[(\mathrm{d}\text{-}\mathrm{c})(\mathrm{k}+1) + (1\text{-}\mathrm{d})^{\mathrm{k}+1} - (1\text{-}\mathrm{c})^{\mathrm{k}+1}], \tag{2.4}$$

and

$$\beta_1^{'''} = \frac{\alpha}{2\mathrm{k}(\mathrm{k}+2)}\{(\mathrm{k}+2)[(1\text{-}\mathrm{c})^2 - (1\text{-}\mathrm{d})^2] + 2(1\text{-}\mathrm{d})^{\mathrm{k}+2} - 2(1\text{-}\mathrm{c})^{\mathrm{k}+2}]\}. \tag{2.5}$$

Some special cases may be obtained from (2.2) as follows:

1. When the upper limit d = 1, the double censoring PPWM reduces to the left censoring PPWM. Thus the left censoring PPWM results when the observations below a fixed threshold c are censored, which is defined as

$$\beta_r^{''} = M^{''}_{1,r,0} = \int_c^1 x(\mathrm{F})(\mathrm{F})^{\mathrm{r}}\, d\mathrm{F} \tag{2.6}$$

where the lower limit c =F(x_{01}), is the fraction of observations which are censored and x_{01} is censoring threshold. While the following PPWM

$$\beta_r^{''} = \frac{1}{1-c^{r+1}}\int_{\mathrm{c}}^1 \mathrm{x}(\mathrm{F})\mathrm{F}^{\mathrm{r}}\, d\mathrm{F} \tag{2.7}$$

was introduced by Wang (1996a) to remain consistent with PWM for complete sample and to simplify the mathematical derivation associated with PPWM sample estimators of distributional parameters. Wang (1990a,b) and Kroll and Stendinger (1996) used the PPWM (2.6) and Wang (1996a) used the PPWM in (2.7) to derive censored quantile estimators for the generalized extreme value and log normal distributions.

The left censoring PPWM estimates of α and k for the GP (1.2) can be obtained by putting d = 1 in equations (2.3), (2.4) and (2.5) respectively

$$\beta_s^{``} = M_{1,0,s}^{``} = \frac{\alpha}{\mathrm{k}\,(\mathrm{s}+1)}[\,(1\text{-}\mathrm{c})^{\mathrm{s}+1}] - \frac{\alpha}{\mathrm{k}\,(\mathrm{k}+s+1)}[(1\text{-}\mathrm{c})^{\mathrm{k+s+1}}], \qquad (2.8)$$

from (2.8), we have

$$\beta_0^{``} = \frac{\alpha}{\mathrm{k}(\mathrm{k}+1)}[\,(\mathrm{k}+1)(1\text{-}\mathrm{c}) \text{-} (1\text{-}\mathrm{c})^{\mathrm{k}+1}], \qquad (2.9)$$

and

$$\beta_1^{``} = \frac{\alpha}{2\mathrm{k}(\mathrm{k}+2)}[\,(\mathrm{k}+2)(1\text{-}\mathrm{c})^2 \text{ - } 2(1\text{-}\mathrm{c})^{\mathrm{k}+2}]. \qquad (2.10)$$

2. When the lower limit c = 0, the double buond censoring PPWM reduces to the right censoring PPWM. Thus the right censoring PPWM results when the observations above a fixed threshold d are censored, which is defined as

$$\beta_r^{`} = \int_0^d x(\mathrm{F})(\mathrm{F})^{\mathrm{r}}\, d\mathrm{F} \qquad (2.11)$$

where the upper limit d = F(x_{02}), x_{02} being censoring threshold. The right censoring PPWM estimates of α and k for the GP (1.2) can be obtained by putting c = 0 in equations (2.3), (2.4) and (2.5) respectively. Thus

$$\beta_s^{`} = M_{1,0,s}^{`} = \frac{\alpha}{\mathrm{k}\,(\mathrm{s}+1)}[1-(1\text{-}\mathrm{d})^{\mathrm{s}+1}] - \frac{\alpha}{\mathrm{k}\,(\mathrm{k}+s+1)}[1-(1\text{-}\mathrm{d})^{\mathrm{k+s+1}}],$$

$$\beta_0^{`} = \frac{\alpha}{\mathrm{k}(\mathrm{k}+1)}[\,(\mathrm{k}+1)(\mathrm{d}) + (1\text{-}\mathrm{d})^{\mathrm{k}+1} - 1], \qquad (2.12)$$

and

$$\beta_1^{`} = \frac{\alpha}{2\mathrm{k}(\mathrm{k}+2)}[\,(\mathrm{k}+2)\{1\text{-}(1\text{-}\mathrm{d})^2\} + 2(1\text{-}\mathrm{d})^{\mathrm{k}+2} - 2]. \qquad (2.13)$$

3. When the upper limit d = 1 and the lower limit c = 0, the PPWM reduces to the ordinary PWM defined by Greenwood *et al* (1979). Thus the PWM of α and k for the GP (1.2) given by Hosking and Wallis (1987) are the quantities

$$\beta_s = M_{1,0,s} = \frac{\alpha}{(\mathrm{s}+1)\,(\mathrm{k}+1+\mathrm{s})} \qquad (2.14)$$

which yield

$$\alpha = \frac{2\beta_0\beta_1}{\beta_0 - 2\beta_1}, \qquad k = \frac{\beta_0}{\beta_0 - 2\beta_1} - 2. \qquad (2.15)$$

We obtain the same result of equations (2.14) and (2.15), by putting c = 0 and d =1 in equations (2.3), (2.4) and (2.5) respectively.

3. Sample estimators of PPWM

Wang (1990a) derived sample estimators $b_r^{```}$ of PPWM for double bound censoring observations. Such estimators are unbiased estimators of their theoretical expression $\beta_r^{```}$. Given a random sample of size n from the distribution F, the following statistic based on the ordered sample $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$

$$b_r^{```} = n^{-1} \sum_{j=1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} x_{(j)}^{*}, \qquad (3.1)$$

where

$$x_{(j)}^{*} = \begin{cases} 0 & x_{(j)} \le x_{01} \\ x_{(j)} & x_{01} < x_{(j)} \le x_{02} \\ 0 & x_{(j)} > x_{02}. \end{cases}$$

is an unbiased estimator for $\beta_r^{```}$.

The doubly censoring PPWM estimators $\hat{\alpha}$ and $\hat{k}$ of the parameters are the solutions of equations (2.4) and (2.5) for α and k when the $\beta_0^{```}$ and $\beta_1^{```}$ are replaced by their sampling estimators $b_0^{```}$ and $b_1^{```}$ given by equation (3.1). To obtain $\hat{k}$, divided equation (2.4) and equation (2.5). Therefore the doubly censoring PPWM for $\hat{k}$ will be

$$\begin{aligned} &(\hat{k}+1)(\hat{k}+2)\{b_0^{```}[(1-c)^2 - (1-d)^2] - 2b_1^{```}[d-c]\} = \\ &\quad 2(\hat{k}+2)b_1^{```}[(1-d)^{\hat{k}+1} - (1-c)^{\hat{k}+1}] - 2(\hat{k}+1)b_0^{```}[(1-d)^{\hat{k}+2} - (1-c)^{\hat{k}+2}]. \end{aligned} \qquad (3.2)$$

The exact solution for (3.2) requires iterative method. Subsituting $\hat{k}$ in equation (2.4), the scale parameter α can be estimated as

$$\hat{\alpha} = \frac{\hat{k}(\hat{k}+1)b_0^{```}}{(\hat{k}+1)(d-c) + (1-d)^{\hat{k}+1} - (1-c)^{\hat{k}+1}}. \qquad (3.3)$$

For the left censoring PPWM, Wang (1990a) proved that the statistic

$$b_r^{``} = n^{-1} \sum_{j=1}^{n} \frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)} x_{(j)}^{*}, \qquad (3.4)$$

where

$$x_{(j)}^{*} = \begin{cases} 0 & x_{(j)} \le x_{01} \\ x_{(j)} & x_{(j)} > x_{01} \end{cases}.$$

is an unbiased estimator for $\beta_r^{``}$, which based on the ordered sample $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$ of size n from a distribution function F. Thus the PPWM estimators $\hat{\alpha}$ and $\hat{\mathrm{k}}$ of $\alpha$ and k given in equations (2.9) and (2.10) are obtained by replacing $\beta_0^{``}$ and $\beta_1^{``}$ by their sampling estimates $b_0^{``}$ and $b_1^{``}$ given in equation (3.4). To obtain $\hat{\mathrm{k}}$, divided equation (2.9) and equation (2.10), therefore left censoring PPWM for $\hat{\mathrm{k}}$ is obtained as

$$\left(\hat{\mathrm{k}}+1\right)\left(\hat{\mathrm{k}}+2\right)\{b_0^{``}(1-c)-2b_1^{``}\} = \\ 2(1-\mathrm{c})^{\hat{\mathrm{k}}}[\left(\hat{\mathrm{k}}+1\right)b_0^{``}(1-\mathrm{c}) \ -\left(\hat{k}+2\right)b_1^{``}]. \tag{3.5}$$

subsituting $\hat{k}$ in equation (2.9), the scale parameter α can be estimated as

$$\hat{\alpha} = \frac{\hat{k}\left(\hat{k}+1\right)b_0^{``}}{\left(\hat{k}+1\right)(1-c)-(1-c)^{\hat{k}+1}}. \tag{3.6}$$

The exact solution of equation (3.5) requires iterative method.

In addition, An unbiased estimate for the right censoring PPWM of $\beta_r^{`}$ based on the observed sample given by Wang (1990a) is the statistic

$$b_r^{`} = n^{-1}\sum_{j=1}^{n}\frac{(\mathrm{j}-1)(\mathrm{j}-2)...(\mathrm{j}-\mathrm{r})}{(\mathrm{n}-1)(\mathrm{n}-2)...(\mathrm{n}-\mathrm{r})}x_{(\mathrm{j})}^{*}, \tag{3.7}$$

where

$$x_{(\mathrm{j})}^{*} = \begin{cases} 0 & x_{(\mathrm{j})} > x_{02} \\ x_{(\mathrm{j})} & x_{(\mathrm{j})} \le x_{02}. \end{cases}.$$

and $x_{(\mathrm{j})}$, j =1,2,…n have been rank ordered from $x_{(1)}$, the smallest to $x_{(\mathrm{n})}$, the largest. Thus the PPWM estimators $\hat{\alpha}$ and $\hat{\mathrm{k}}$ of α and k given in equations (2.12) and (2.13) are obtained by replacing $\beta_0^{`}$ and $\beta_1^{`}$ by their sampling estimates $b_0^{`}$ and $b_1^{`}$, which can be obtained from equation (3.7). To obtain $\hat{k}$, divide equation (2.12) and equation (2.13)

$$\left(\hat{\mathrm{k}}+1\right)\left(\hat{\mathrm{k}}+2\right)\{b_0^{`}[1-(1-d)^2]-2b_1^{`}\mathrm{d}\} = \\ 2\left(\hat{\mathrm{k}}+2\right)b_1^{`}[(1-\mathrm{d})^{\hat{\mathrm{k}}+1}-1]-2b_0^{`}\left(\hat{\mathrm{k}}+1\right)[(1-d)^{\hat{\mathrm{k}}+2}-1] \tag{3.8}$$

Given $\hat{k}$, the scale parameter α can be estimated as

$$\hat{\alpha} = \frac{\hat{k}\left(\hat{k}+1\right)b_0^{`}}{\left(\hat{k}+1\right)d+(1-d)^{\hat{k}+1}-1}. \tag{3.9}$$

The exact solution of equation (3.8) requires iterative method.

While an unbiased estimate for the ordinary PWM for a given random sample of size n from the distribution F is the statistic

$$b_r = n^{-1}\sum_{j=}^{n}\frac{(j-1)(j-2)...(j-r)}{(n-1)(n-2)...(n-r)}x_{(j)} \tag{3.10}$$

(Landwehr *et al*. 1979 a). Hosking and Wallis (1985) introduced the following statistic

$$\widehat{\beta}_r[p_{j,n}] = n^{-1}\sum_{j=1}^{n} p_{j,n}^{r} x_j \tag{3.11}$$

where $p_{j,n}$ is a plotting position–that is, a distribution free estimate of $F(x_j)$. Reasonable choices of $p_{j,n}$, such as $p_{j,n} = (j-a)/n$, 0<a<1, or $(j-a)/(n+1-2a)$, -0.5<a<0.5 , yield estimators $\widehat{\beta}_r[p_{j,n}]$, which are asymptotically equivalent to b_r and therefore consistent estimators of β_r. The PWM estimates $\hat{\alpha}$ and $\hat{k}$ of α and k given in equations(2.14) and (2.15) are obtained by Hosking and Wallis (1985) by replacing β_0 and β_1 by their estimates defined in equations (3.10) and (3.11).

4. Nnumerical results

Monte Carlo experiments have been performed to investigate the properties of PPWM of the GP distribution from censoerd samples. Simulations were performed for sample sizes n = 15, 20, 30 and 50. Following Hosking and Wallis (1985), we take shape parameter of the distribution ranging from -0.4 to 0.4, the scale parameter α was set to 1 throughout. All of the estimation methods considered are equivariant under scale changes of data, so setting α =1 involves no loss of generality. Different levels of censoring thershold are considered, namely, c = 0.1 (0.4) 0.1 and d = 0.6 (0.9) 0.1. when c = 0 and d = 1, PPWM become the ordinary PWM. For each combination of values of n and k, 1000 random samples were generated from the GP distribution, and for each sample the parameters α and k, were estimated under the three cases: doubly censoring PPWM, left and right censoring PPWM. In the three cases, the exact solutions of equations (3.2), (3.5) and (3.8) for the scale parameter require iterative technique. All simulations studies are obtained via the Mathcad (2001) package. Simulation results are summarized in Tabels 1- 6, which give bias and root mean sequared error (RMSE) of estimators of the parameters α and k from the GP distribution.

Table 1

Bias of PPWM estimates of parameters from doubly censored samples for GP distribution

n	Censoring level	Censoring level	$\hat{\alpha}$					$\hat{k}$				
	c	d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
15	0.1	0.9	-.021	-.164	-.513	-.543	-.708	.391	.192	-.037	-.2	-.402
	0.2	0.9	-.209	-.363	-.585	-.651	-.734	.393	.194	.059	-.202	-.402
	0.1	0.8	-.739	-.886	-.718	-.844	-.896	.396	.2	-.09	-.209	-.408
	0.2	0.8	-.824	-.955	-.74	-.973	-.948	.398	.210	-.018	-.215	-.413
20	0.1	0.9	-.03	-.134	-.509	-.36	-.488	.391	.191	-.034	-.2	-.4
	0.2	0.9	.046	-.227	-.578	-.438	-.516	.393	.193	-.044	.201	.402
	0.1	0.8	-.636	-.688	-.679	-.745	-.853	.394	.196	-.06	-.206	-.405
	0.2	0.8	-.811	-.72	-.699	-.824	-.89	.396	.198	-.017	-.212	-.412
30	0.1	0.9	-.005	-.106	-.403	-.277	-.440	.371	.193	-.026	-.2	-.4
	0.2	0.9	-.033	-.168	-.548	-.261	-.255	.377	.178	-.034	-.201	-.4
	0.1	0.8	-.546	-.527	-.65	-.724	-.737	.391	.193	-.06	-.205	-.402
	0.2	0.8	-.613	-.66	-.680	-.813	-.741	.393	.196	-.016	-.21	-.409
50	0.1	0.9	-.004	-.106	-.393	-.188	-.281	.370	.174	-.025	-.199	-.387
	0.2	0.9	-.02	-.152	-.517	-.196	-.278	.371	.193	-.028	-.202	-.4
	0.1	0.8	-.431	-.434	-.530	-.603	-.621	.38	.194	-.011	-.204	-.4
	0.2	0.8	-.532	-.520	-.585	-.653	-.613	.381	.189	.012	-.209	-.403

Table 2

RMSE of PPWM estimates of parameters from doubly censored samples for GP distribution

n	Censoring level	Censoring level	$\hat{\alpha}$					$\hat{k}$				
	c	d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
15	0.1	0.9	.738	.717	.725	.729	.806	.398	.199	.094	.202	.4
	0.2	0.9	.749	.731	.757	.781	.821	.397	.199	.112	.203	.4
	0.1	0.8	.920	.943	.849	.922	.941	.394	.2	.124	.217	.415
	0.2	0.8	.940	.955	.861	.927	.970	.396	.2	.132	.225	.419
20	0.1	0.9	.641	.588	.718	.563	.635	.386	.196	.078	.201	.391
	0.2	0.9	.653	.63	.731	.636	.66	.394	.197	.091	.196	.398
	0.1	0.8	.92	.943	.827	.863	.915	.394	.199	.110	.215	.412
	0.2	0.8	.944	.940	.854	.91	.94	.398	.2	.122	.219	.417
30	0.1	0.9	.641	.522	.634	.471	.394	.393	.195	.073	.2	.383
	0.2	0.9	.644	.522	.675	.444	.396	.381	.197	.088	.195	.385
	0.1	0.8	.804	.831	.81	.838	.397	.393	.198	.099	.21	.404
	0.2	0.8	.844	.864	.825	.894	.398	.382	.198	.117	.212	.405
50	0.1	0.9	.493	.339	.365	.328	.381	.387	.195	.059	.199	.380
	0.2	0.9	.567	.507	.580	.34	.382	.388	.197	.068	.192	.382
	0.1	0.8	.631	.620	.607	.362	.393	.390	.198	.087	.199	.391
	0.2	0.8	.789	.732	.75	.370	.393	.398	.199	.112	.21	.393

Table 3

Bias of PPWM estimates of parameters from left censored samples for GP distribution

n	Censoring level	$\hat{\alpha}$					$\hat{k}$				
	c	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
15	0.1	.34	-.002	-.203	-.336	-.431	.303	.102	-.066	-.243	-.418
	0.2	.354	-.004	-.205	-.341	-.438	.304	.106	-.074	-.247	-.43
	0.3	.362	-.008	-.206	-.345	-.449	.319	.122	-.076	-.249	-.437
	0.4	.377	.05	-.216	-.358	-.459	.332	.139	-.081	-.257	-.454
20	0.1	.336	-.002	-.201	-.336	-.431	.292	.085	-.064	-.242	-.407
	0.2	.351	-.004	-.201	-.339	-.437	.302	.095	-.074	-.243	-.42
	0.3	.359	-.006	-.203	-.343	-.445	.311	.104	-.075	-.248	-.429
	0.4	.367	-.014	-.214	-.358	-.458	.321	.12	-.079	-.257	-.44
30	0.1	.333	.002	-.199	-.335	-.431	.279	.073	-.060	-.217	-.401
	0.2	.35	-.003	-.201	-.339	-.432	.294	-.087	-.062	-.225	-.408
	0.3	.356	-.006	-.204	.342	-.442	.298	.02	-.074	-.236	-.422
	0.4	.366	-.01	-.212	-.353	-.453	.319	.113	-.079	-.253	-.436
50	0.1	.327	-.001	-.188	-.332	-.428	.256	.059	-.053	-.201	-.397
	0.2	.346	-.002	-.192	-.334	-.432	.294	.062	-.071	-.212	-.402
	0.3	.354	-.005	-.196	-.34	-.44	.272	.074	-.073	-.221	-.408
	0.4	.361	-.009	-.21	-.348	-.552	.309	.096	-.078	-.250	-.415

Table 4

RMSE of PPWM estimates of parameters from left censored samples for GP distribution

n	Censoring level	$\hat{\alpha}$					$\hat{k}$				
	c	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
15	0.1	.756	.337	.281	.368	.446	.332	.169	.139	.26	.423
	0.2	.758	.33	.29	.373	.452	.332	.17	.141	.268	.438
	0.3	.832	.345	.3	.376	.466	.345	.175	.143	.271	.447
	0.4	.958	.37	.318	.39	.478	.359	.175	.148	.28	.467
20	0.1	.683	.311	.271	.354	.438	.324	.16	.133	.242	.411
	0.2	.729	.301	.272	.359	.444	.331	.166	.138	.261	.427
	0.3	.74	.321	.281	.367	.453	.338	.169	.141	.268	.437
	0.4	.731	.307	.3	.383	.468	.354	.172	.144	.275	.45
30	0.1	.710	.227	.247	.35	.430	.294	.153	.127	.227	.402
	0.2	.720	.231	.254	.355	.437	.329	.156	.139	.238	.411
	0.3	.729	.246	.256	.36	.452	.337	.162	.140	.252	.428
	0.4	.737	.264	.267	.379	.469	.346	.165	.142	.273	.445
50	0.1	.56	.186	.233	.345	.435	.294	.139	.11	.206	.398
	0.2	.502	.19	.236	.346	.443	.308	.145	.122	.219	.403
	0.3	.544	.194	.246	.356	.447	.319	.148	.131	.233	.411
	0.4	.564	.214	.253	.363	.458	.326	.155	.134	.264	.42

Table 5

Bias of PPWM estimates of parameters from right censored samples for GP distribution

n	Censoring level	$\hat{\alpha}$					$\hat{k}$				
	d	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
15	0.9	-.16	-.278	-.517	-.537	-.635	.398	.197	-.013	-.2	-.4
	0.8	-.802	-.837	-.698	-.877	-.889	.395	.199	-.049	-.202	-.402
	0.7	-.887	-.907	-.794	-.959	-.942	.395	.201	-.037	-.201	-.404
	0.6	-.964	-.974	-.905	-.985	-.984	.398	.2	-.12	-.202	-.4
20	0.9	-152	-.176	-.492	-.333	-.449	.392	.192	-.012	-.202	-.4
	0.8	-.668	-.686	-.62	-.815	-.852	.389	.193	-.026	-.206	-.44
	0.7	-.873	-.878	-.777	-.924	-.931	.394	.194	-.033	-.201	-.403
	0.6	-.894	-.914	-.877	-.95	-.949	.398	.199	-.07	-.201	-.404
30	0.9	-.02	-.143	-.471	-.234	-.314	.374	.169	-.009	-.204	-.403
	0.8	-.51	-.534	-.573	-.816	-.733	.388	.192	-.031	-.206	-.406
	0.7	-.796	-.791	-.749	-.808	-.876	.387	.188	-.023	-.206	-.405
	0.6	-.879	-.903	-.869	-.916	-.919	.398	.193	-.031	-.202	-.404
50	0.9	-.001	-.078	-.468	-.231	-.265	.356	.165	-.005	-.208	-.43
	0.8	-.674	-.339	-.558	-.518	-.564	.384	.18	-.021	-.211	-.411
	0.7	-.663	-.726	-.692	-.744	-.764	.386	.184	-.022	-.213	-.412
	0.6	-.787	-.855	-.828	-.878	-.851	.393	.191	-.024	-.202	-.4

Table 6

RMSE of PPWM estimates of parameters from right censored for GP distribution

n	Censoring level	$\hat{\alpha}$					$\hat{k}$				
	c	-0.4	-0.2	0	0.2	0.4	-0.4	-0.2	0	0.2	0.4
15	0.9	.71	.666	.725	.716	.735	.397	.199	.072	.204	.404
	0.8	.934	.936	.839	.946	.937	.397	.199	.094	.208	.406
	0.7	.966	.967	.9	.978	.968	.398	.2	.108	.211	.407
	0.6	.986	.991	.955	.993	.991	.399	.201	.148	.223	.418
20	0.9	.616	.561	.7	.534	.601	.395	.199	.066	.203	.4
	0.8	.881	.869	.792	.898	.917	.394	.198	.093	.207	.403
	0.7	.948	.95	.777	.961	.964	.396	.199	.102	.203	.404
	0.6	.966	.968	.937	.978	.973	.398	.2	.131	.219	.415
30	0.9	.564	.533	.679	.411	.435	.384	.194	.058	.202	.4
	0.8	.798	.689	.752	.816	.839	.392	.196	.062	.203	.403
	0.7	.927	.868	.749	.903	.937	.398	.197	.086	.21	.404
	0.6	.953	.933	.931	.961	.96	.397	.198	.127	.214	.41
50	0.9	.617	.47	.665	.4	.353	.356	.191	.042	.202	.4
	0.8	.832	.659	.735	.698	.711	.393	.195	.061	.202	.402
	0.7	.831	.904	.828	.863	.859	.387	.197	.085	.203	.403
	0.6	.911	.963	.911	.936	.921	.397	.197	.101	.204	.408

From a numerical results, it is clear that as the censoring threshold increases the bias and RMSE of the estimated parameters increases, while as The sample size increases the bias and RMSE of the estimated parameters decreases. For all simulated cases , we observe that the bias and RMSE for the left censoring PPWM is smaller than that of the right and doubly censoring PPWM.

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