Modified Goodness of Fit Tests for the Weibull Distribution Based on Moving Extreme Ranked Set Sampling

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Abstract

Moving extreme ranked set sampling is a very useful modified version of the usual ranked set sampling that allows for increasing the set size without introducing too much ranking error. This article deals with modified empirical distribution function goodness of fit tests for Weibull distribution based on moving extreme ranked set sampling. Tables of critical values for the modified Kolmogrov-Smirnov, Cramer-von-Mises, Anderson-Darling, Watson and Kuiper goodness of fit tests for Weibull distribution with unknown parameters are created. Functional relationship between the critical values of these test statistics are examined for each set size, number of cycles and significance level. Powers of test statistics for a number of alternative distributions are given through a simulation. Furthermore, power efficiencies of these test statistics based on moving extreme ranked set sampling relative to simple random sampling are created for the same sample size. The resulting of power efficiencies showed that the modified tests under moving extreme ranked set sampling are more efficient than their corresponding in simple random sampling. In addition the Watson test statistics has the highest efficiency for all alternative hypotheses.

Keywords: Moving extreme Ranked Set Sampling; Simple Random Sample; Anderson-Darling test statistic; Cramer-von Mises test statistic; Critical values; Power test; Kolmogrov-Smirnov statistic, Watson statistic.

1. Introduction

Ranked set sampling (RSS) is a development that enables one to provide more structure to the collected sample items. RSS was introduced by McIntyre (1952) for estimating pasture yields. This technique is useful for cases when the variable of interest can be more easily ranked than quantified. RSS has many applications in different fields such as agricultural, medical and biological areas.

There are two factors that affect the efficiency of RSS, the set size and the ranking errors. The larger the set size, the more the efficiency of RSS. Thus the larger the set size, the more troublesome in visual ranking and ranking errors for this reason several authors modified RSS to reduce the error in ranking and to make visual ranking tractable by an experimenter. Samawi *et al.* (1996) investigated extreme ranked set sample (ERSS), i.e., they quantify the smallest and the largest order statistics instead of detailed ranking. Al-Odat and Al-Saleh (2001) introduced moving extreme ranked set sampling (MERSS). This method uses only extremes with varied set size to reduce error in ranking. MERSS does not need a complete ranking but RSS needs the ranking of all elements of each set. Samawi and Al-Saleh (2013) introduced two types of MERSS which are based on the maximum and minimum order observations and denoted by $MERSS_{max}$ and $MERSS_{min}$ respectively. They provided estimation of the odds ratio between two independent groups using two types of MERSS. Theoretical properties of the suggested estimator are derived and compared with its counterpart estimator using simple random sampling (SRS). They found that the estimator based on MERSS is always valid and has some advantages over that based on SRS. The MERSS procedure as described by Samawi and Al-Saleh (2013)can be summarized as follows:

Step 1: Select *m* random samples of size 1, 2, 3… *m*, respectively.

Step 2: Identify the maximum of each set, visually or by any cost free method without actual measurement of the variable of interest.

Step 3: Measure accurately the selected judgment identified maxima.

Step 4: Repeat the above steps *r* times in order to obtain a sample of large size $n = rm$. This sample will be denoted by $\text{MERSS}_{\text{max}}$.

Step 5: Repeat the above steps through identifying the minimum of each set instead of the maximum to have a moving extreme ranked set sample as $\text{MERSS}_{\text{min}}$.

Goodness-of-fit (GOF) tests are designed to measure how well the observed sample data fits some proposed model. One class of GOF tests that can be used consists of tests based on the distance between the empirical and hypothesized distribution functions. Five of the known tests in this class are Kolmogrov-Smirnov, Cramer-von-Mises, Anderson-Darling, Watson and Kuiper tests. These tests are valid when there are no unknown parameters in the hypothesized distribution. These tests become extremely conservative if they are used in case where unknown parameters must be estimated from the sample data.

Through the last two decades, the goodness of fit tests based on data collected via RSS technique and its modifications have not taken the attention of authors. Stockes and Sager (1988) showed that the empirical distribution function of RSS is an unbiased estimator for the population distribution function and has greater precision even if the ranking is imperfect. Then, they proposed a Kolmogrov-Smirnov GOF test based on the empirical distribution function (EDF). Al-Subh *et al*. (2009) gave a comparison study for the power of a set of empirical distribution function goodness of fit tests for the logistic distribution under SRS and RSS. Ibrahim *et al.* (2009) proposed ERSS method to improve the power of empirical distribution function GOF tests for logistic distribution through simulation study. Shahabuddin *et al.* (2009) investigated the performance of several GOF tests under SRS and RSS. Hassan (2012) established tables of critical values for the exponentiated Pareto distribution under ERSS. She investigated the power of the modified test statistics under ERSS and SRS for a number of alternative distributions and showed that the modified tests under ERSS are more efficient than their corresponding in SRS.

In the literature, there were no studies that had been performed about the GOF tests based on MERSS. Therefore, the main objective in this article is to create Tables of critical values for the Weibull distribution under $\text{MERSS}_{\text{min}}$ and $\text{MERSS}_{\text{max}}$. The power comparisons and the efficiency of a set of modified EDF tests are investigated for a number of alternative distributions based on $\text{MERSS}_{\text{min}}$, $\text{MERSS}_{\text{max}}$ and SRS.

This article is organized as follows. In section 2, maximum likelihood estimator of the unknown parameters from Weibull distribution based on $\text{MERSS}_{\text{min}}$ and $\text{MERSS}_{\text{max}}$ is obtained. Section 3 deals with the set of modified EDF goodness of fit tests under SRS and two types of MERSS. Section 4 generates the percentage points for the modified test statistics through simulation technique. In the same section response functions that give the percentage points for the modified test statistics are obtained to avoid using a large number of tables of critical values.

Section 5 deals with the power efficiency for the modified test statistics under $MERSS_{min}$ and $MERSS_{max}$ relative to SRS. Finally concluding remarks are presented in Section 6.

2. Maximum Likelihood Estimation

In this section the maximum likelihood estimators (MLEs) of the unknown parameters for the Weibull distribution will be obtained based on $MERSS_{max}$ and $MERSS_{min}$.

The Weibull distribution is an important distribution for modeling and lifetime data analysis in biological, medical and engineering sciences. It can therefore model a great variety of data and life characteristics. It is used extensively in reliability applications to model failure times. The cumulative distribution function (CDF) and probability density function (PDF) of the Weibull distribution are given, respectively, by

$$F(x\ ;\lambda,\alpha)=1-e^{-(\frac{x}{\lambda})^{\alpha}}\quad ;x>0,\alpha>0,\lambda>0, \tag{2.1}$$

and,

$$f\ (x\ ;\lambda,\alpha)=\frac{\alpha}{\lambda^{\alpha}}\ x^{\ \alpha-1}e^{-(\frac{x}{\lambda})^{\alpha}}\quad ;x\ >0,\alpha>0,\lambda>0, \tag{2.2}$$

2.1 MLE Based on $MERSS_{max}$

Let $\{X_{(m:m)j},X_{(m-1:m-1)j},...,X_{(1:1)j}\}$, for $j=1,...,r$, be a $MERSS_{max}$ of size $n = rm$, where m is the set size with r number of cycles. If judgement ranking error are accurate then $x_{i:i}$, for $i=1,...,m$, has the same distribution as the largest order statistic of a SRS of size i from PDF (2.1), therefore

$$f\ (x_{(i:i)j})=\frac{i\ \alpha}{\lambda^{\alpha}}x_{(i:i)j}^{\alpha-1}\ e^{-(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}\ [1-e^{-(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}]^{i-1},$$

The likelihood function of the sample $\{X_{(m:m)j},X_{(m-1:m-1)j},...,X_{(1:1)j}\}$ is given by

$$l(\alpha,\lambda)=\prod_{j=1}^{r}\prod_{i=1}^{m}\ \frac{i\ \alpha}{\lambda^{\alpha}}\ x_{(i:i)j}^{\alpha-1}\ e^{-(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}\ [1-e^{-(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}]^{i-1}, \tag{2.3}$$

The log likelihood function $\ell\equiv\ln l(\alpha,\lambda)$ is given by

$$\ell=\sum_{j=1}^{r}\sum_{i=1}^{m}\{\ln i+\ln\alpha-\alpha\ln\lambda+(\alpha-1)\ln x_{(i:i)j}-(\frac{x_{(i:i)j}}{\lambda})^{\alpha} \tag{2.4}$$

$$+(i-1)\ln[1-e^{-(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}]\}.$$

The first partial derivatives of the logarithm of the likelihood function with respect to α and λ , and the following normal equations, are obtained as follows

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+\sum_{j=1}^{r}\sum_{i=1}^{m}\ln(\frac{x_{(i:i)j}}{\lambda})-\sum_{j=1}^{r}\sum_{i=1}^{m}(\frac{x_{(i:i)j}}{\lambda})^{\alpha}\ln(\frac{x_{(i:i)j}}{\lambda})$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{m}\frac{(i-1)}{(e^{(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}-1)}(\frac{x_{(i:i)j}}{\lambda})^{\alpha}\ln(\frac{x_{(i:i)j}}{\lambda})=0, \tag{2.5}$$

$$\frac{\partial \ell}{\partial \lambda}=-\frac{n\,\alpha}{\lambda}+\sum_{j=1}^{r}\sum_{i=1}^{m}\{\alpha\,(\frac{x_{(i:i)j}^{\alpha}}{\lambda^{\alpha+1}})\}$$
$$-\sum_{j=1}^{r}\sum_{i=1}^{m}\{\frac{(i-1)}{(e^{(\frac{x_{(i:i)j}}{\lambda})^{\alpha}}-1)}\alpha\,(\frac{x_{(i:i)j}^{\alpha}}{\lambda^{\alpha+1}})\}=0. \tag{2.6}$$

Obviously, it is difficult to obtain a closed form solution to the non linear equations (2.5) and (2.6). Therefore, an iterative procedure is applied to solve these equations numerically using MathCAD (14) program.

2.2 MLE Based on MERSS$_{min}$

Let $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$ be MERSS$_{min}$ of size $n = rm$. If judgement ranking error are accurate then, for $i=1,...,m$, $y_{1:i}$ has the same distribution as the 1st order statistics of a SRS of size i from PDF (2.1), therefore

$$f\ (y_{(1:i)j})=\frac{i\ \alpha}{\lambda^{\alpha}}\,y_{(1:i)j}^{\alpha-1}\ e^{-i\,(\frac{y_{(1:i)j}}{\lambda})^{\alpha}}$$

The likelihood function of the sample $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$ is given by

$$l(\alpha,\lambda)=\prod_{j=1}^{r}\prod_{i=1}^{m}\frac{i\ \alpha}{\lambda^{\alpha}}\,y_{(1:i)j}^{\alpha-1}\ e^{-i\,(\frac{y_{(1:i)j}}{\lambda})^{\alpha}}$$

The log likelihood function $\ell \equiv \ln l(\alpha,\lambda)$ is given by

$$\ell=\sum_{j=1}^{r}\sum_{i=1}^{m}\{\ln i+\ln\alpha-\alpha\ln\lambda+(\alpha-1)\ln y_{(1:i)j}-i\,(\frac{y_{(1:i)j}}{\lambda})^{\alpha}\}.$$

Furthermore, the normal equations from maximum likelihood for MERSS$_{min}$ are as follows

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+\sum_{j=1}^{r}\sum_{i=1}^{m}\ln(\frac{y_{(1:i)j}}{\lambda})-\sum_{j=1}^{r}\sum_{i=1}^{m}i\,(\frac{y_{(1:i)j}}{\lambda})^{\alpha}\ln(\frac{y_{(1:i)j}}{\lambda})=0, \tag{2.7}$$

$$\frac{\partial \ell}{\partial \lambda}=\sum_{j=1}^{r}\sum_{i=1}^{m}(i\ \ \alpha\,(\frac{y_{(1:i)j}^{\alpha}}{\lambda^{\alpha+1}}))-\frac{n\,\alpha}{\lambda}=0. \tag{2.8}$$

An iterative procedure is applied to solve equations (2.7) and (2.8), numerically using MathCAD (14) program.

3. Modified EDF Goodness of Fit Tests

In this section modified EDF goodness of fit tests based on SRS, $MERSS_{min}$ and $MERSS_{max}$ will be discussed

3.1 Tests Based on SRS

A goodness of fit test based on the EDF, where the parameters are estimated is called modified goodness of fit test. The objective is to test the statistical hypothesis

$$H_0 : F(x) = F_0(x) \ \forall x, \quad \text{vs} \quad H_1 : F(x) \neq F_0(x) \quad \text{for some } x, \tag{3.1}$$

where $F_0(x)$ is a hypothesized distribution function based on a random sample $X_1, X_2, ..., X_n$ from the distribution function for Weibull distribution with two unknown parameters defined in (2.2).

The following set of the modified EDF goodness of fit tests defined as follows

a) The Kolmogrov smirnov (KS) test statistic D is

$$D = \max_{1\le i\le n}\{\max_{1\le i\le n}[\frac{i}{n} - F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda})], \max_{1\le i\le n}[F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) - \frac{i-1}{n}]\} \tag{3.2}$$

b) The Kuiper test statistic V is a modification of (KS) and takes the following form

$$V = \{\max_{1\le i\le n}[\frac{i}{n} - F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda})] + \max_{1\le i\le n}[F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) - \frac{i-1}{n}]\} \tag{3.3}$$

c) The Cramer-von Mises (CvM) statistic W^2 is presented by the following formula

$$W^2 = (\frac{1}{12n}) + \sum_{i=1}^{n}[F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}) - (\frac{2i-1}{2n})^2] \tag{3.4}$$

d) The Watson statistic U^2 is a modification of (CvM) statistic and takes the following form

$$U^2 = W^2 - n(\frac{\sum_{i=1}^{n} F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda})}{n} - \frac{1}{2})^2 \tag{3.5}$$

e) The Anderson-Darling (AD) statistic A^2 takes the following form

$$A^2 = -n - \frac{1}{n}\{\sum_{i=1}^{n}(2i-1)[\ln(F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}))] + [\ln(1 - F_0(x_{(i)}, \hat{\alpha}, \hat{\lambda}))]]\} \tag{3.6}$$

Let us denote the test statistics (3.2)-(3.6) by T, under SRS.

3.2 Tests Based on MERSS$_{max}$

To test the hypothesis based on MERSS$_{max}$, Let $\{X_{(m:m)j}, X_{(m-1:m-1)j}, ..., X_{(1:1)j}\}$ be a random sample selected via the largest order statistic. According to Al-Subh *et al.* (2009), testing the hypotheses

$H_0 : F(x) = F_0(x)\ \forall x$, vs $H_1 : F(x) \neq F_0(x)$ for some x, is equivalent to testing the hypotheses,

$$H_0^* : G(x) = G_0(x)\ \forall x, \quad \text{vs} \quad H_1^* : G(x) \neq G_0(x) \quad \text{for some } x, \tag{3.7}$$

where, $G(x)$ and $G_0(x)$ are the CDF's of the MERSS$_{max}$ of random samples chosen from $F(x)$ and $F_0(x)$ respectively, $i = 1,...,n$.

3.3 Tests Based on MERSS$_{min}$

To test the hypothesis based on MERSS$_{min}$, Let $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$ be a random sample selected via the first order statistic. According to Al-Subh et al. (2009), testing the hypotheses

$H_0 : F(y) = F_0(y)\ \forall y$, vs $H_1 : F(y) \neq F_0(y)$ for some y, is equivalent to testing the hypotheses,

$$H_0^{**} : K(y) = K_0(y)\ \forall y, \quad \text{vs} \quad H_1^{**} : K(y) \neq K_0(y) \quad \text{for some } y, \tag{3.8}$$

where, $K(y)$ and $K_0(y)$ are the CDF's of the MERSS$_{min}$ of random samples chosen from $F(y)$ and $F_0(y)$ respectively, $i = 1,...,n$.

Thus the goodness of fit tests for the hypothesis (3.7) and (3.8), denoted by T_1^* for MERSS$_{max}$ and T_2^* for MERSS$_{min}$, can be performed using the test statistic T as defined in the beginning Section, but by using the data $\{X_{(m:m)j}, X_{(m-1:m-1)j}, ..., X_{(1:1)j}\}$ and $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$ respectively.

4. Percentage Points of the Modified Test Statistics

The aim in this section is to obtain percentage points of T_i^*, $i = 1,2$, for the Weibull distribution based on MERSS$_{min}$ and MERSS$_{max}$. Monte Carlo simulation is used to create critical values for the proposed test statistics mentioned in Section 3 for the Weibull distribution with unknown parameters using the following steps:

Step 1: Generate a sample from the Weibull distribution with $\alpha = 0.5$ and $\lambda = 1$, for set sizes $m = 2,3,4,5$, and cycles $r = 3,7,9$ using the i^{th} order statistic, then $\{X_{(m:m)j}, X_{(m-1:m-1)j}, ..., X_{(1:1)j}\}$ be a random sample generated based on MERSS$_{max}$ and $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$ be a random sample generated based on MERSS$_{min}$.

Step 2: This random sample is used to estimate the unknown parameters α, λ using maximum likelihood method of estimation by solving the non linear equations (2.5) and (2.6) based on MERSS$_{max}$ and equations (2.7) and (2.8) based on MERSS$_{min}$.

Step 3: The resulting MLEs of the unknown parameters are used to determine the hypothesized CDF of the Weibull distribution.

Step 4: Obtain the EDF for $MERSS_{max}$, as

$$\hat{F}_{MERSS_{\max}}(x)=\frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} I(X_{(i:i)j}\le x), \qquad I(X_{(i:i)j})=\begin{cases}1 & X_{(i:i)j}\le x\\ 0 & otherwise\end{cases} \tag{4.1}$$

Similarly, obtain the EDF for $MERSS_{min}$, as

$$\hat{F}_{MERSS_{\min}}(y)=\frac{1}{n}\sum_{j=1}^{r}\sum_{i=1}^{m} I(Y_{(1:i)j}\le y), \qquad I(Y_{(1:i)j})=\begin{cases}1 & Y_{(1:i)j}\le y\\ 0 & otherwise\end{cases} \tag{4.2}$$

Step 5: Use steps 3 and 4 to calculate the modified KS, V, CvM, AD and U test statistics.

Step 6: This procedure is repeated 5000 times, thus generating 5000 independent values of the appropriate test statistics. These 5000 values are then arranged in an ascending order and the values of these test statistics are calculated at various significance levels, i.e., γ = 0.01, 0.025, 0.05, 0.10, 0.15, 0.20 and 0.25. The obtained values are the critical values for that particular test under each sample size used. (Tables of critical values under needed).

To avoid using a large number of tables of critical values, response functions are estimated which give the predicted critical values for each combination of the set size, number of cycles and significance levels. The response function for the critical values are obtained using the following equation

$$critical\ value_{test\ statistic}=b_0+b_1(set\ size)+b_2(number\ of\ cycles)+b_3(significance\ level\ \gamma) \tag{4.3}$$

The estimated coefficients of the response functions, t-statistics and their R^2 values are given in the following tables under $MERSS_{min}$ and $MERSS_{max}$ respectively using (4.3).

Table (1): Response function for different test statistics under $MERSS_{min}$

Test statistic	Estimated coefficients				
	b_0	b_1	b_2	b_3	R^2
D	0.412(64.9)	-0.025(-19.6)	-0.161(-25.2)	-0.284(-16.8)	0.923
V	0.690(68.3)	-0.039(-19.2)	-0.025(-24.9)	-0.501(-18.7)	0.925
W^2	0.129(20.6)	0.007(5.3)	0.002(3.02)	-0.403(-24.1)	0.851
A^2	0.745(21.8)	0.041(6.01)	0.013(3.8)	-2.203(-24.2)	0.855
U^2	0.123(20.3)	0.006(5.2)	0.002(2.9)	-0.381(-23.6)	0.846

The number in parentheses is the value of the t-statistic used to test the hypothesis that the true value of the coefficient is different from zero.

Table (2): Response function for different test statistics under $MERSS_{max}$

Test statistic	Estimated coefficients				R^2
	b_0	b_1	b_2	b_3	
D	0.439(59.6)	-0.020(-13.4)	-0.015(-19.9)	-0.369(-18.9)	0.896
V	0.716(57.9)	-0.026(-10.4)	-0.021(-16.6)	-0.688(-20.9)	0.884
W^2	0.080(5.7)	0.038(13.6)	0.015(10.8)	-0.753(-20.4)	0.869
A^2	0.026(0.16)	0.295(8.9)	0.210(12.6)	-6.094(-13.9)	0.800
U^2	0.062(5.4)	0.035(15.2)	0.014(11.7)	-0.653(-21.2)	0.883

The number in parentheses is the value of the t-statistic used to test the hypothesis that the true value of the coefficient is different from zero.

5. Power efficiency

In this section a power study is carried out to investigate the power of the modified test statistics to the null hypothesis under SRS, $MERSS_{min}$ and $MERSS_{max}$. The power of a test is useful in assessing the goodness of a test or in comparing competing tests. Power comparisons are made among KS, Kuiper, CvM, AD and Watson test statistics for the Weibull distribution with unknown parameters. In this study the null hypothesis H_0 is that the random sample comes from the Weibull distribution and the alternative hypothesis H_1 is that the sample follows some other distributions.

The power is determined by generating 5000 random samples for each of the four alternatives:

1. Normal distribution, denoted by N (3, 1).
2. Lognormal distribution, denoted by LN (3, 1).
3. Uniform distribution, denoted by U (0, 1).
4. Exponential distribution, denoted by Exp (2).

For each test, all test statistics are calculated and compared to its respective critical values and counted the number of rejections of the null hypothesis. The power results for the tests at the significance level γ =0.01 and 0.05.

The procedure for calculating the power of $T_i^*,\ i=1,2$, under the alternative distributions are as follows

Step (1): Let $\{X_{(m:m)j}, X_{(m-1:m-1)j}, ..., X_{(1:1)j}\}$ and $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$ be a random sample from the alternative distributions for largest and smallest order statistics respectively.

Step (2): Obtain the EDF as defined in (4.1) and (4.2).

Step (3): Calculate the value of $T_i^*,\ i=1,2$, as defined in tests from (3.2)-(3.6) but using data $\{X_{(m:m)j}, X_{(m-1:m-1)j}, ..., X_{(1:1)j}\}$ and $\{Y_{(1:m)j}, Y_{(1:m-1)j}, Y_{(1:m-2)j}, ..., Y_{(1:1)j}\}$, If the critical values of the test statistics for the alternative distribution exceed the corresponding critical values, then the null hypothesis H_0 will be rejected at the significance level (0.05 and 0.01).

Step (4): Repeat the above steps from (1-3) 5000 times to generate 5000 independent sets of the test statistics.

Step (5): The power of each test is obtained by counting the number of rejections of the null hypothesis divided by 5000. Figures 1 and 2 represent the power of each test statistics for each of four alternative distributions based on $MERSS_{min}$ and $MERSS_{max}$

Step (6): By the similar way the power of each test statistics is obtained under SRS but using the data $X_1, X_2, ..., X_n$.

Step (7): The efficiency of test statistics, T_i^*, $i = 1, 2$, under $MERSS_{max}$ and $MERSS_{min}$ relative to test statistics ,T , under SRS is calculated, where the relative efficiency is defined as the ratio of the powers ,

$$eff\ (T^*, T) = \frac{power\ of\ T_i^*}{power\ of\ T}, \quad i = 1, 2$$

The efficiency values of tests at the significance level $\gamma = 0.01$ and 0.05 are presented in Table (3) for $MERSS_{min}$ and in Table (4) for $MERSS_{max}$.

From the simulation results given in Tables (1-4), the following remarks may be observed.

1. For different significance levels and the same sample sizes, the change of critical values for all test statistics for $MERSS_{max}$ are greater than that the corresponding for $MERSS_{min}$. As the set size increases, the critical values for test statistics decrease monotonically, for Kolmogrov-Smirnov and Kuiper test statistics.
2. Based on $MERSS_{max}$ and $MERSS_{min}$ the efficiency of the all modified tests is greater than one which indicated that the power for test statistics under $MERSS_{max}$ and $MERSS_{min}$ is larger than the corresponding under SRS.
3. As the number of cycles increases, the efficiencies for each test statistics increase.
4. Anderson-Darling test statistics has the highest power and Watson test has the smallest power for all alternative hypotheses based on $MERSS_{max}$ and $MERSS_{min}$.
5. The power for $MERSS_{max}$ and $MERSS_{min}$ are the same as the number of cycles increase, which leads to their equality in their efficiencies. (See Figures (1) and (2))
6. Powers efficiency of modified EDF tests are broadly in the following order of descending power $U^2 \to V \to D \to W^2 \to A^2$

6. Conclusion

For different significance levels and sample sizes, the modified EDF tests of fit for Weibull distribution under $MERSS_{min}$, $MERSS_{max}$ and SRS are investigated. A power study is made using four alternative families of distributions based on SRS, $MERSS_{min}$ and $MERSS_{max}$. This study shows that the efficiency of GOF tests can be much improved if the sample is collected via the $MERSS_{min}$ or $MERSS_{max}$.

Furthermore, the modified EDF tests under $MERSS_{min}$ and $MERSS_{max}$ are more efficient than their corresponding in SRS. The efficiency for all tests varies for different values of set sizes and number of cycles. The critical values for the Anderson-Darling test statistic are greater than the other GOF tests in case of $MERSS_{max}$ and $MERSS_{min}$.

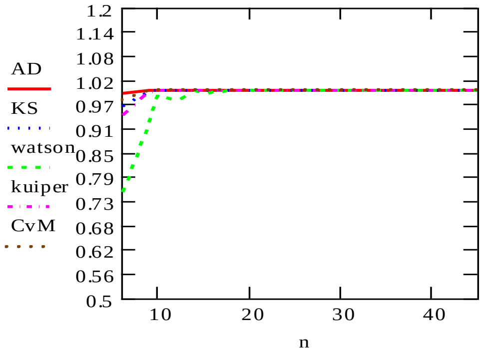
In general, the Watson statistic, Kolmogrov-Smirnov, and Kuiper tests appear to be the best EDF test statistics. The Anderson-Darling test statistic tends to be least powerful among the five EDF considered here. On the other hand, the Watson test statistic is the most powerful goodness-of-fit test among the competitors. The efficiency of the modified EDF tests increases as the sample size increases in most cases for both sampling schemes; $MERSS_{min}$ and $MERSS_{max}$.

Table (3): Efficiency of test statistics for Weibull distribution with estimated parameters based on $MERSS_{min}$.

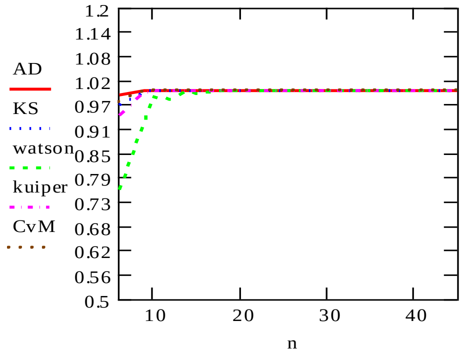
Number of Cycles (r)	Set Size (m)	Test Statistics	Significance level γ							
			0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
			Alternatives							
			exp (2)		LN (3, 1)		N (3, 1)		U (0,1)	
3	2	D	3.125	2.096	2.994	2.071	3.128	2.075	3.163	2.109
		V	3.432	2.424	3.295	2.339	3.462	2.391	3.485	2.442
		W^2	2.499	1.904	2.444	1.848	2.479	1.840	2.514	1.908
		A^2	2.427	1.832	2.442	1.807	2.381	1.784	2.432	1.830
		U^2	5.692	3.322	5.505	3.184	5.664	3.333	5.701	3.339
	3	D	3.525	2.316	3.529	2.269	3.468	2.259	3.543	2.277
		V	3.621	2.513	3.612	2.448	3.601	2.468	3.644	2.473
		W^2	2.706	1.957	2.701	1.930	2.664	1.910	2.743	1.928
		A^2	2.657	1.866	2.622	1.848	2.622	1.826	2.643	1.838
		U^2	8.010	4.348	7.396	4.182	7.546	4.178	7.923	4.300
	4	D	3.734	2.268	3.864	2.291	3.857	2.288	3.809	2.291
		V	3.777	2.431	3.895	2.452	3.857	2.500	3.824	2.437
		W^2	2.775	2.000	2.879	2.022	2.846	2.016	2.806	2.002
		A^2	2.571	1.876	2.625	1.894	2.584	1.883	2.632	1.901
		U^2	10.609	4.793	10.644	4.866	9.809	4.771	9.840	4.660
	5	D	3.968	2.463	3.876	2.410	3.906	2.370	3.968	2.463
		V	3.937	2.577	3.846	2.525	3.857	2.494	3.937	2.577
		W^2	3.021	2.041	2.924	2.033	2.956	2.045	3.021	2.041
		A^2	2.747	1.946	2.688	1.927	2.717	1.912	2.747	1.946
		U^2	11.805	5.161	11.435	4.892	11.926	5.151	11.866	5.172
7	2	D	3.831	2.227	3.774	2.262	3.876	2.237	3.831	2.309
		V	3.922	2.320	3.831	2.370	3.906	2.353	3.937	2.433
		W^2	2.695	1.916	2.710	1.957	2.725	1.916	2.747	1.976
		A^2	2.653	1.808	2.674	1.842	2.667	1.869	2.695	1.862
		U^2	9.881	4.464	9.980	4.464	9.784	4.566	10.396	4.566
	3	D	3.891	2.433	3.906	2.404	3.774	2.364	3.891	2.433
		V	3.968	2.445	4.000	2.415	3.861	2.387	3.968	2.445
		W^2	3.058	2.037	2.985	2.020	2.941	1.996	3.058	2.037
		A^2	2.959	1.916	2.890	1.898	2.874	1.845	2.959	1.916
		U^2	10.989	5.025	11.494	4.950	11.236	5.000	10.989	5.025
	4	D	4.237	2.525	4.098	2.481	4.049	2.469	4.237	2.525
		V	4.310	2.564	4.167	2.506	4.098	2.488	4.310	2.564
		W^2	2.985	2.110	2.933	2.088	2.907	2.045	2.985	2.110
		A^2	2.770	1.972	2.717	1.938	2.674	1.908	2.770	1.972
		U^2	12.987	5.435	12.500	5.348	12.346	5.076	12.987	5.435
	5	D	3.984	2.519	4.115	2.532	4.219	2.513	3.984	2.519
		V	4.049	2.564	4.167	2.604	4.274	2.584	4.049	2.564
		W^2	3.268	2.045	3.279	2.041	3.322	2.058	3.268	2.045
		A^2	2.994	1.946	3.058	1.957	3.106	1.953	2.994	1.946
		U^2	13.158	4.525	14.085	4.975	14.286	4.926	13.158	4.525
9	2	D	3.745	2.347	3.937	2.421	3.906	2.375	3.745	2.347
		V	3.759	2.404	3.984	2.519	3.891	2.457	3.759	2.404
		W^2	2.849	2.020	2.950	2.079	2.907	2.037	2.849	2.020
		A^2	2.770	1.916	2.841	1.946	2.717	1.912	2.770	1.916
		U^2	11.494	4.950	11.905	5.051	11.111	4.785	11.494	4.950
	3	D	4.167	2.532	4.098	2.457	4.202	2.500	4.167	2.532
		V	4.219	2.591	4.115	2.506	4.237	2.564	4.219	2.591
		W^2	3.021	2.096	2.985	2.088	3.077	2.096	3.021	2.096
		A^2	2.976	1.934	2.976	1.927	3.030	1.938	2.976	1.934
		U^2	13.333	5.051	12.821	5.464	12.346	5.128	13.333	5.051
	4	D	4.255	2.439	4.274	2.410	4.484	2.451	4.255	2.439
		V	4.292	2.451	4.274	2.421	4.505	2.457	4.292	2.451
		W^2	3.003	2.037	2.994	2.033	3.115	2.058	3.003	2.037
		A^2	2.857	1.934	2.882	1.938	3.003	1.957	2.857	1.934
		U^2	11.364	4.878	11.628	4.831	12.346	5.102	11.364	4.878
	5	D	3.922	2.273	4.049	2.381	3.953	2.326	3.922	2.273
		V	3.953	2.309	4.115	2.427	3.984	2.370	3.953	2.309
		W^2	3.067	1.965	3.215	2.070	3.175	2.024	3.067	1.965
		A^2	2.825	1.802	3.040	1.905	2.915	1.862	2.825	1.802
		U^2	13.889	4.651	14.085	4.878	12.821	4.630	13.889	4.651

Table (4): Efficiency of test statistics for Weibull distribution with estimated parameters based $MERSS_{max}$.

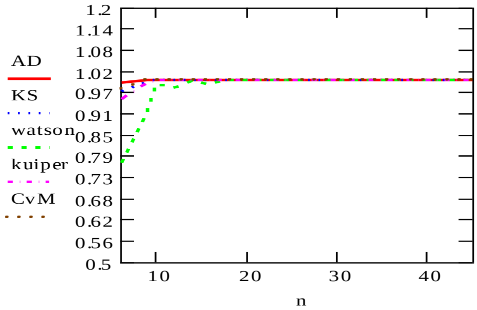
Number of Cycles (r)	Set Size (m)	Test Statistics	Significance level γ							
			0.01	0.05	0.01	0.05	0.01	0.05	0.01	0.05
			Alternatives							
			exp (2)		LN (3, 1)		N (3, 1)		U (0,1)	
3	2	D	3.034	2.059	2.909	2.028	2.987	2.013	2.983	2.037
		V	3.367	2.419	3.216	2.332	3.316	2.356	3.314	2.395
		W^2	2.436	1.886	2.379	1.835	2.399	1.819	2.420	1.885
		A^2	2.402	1.826	2.424	1.803	2.342	1.772	2.398	1.821
		U^2	4.860	3.101	4.703	2.929	4.745	3.048	4.879	3.066
	3	D	3.421	2.307	3.407	2.253	3.359	2.250	3.432	2.268
		V	3.518	2.503	3.491	2.433	3.495	2.460	3.533	2.463
		W^2	2.670	1.957	2.658	1.923	2.632	1.906	2.704	1.923
		A^2	2.644	1.864	2.612	1.847	2.609	1.826	2.630	1.836
		U^2	6.324	4.029	5.730	3.818	6.028	3.863	6.154	3.944
	4	D	3.715	2.261	3.845	2.284	3.834	2.284	3.790	2.289
		V	3.754	2.426	3.871	2.447	3.830	2.495	3.801	2.434
		W^2	2.769	1.996	2.873	2.020	2.840	2.014	2.801	2.002
		A^2	2.571	1.876	2.625	1.894	2.584	1.883	2.632	1.901
		U^2	9.230	4.502	9.149	4.537	8.436	4.439	8.457	4.344
	5	D	3.960	2.461	3.864	2.407	3.898	2.367	3.960	2.461
		V	3.929	2.575	3.835	2.523	3.853	2.491	3.929	2.575
		W^2	3.018	2.041	2.924	2.033	2.956	2.045	3.018	2.041
		A^2	2.747	1.946	2.688	1.927	2.717	1.912	2.747	1.946
		U^2	10.402	4.953	9.965	4.655	10.506	4.927	10.256	4.938
7	2	D	3.831	2.227	3.774	2.262	3.876	2.237	3.831	2.309
		V	3.922	2.320	3.831	2.370	3.906	2.353	3.937	2.433
		W^2	2.695	1.916	2.710	1.957	2.725	1.916	2.747	1.976
		A^2	2.653	1.808	2.674	1.842	2.667	1.869	2.695	1.862
		U^2	9.782	4.460	9.910	4.460	9.696	4.557	10.323	4.557
	3	D	3.891	2.433	3.906	2.404	3.774	2.364	3.891	2.433
		V	3.968	2.445	4.000	2.415	3.861	2.387	3.968	2.445
		W^2	3.058	2.037	2.985	2.020	2.941	1.996	3.058	2.037
		A^2	2.959	1.916	2.890	1.898	2.874	1.845	2.959	1.916
		U^2	10.989	5.025	11.483	4.950	11.236	5.000	10.989	5.025
	4	D	4.237	2.525	4.098	2.481	4.049	2.469	4.237	2.525
		V	4.310	2.564	4.167	2.506	4.098	2.488	4.310	2.564
		W^2	2.985	2.110	2.933	2.088	2.907	2.045	2.985	2.110
		A^2	2.770	1.972	2.717	1.938	2.674	1.908	2.770	1.972
		U^2	12.987	5.435	12.500	5.348	12.346	5.076	12.987	5.435
	5	D	3.984	2.519	4.115	2.532	4.219	2.513	3.984	2.519
		V	4.049	2.564	4.167	2.604	4.274	2.584	4.049	2.564
		W^2	3.268	2.045	3.279	2.041	3.322	2.058	3.268	2.045
		A^2	2.994	1.946	3.058	1.957	3.106	1.953	2.994	1.946
		U^2	13.158	4.525	14.085	4.975	14.286	4.926	13.158	4.525
9	2	D	3.745	2.347	3.937	2.421	3.906	2.375	3.745	2.347
		V	3.759	2.404	3.984	2.519	3.891	2.457	3.759	2.404
		W^2	2.849	2.020	2.950	2.079	2.907	2.037	2.849	2.020
		A^2	2.770	1.916	2.841	1.946	2.717	1.912	2.770	1.916
		U^2	11.494	4.950	11.905	5.051	11.111	4.785	11.494	4.950
	3	D	4.167	2.532	4.098	2.457	4.202	2.500	4.167	2.532
		V	4.219	2.591	4.115	2.506	4.237	2.564	4.219	2.591
		W^2	3.021	2.096	2.985	2.088	3.077	2.096	3.021	2.096
		A^2	2.976	1.934	2.976	1.927	3.030	1.938	2.976	1.934
		U^2	13.333	5.051	12.821	5.464	12.346	5.128	13.333	5.051
	4	D	4.255	2.439	4.274	2.410	4.484	2.451	4.255	2.439
		V	4.292	2.451	4.274	2.421	4.505	2.457	4.292	2.451
		W^2	3.003	2.037	2.994	2.033	3.115	2.058	3.003	2.037
		A^2	2.857	1.934	2.882	1.938	3.003	1.957	2.857	1.934
		U^2	11.364	4.878	11.628	4.831	12.346	5.102	11.364	4.878
	5	D	3.922	2.273	4.049	2.381	3.953	2.326	3.922	2.273
		V	3.953	2.309	4.115	2.427	3.984	2.370	3.953	2.309
		W^2	3.067	1.965	3.215	2.070	3.175	2.024	3.067	1.965
		A^2	2.825	1.802	3.040	1.905	2.915	1.862	2.825	1.802
		U^2	13.889	4.651	14.085	4.878	12.821	4.630	13.889	4.651



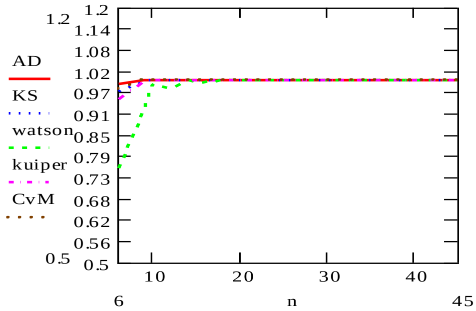
The exponential distribution (2)



The lognormal distribution (3,1)

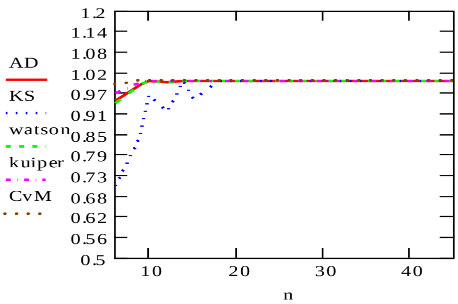


The normal distribution (3,1)

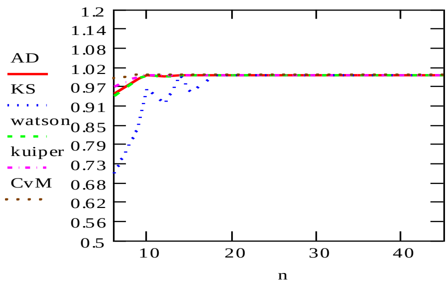


The uniform distribution (0,1)

Figure (1): Power comparison for Weibull distribution for different alternative distributions under $MERSS_{min}$



The exponential distribution (2)



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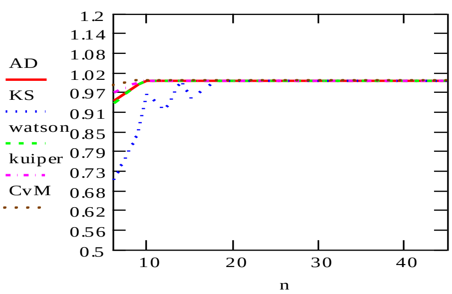
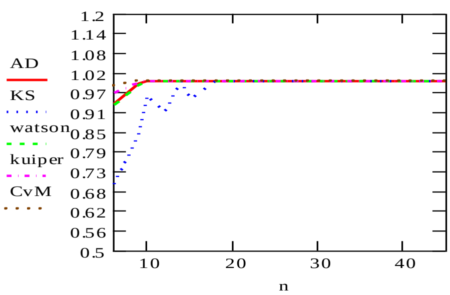


The normal distribution (3,1)



The uniform distribution (0,1)

Figure (2): Power comparison for Weibull distribution for different alternative distributions under $MERSS_{max}$

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