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The Compound Family of Generalized Inverse Weibull Power Series Distributions

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Authors' contributions

This work was carried out in collaboration between all authors. Author ASH designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Authors SMA and KAA managed the analyses of the study and literature searches. All authors read and approved the final manuscript.

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ABSTRACT

Compounding a continuous lifetime distribution with a discrete one is a useful technique for constructing flexible distributions to facilitate better modeling of lifetime data. In this paper, a new family of lifetime distributions, called the generalized inverse Weibull power series distribution is introduced. This new family is obtained by compounding the generalized inverse Weibull and truncated power series distributions. This compounding procedure follows the same way that was previously carried out by [1]. This family contains several new distributions such as generalized inverse Weibull Poisson; inverse Weibull Poisson; inverse Rayleigh Poisson; inverse exponential Poisson; generalized inverse Weibull logarithmic; inverse Weibull logarithmic; inverse Rayleigh logarithmic; inverse exponential logarithmic; generalized inverse Weibull geometric; inverse Weibull geometric; inverse Rayleigh geometric and inverse exponential geometric as special cases.
The hazard rate function of the new family of distributions can be increasing, decreasing and bathtub-shaped. Several properties of the new family including; quantile, entropy, moments and distribution of order statistics are provided. The model parameters of the new family are estimated

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by the maximum likelihood method. The two new models namely; generalized inverse Weibull Poisson and the generalized inverse Weibull geometric distributions are studied in some details. Finally, applications to two real data sets are analyzed to illustrate the flexibility and potentiality of the new family.

Keywords: Generalized inverse Weibull distribution; power series distribution; distribution of minimum; entropy; quantile function; estimation.

1. INTRODUCTION

Numerous probability distributions do not provide adequate fits to real data in many practical situations. So, several distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions. Compounding lifetime distributions have been obtained by mixing the distribution when the lifetime can be expressed as the minimum of a sequence of independent and identically distributed random variables together with a discrete random variable. This idea was first pioneered in [1] by compounding the exponential random variable together with a geometric random variable. This distribution is known as the exponential geometric distribution. In the same manner, an exponential Poisson distribution has been introduced in [2] by compounding an exponential distribution together with Poisson distribution. While, in [3] generalized the exponential Poisson by including a power parameter in his distribution. The Weibull-geometric and Weibull-Poisson distributions which naturally extend the exponential geometric and exponential Poisson have been provided in [4] and [5].

In the last few years, several families of distributions have been proposed by compounding some useful lifetime and power series distributions. The exponential power series family of distributions with decreasing failure rate has been introduced in [6]; which contains as special cases the exponential Poisson, exponential geometric and exponential logarithmic distributions. A three-parameter Weibull power series distribution with decreasing, increasing, upside-down bathtub failure rate functions has been introduced in [7]. The generalized exponential power series distributions have been proposed in [8]. The compound class of extended Weibull power series distributions has been proposed in [9]. The class of Lindley power series distributions has been introduced in [10]. A new family of distributions has been defined in [11] by compounding the Burr XII and truncated power series distributions.

The inverse Weibull (IW) distribution is an important life time model in reliability and survival analysis (see [12]). The IW distribution can be used to model a variety of failure characteristics such as infant mortality, useful life, wear out period, relays, ball bearings, etc. The three-parameter generalized inverse Weibull (GIW) distribution is commonly used in the lifetime-literature and more flexible than the inverse Weibull distribution. The GIW distribution was introduced in [13] which extends to several distributions. The probability density function (pdf) of the GIW with shape parameter α and scale parameters γ and λ takes the following form

$$f(w;\gamma,\lambda,\alpha)=\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}},\quad w>0,\ \ \lambda,\gamma \text{ and } \alpha>0. \qquad (1)$$

The corresponding cumulative distribution function (cdf) is

$$F(w;\gamma,\lambda,\alpha)=e^{-\gamma(\frac{\lambda}{w})^{\alpha}}. \qquad (2)$$

The rth moment about zero for the GIW distribution is given by

$$\mu_r{}'=E(W^r)=\lambda^r\gamma^{\frac{r}{\alpha}}\ \Gamma\left(1-\frac{r}{\alpha}\right),\ r=1,2,\ldots \qquad (3)$$

where, $\Gamma(.)$ is the standard gamma function.

In this article, a new family of the generalized inverse Weibull power series (GIWPS) models is introduced by compounding the generalized inverse Weibull distribution together with power series distribution. Some of statistical properties of the GIWPS distribution such as moments, quantiles, Rényi entropy and order statistics are studied. In particular, two sub-models of the new family are derived and studied in some details. An application of the proposed family is illustrated using a real data set.

This paper is organized as follows. In Section 2, the generalized inverse Weibull power series distribution with its probability density, cumulative distribution, reliability and hazard rate functions are introduced. In the same section, some special sub-models are derived and two important propositions are introduced. In Section 3, some mathematical properties of the new family are derived. Estimation of the model parameters involved using the maximum likelihood method and some related inferences are discussed in Section 4. Two special models which are; the generalized inverse Weibull Poisson and the generalized inverse Weibull geometric distributions are investigated in Sections 5 and 6 respectively. Applications to two real data sets are presented in Section 7. Finally, some concluding remarks are addressed in Section 8.

2. CONSTRUCTION OF THE NEW FAMILY

In this section, the generalized inverse Weibull power series family of distributions is created. This new family is derived by compounding the generalized inverse Weibull and power series distributions.

Let $W_1, W_2, \ldots, W_z$ be Z identically independent distributed random variables having the GIW with probability density function (1). Suppose that Z has a zero truncated power series distribution with the following probability mass function

$$P_Z(Z=z;\theta)=\frac{a_z\theta^z}{c(\theta)},\quad z=1,2,3,\ldots\quad, \qquad (4)$$

where, $\theta>0$ is the scale parameter. The coefficients a_z's depend only on z, $c(\theta)=\sum_{z=1}^{\infty}a_z\theta^z$ is such that $c(\infty)$ is finite, $c'(.)$ and $c''(.)$ denote its first and second derivatives, respectively. Table 1 shows useful quantities of some power series distributions (truncated at zero) according to (4); such as the Poisson, logarithmic, geometric and binomial distributions. The properties of the power series class of distributions can be seen in [14].

Let $W_{(1)}$ = min{ $W_1, W_2, \ldots, W_z$}, and by assuming that; W and $Z's$ are independent, then the conditional probability density function of $W_{(1)}|Z=z$ is given by

$$f_{W_{(1)}|Z}(w|z;\gamma,\lambda,\alpha)=z\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^{z-1}. \qquad (5)$$

Table 1. Useful quantities for some power series distributions

Distribution	a_z	$c(\theta)$	$c'(\theta)$	$c''(\theta)$	$c^{-1}(\theta)$	θ
Poisson	$z!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\ln(1+\theta)$	$0<\theta<\infty$
Logarithmic	z^{-1}	$-\ln(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$0<\theta<1$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(1+\theta)^{-1}$	$0<\theta<1$
Binomial	$\binom{m}{z}$	$(1+\theta)^m-1$	$m(1+\theta)^{m-1}$	$\frac{m(m-1)}{(1+\theta)^{2-m}}$	$(\theta-1)^{\frac{1}{m}}-1$	$0<\theta<1$

The joint probability density function of $W_{(1)}$ and Z is obtained as follows

$$f_{W_{(1)}Z}(w,z;\gamma,\lambda,\alpha,\theta)=z\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^{z-1}\frac{a_z\theta^z}{c(\theta)}.$$

The probability density of the generalized inverse Weibull power series family of distributions is defined by the marginal density of $W_{(1)}$

$$g(w;\gamma,\lambda,\alpha,\theta)=\frac{\theta\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\,c'\left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right)}{c(\theta)},w>0,\alpha,\theta,\gamma,\lambda>0\,. \qquad (6)$$

The cumulative distribution function of the GIWPS distribution corresponding to (6) is obtained as follows

$$G(w;\gamma,\lambda,\alpha,\theta)=1-\frac{c\left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right)}{c(\theta)},\qquad w>0. \tag{7}$$

The random variable *W* following (6) with the set of parameters $\psi=(\gamma,\lambda,\alpha,\theta)$ is denoted by $W\sim GIWPS(\psi)$.

In addition, the reliability and hazard rate functions for the GIWPS distribution take, respectively, the following forms

$$R(w;\gamma,\lambda,\alpha,\theta)=\frac{c\left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right)}{c(\theta)}, \tag{8}$$

and

$$h(w;\gamma,\lambda,\alpha,\theta)=\frac{\theta\alpha\gamma\lambda^{\alpha}w^{-(\alpha+1)}e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\mathrm{c}'\left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right)}{c\left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right)}. \tag{9}$$

2.1 Useful Expansion

In this subsection, two important propositions will be provided. The first proposition indicates that the new family has the GIW distribution as a limiting case, whereas the second proposition provides a useful expansion for the pdf of the GIWPS family of distributions (6).

Proposition 1: *The GIW distribution with parameters* γ,λ and α *is a limiting special case of the GIWPS family of distributions as* $\theta\to 0^{+}$.

Proof:

$$\lim_{\theta\to 0^{+}}G(w;\gamma,\lambda,\alpha,\theta)=1-\lim_{\theta\to 0^{+}}\frac{\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)+{a_1}^{-1}\sum_{z=2}^{\infty}a_z\theta^{z-1}\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^{z-1}}{1+{a_1}^{-1}\sum_{z=2}^{\infty}a_z\theta^{z-1}},$$

and by using L'Hopital's rule, it follows that

$$\lim_{\theta\to 0^{+}}G(w;\gamma,\lambda,\alpha,\theta)=e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}},$$

which is the distribution function of the generalized inverse Weibull distribution as defined in (2).

Proposition 2: *The probability density function of the GIWPS can be expressed as an infinite mixture of the GIW with parameters* $\gamma j,\lambda$ *and* α, *which is given by:*

$$g(w;\gamma,\lambda,\alpha,\theta)=\sum_{z=1}^{\infty}\sum_{j=1}^{\infty}\binom{z}{j}P_Z(z;\theta)\,(-1)^{j-1}f(w,\gamma j,\lambda,\alpha),$$

where, $f(w,\gamma j,\lambda,\alpha)$ is the pdf of the GIW distribution with parameters, $\gamma j,\lambda$ and α.

Proof:

The pdf (6) can be rewritten as following

$$g(w;\gamma,\lambda,\alpha,\theta)=\frac{\theta\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}\sum_{z=1}^{\infty}a_z z\theta^{z-1}\sum_{i=0}^{\infty}\binom{z-1}{i}(-1)^i e^{-\gamma(i+1)\left(\frac{\lambda}{w}\right)^{\alpha}}}{c(\theta)},$$
$$=\sum_{z=1}^{\infty}\frac{a_z\theta^z}{c(\theta)}\sum_{i=0}^{\infty}z\binom{z-1}{i}(-1)^i(i+1)^{-1}\alpha\gamma(i+1)\lambda^{\alpha}w^{-\alpha-1}e^{-\gamma(i+1)\left(\frac{\lambda}{w}\right)^{\alpha}},$$
$$=\sum_{z=1}^{\infty}\sum_{j=1}^{\infty}\binom{z}{j}P_Z(z;\theta)(-1)^{j-1}f\,(w,\gamma j,\lambda,\alpha), \tag{10}$$

where, $f(w,\gamma j,\lambda,\alpha)$ is the pdf of the GIW distribution with parameters $\gamma j,\lambda$ and α.

2.2 Special Sub-models

The cdf (7) extends some distributions which have not been studied in the literature, the new sub models are; generalized inverse Weibull Poisson (GIWP); inverse Weibull Poisson (IWP); inverse Rayleigh Poisson (IRP); inverse exponential Poisson (IEP); generalized inverse Weibull logarithmic (GIWL); inverse Weibull logarithmic (IWL); inverse Rayleigh logarithmic (IRL); inverse exponential logarithmic (IEL), generalized inverse Weibull geometric (GIWG); inverse Weibull geometric (IWG); inverse Rayleigh geometric (IRG) and inverse exponential geometric (IEG). Table 2 gives the cdf of the new sub-models.

3. SOME MATHEMATICAL PROPERTIES

In this section, some mathematical properties of the GIWPS distribution including, quantile function, rth moment, Rényi entropy and distribution of order statistics will be derived.

3.1 Quantiles and Moments

The quantile function has been used in several statistical aspects such as the generating random numbers. The quantile function, say $Q(u)$ of W is given by

$$w=Q(u)=\lambda\left[\frac{-1}{\gamma}\ln\left\{1-\frac{c^{-1}[c(\theta)(1-u)]}{\theta}\right\}\right]^{\frac{-1}{\alpha}}, \tag{11}$$

where, u is a uniform random variable on the unit interval (0,1) and $c^{-1}(\theta)$ is the inverse function of $c(\theta)$.

Some of the most important features and characteristics of a distribution can be studied through its moments such as tendency, dispersion, skewness and kurtosis. Therefore, a general expression for the rth moment of the GIWPS distribution will be derived.

Proposition 3: *The rth moment about zero for a GIWPS distribution is given by*

$$\mu_r{}'=E(W^r)=\sum_{z=1}^{\infty}\sum_{j=1}^{\infty}P_Z(z;\theta)\binom{z}{j}(-1)^{j-1}(\gamma j)^{\frac{r}{\alpha}}\lambda^r\Gamma\left(1-\frac{r}{\alpha}\right),\alpha>r,\ r=1,2,\dots. \tag{12}$$

Proof: The rth moment of the GIWPS distribution is easily obtained by substituting the rth moment of the GIW distribution defined in (3), but with parameters $\gamma j,\lambda$ and α, in expression (10), then we obtain the result in (12).

Based on the first four moments of the GIWPS distribution, the measures of skewness (SK) and kurtosis (K) can be obtained from following relations respectively

$$SK=\frac{\mu_3{}'-3\mu_1{}'\mu_2{}'+2(\mu_1{}')^3}{(\mu_2{}'-(\mu_1{}')^2)^{\frac{3}{2}}},$$

and

$$K=\frac{\mu_4{}'-4\mu_1{}'\mu_3{}'+6(\mu_1{}')^2\mu_2{}'-3(\mu_1{}')^4}{(\mu_2{}'-(\mu_1{}')^2)^2},$$

where, $\mu_1{}',\mu_2{}',\ \mu_3{}'$ and $\mu_4{}'$ can be obtained from (12), by substituting $r=1,2,3$ and 4.

Table 2. Useful new lifetime distributions from the GIWPS family

$c(\theta)$	λ	α	γ	θ	Models	cdf
$e^{\theta}-1$	$\lambda>0$	$\alpha>0$	$\gamma>0$	$\theta>0$	GIWP	$\frac{e^{\theta}-\exp\left\{\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\}}{e^{\theta}-1}$
$e^{\theta}-1$	$\lambda>0$	$\alpha>0$	$\gamma=1$	$\theta>0$	IWP	$\frac{e^{\theta}-\exp\left\{\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\}}{e^{\theta}-1}$
$e^{\theta}-1$	$\lambda>0$	$\alpha=2$	$\gamma=1$	$\theta>0$	IRP	$\frac{e^{\theta}-\exp\left\{\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{2}}\right)\right\}}{e^{\theta}-1}$
$e^{\theta}-1$	$\lambda>0$	$\alpha=1$	$\gamma=1$	$\theta>0$	IEP	$\frac{e^{\theta}-\exp\left\{\theta\left(1-e^{\frac{-\lambda}{W}}\right)\right\}}{e^{\theta}-1}$
$-\ln(1-\theta)$	$\lambda>0$	$\alpha>0$	$\gamma>0$	$0<\theta<1$	GIWL	$1-\frac{\ln\left[1-\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]}{\ln[1-\theta]}$
$-\ln(1-\theta)$	$\lambda>0$	$\alpha>0$	$\gamma=1$	$0<\theta<1$	IWL	$1-\frac{\ln\left[1-\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]}{\ln[1-\theta]}$
$-\ln(1-\theta)$	$\lambda>0$	$\alpha=2$	$\gamma=1$	$0<\theta<1$	IRL	$1-\frac{\ln\left[1-\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{2}}\right)\right]}{\ln[1-\theta]}$
$-\ln(1-\theta)$	$\lambda>0$	$\alpha=1$	$\gamma=1$	$0<\theta<1$	IEL	$1-\frac{\ln\left[1-\theta\left(1-e^{\frac{-\lambda}{W}}\right)\right]}{\ln[1-\theta]}$
$\frac{\theta}{(1-\theta)}$	$\lambda>0$	$\alpha>0$	$\gamma>0$	$0<\theta<1$	GIWG	$1-\frac{(1-\theta)\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}{1-\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}$
$\frac{\theta}{(1-\theta)}$	$\lambda>0$	$\alpha>0$	$\gamma=1$	$0<\theta<1$	IWG	$1-\frac{(1-\theta)\left(1-e^{-\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}{1-\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}$
$\frac{\theta}{(1-\theta)}$	$\lambda>0$	$\alpha=2$	$\gamma=1$	$0<\theta<1$	IRG	$1-\frac{(1-\theta)\left(1-e^{-\left(\frac{\lambda}{w}\right)^{2}}\right)}{1-\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{2}}\right)}$
$\frac{\theta}{(1-\theta)}$	$\lambda>0$	$\alpha=1$	$\gamma=1$	$0<\theta<1$	IEG	$1-\frac{(1-\theta)\left(1-e^{\frac{-\lambda}{w}}\right)}{1-\theta\left(1-e^{-\frac{\lambda}{w}}\right)}$
$(1+\theta)^{m}-1$	$\lambda>0$	$\alpha>0$	$\gamma>0$	$0<\theta<1$	GIWB	$1-\frac{(1+\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right))^{m}-1}{(1+\theta)^{m}-1}$
$(1+\theta)^{m}-1$	$\lambda>0$	$\alpha>0$	$\gamma=1$	$0<\theta<1$	IWB	$1-\frac{(1+\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{\alpha}}\right))^{m}-1}{(1+\theta)^{m}-1}$
$(1+\theta)^{m}-1$	$\lambda>0$	$\alpha=2$	$\gamma=1$	$0<\theta<1$	IRB	$1-\frac{(1+\theta\left(1-e^{-\left(\frac{\lambda}{w}\right)^{2}}\right))^{m}-1}{(1+\theta)^{m}-1}$
$(1+\theta)^{m}-1$	$\lambda>0$	$\alpha=1$	$\gamma=1$	$0<\theta<1$	IEB	$1-\frac{(1+\theta\left(1-e^{\frac{-\lambda}{w}}\right))^{m}-1}{(1+\theta)^{m}-1}$

Also, it is easy to show that,

$$M_W(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r{}' \ ,$$

where, $\mu_r{}'$ is the rth moment, while $M_w(t)$ denotes the moment generating function (mgf) of W. Then by using (12), the mgf of W can be written as follows:

$$M_W(t) = \sum_{r=0}^{\infty}\sum_{z=1}^{\infty}\sum_{j=1}^{\infty} \frac{t^r}{r!} P_Z(z;\theta)\binom{z}{j}(-1)^{j-1}(\gamma j)^{\frac{r}{\alpha}}\lambda^r\Gamma\left(1-\frac{r}{\alpha}\right), \qquad \alpha > r, \qquad r = 1,2,\dots \ .$$

3.2 Order Statistics

In this section, expressions for the pdf of the ith order statistics from the GIWPS distribution are derived. In particular, the distribution of smallest and largest order statistics are obtained.

Let $W_1,\ W_2 \dots, W_n$ be a random sample with probability density function (6) and $W_{1:n} < W_{2:n} < \cdots < W_{n:n}$ be the corresponding order statistics. The pdf of the ith order statistics, say $g_{i:n}(w;\psi), \psi = \{\gamma, \lambda, \alpha, \theta\}$, is obtained as follows:

$$g_{i:n}(w;\psi) = \frac{1}{\beta(i\ ,n-i+1)} g(w;\psi)[G(w;\psi)]^{i-1}[1-G(w;\psi)]^{n-i}, \qquad w > 0. \tag{13}$$

By using, the pdf (6), cdf (7) and applying the binomial expansion in (13), then we get

$$g_{i:n}(w;\psi) = \frac{1}{\beta(i,n-i+1)} g(w;\psi)\sum_{j=0}^{i-1}\binom{i-1}{j}(-1)^j\left[\frac{c(\theta(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}))}{c(\theta)}\right]^{n+j-i}. \tag{14}$$

Now, since an expansion for $\left[c\left(\theta(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}})\right)\right]^{n+j-i}$ can be written as follows:

$$\begin{aligned}\left[c\left(\theta(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}})\right)\right]^{n+j-i} &= \left\{\sum_{z=1}^{\infty} a_z\,\theta^z\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^z\right\}^{n+j-i}, \\ &= \left[a_1\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]^{n+j-i}\left\{[1+\frac{a_2}{a_1}\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)+\frac{a_3}{a_1}\theta^2\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^2+\cdots\right\}^{n+j-i}, \\ &= \left[a_1\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]^{n+j-i}\left\{\sum_{m=0}^{\infty} c_m\theta^m\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^m\right\}^{n+j-i}, \end{aligned} \tag{15}$$

where, $c_m = \frac{a_{m+1}}{a_1}$.

As mentioned in [15], for a positive integer *j*, we have the following relation

$$\left(\sum_{m=0}^{\infty} c_m\, z^m\right)^j = \sum_{m=0}^{\infty} d_{j,m}\, z^m. \tag{16}$$

Hence by applying relation (16) in (15), then (15) can be written as follows

$$\left[c\left(\theta(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}})\right)\right]^{n+j-i} = a_1{}^{n+j-i}\sum_{m=0}^{\infty} d_{n+j-i,m}\,\theta^{n+j+m-i}\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)^{m+n+j-i}, \tag{17}$$

where, $d_{n+j-i,0} = 1$ and the coefficients $d_{n+j-i,m}$ are easily determined from the recurrence equation $d_{n+j-i,t} = t^{-1}\sum_{m=1}^{t}[m(n+j-i+1)-t]\, c_m d_{n+j-i,t-m}$, $t \geq 1$.

By using the expansion $c'(\theta) = \sum_{z=1}^{\infty} z\, a_z \theta^{z-1}$ for $c'\left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]$, then it can be written as;

$$c'\left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right] = \sum_{z=1}^{\infty} a_z z\left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]^{z-1},$$

$$c'\left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right] = a_1\sum_{k=0}^{\infty} c_k(k+1)\left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]^{k}, c_k = \frac{a_{k+1}}{a_1}. \quad (18)$$

Then, the pdf of the ith order statistics from the GIWPS distribution is obtained by substituting expansions (17) and (18) in pdf (14) as follows

$$g_{i:n}(w;\psi) = \frac{1}{\beta(i,n-i+1)}\sum_{j=0}^{i-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\tau_{j,k,m,r}\, f(w;(r+1)\gamma,\lambda,\alpha), \quad w>0, \quad (19)$$

where,

$f(w;(r+1)\gamma,\lambda,\alpha)$ is the pdf of the GIW distribution with parameters $(r+1)\gamma, \lambda$ and α,

$$\tau_{j,k,m,r} = \frac{c_k(k+1)\theta^{k+n+j-i+m}}{c(\theta)^{n+j-i+1}(r+1)}\binom{i-1}{j}\binom{k+n+j-i+m}{r}(-1)^{j+r}{a_1}^{n+j-i+1}\, d_{n+j-i,m}.$$

The pdf of the smallest and the largest order statistics from the GIWPS distribution is obtained by substituting $i = 1$ and n, in (19), respectively, as follows

$$g_{1:n}(w;\psi) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{n\, c_k(k+1)\theta^{k+n-1+m}}{c(\theta)^n(r+1)}\binom{k+n-1+m}{r}(-1)^r {a_1}^n\, d_{n-1,m} f(w;(r+1)\gamma,\lambda,\alpha), \quad w>0,$$

and

$$g_{n:n}(w;\psi) = \sum_{j=0}^{n-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\frac{nc_k(k+1)\theta^{k+j+m}}{c(\theta)^{j+1}(r+1)}\binom{n-1}{j}\binom{k+j+m}{r}(-1)^{j+r}{a_1}^{j+1}d_{j,m} f(w;(r+1)\gamma,\lambda,\alpha), \quad w>0,$$

where, again $f(w;(r+1)\gamma,\lambda,\alpha)$ is the pdf of a GIW distribution with parameters $(r+1)\gamma, \lambda$ and α.

Furthermore, the $v_{i:n}$th moment of ith order statistics from the GIWPS distribution can be obtained from (19) as follows

$$\mu_{v_{i:n}}{}' = \frac{1}{\beta(i,n-i+1)}\sum_{j=0}^{i-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\tau_{j,k,m,r}\int_0^{\infty} w^v f(w;(r+1)\gamma,\lambda,\alpha)\, dw, \quad (20)$$

where, $\int_0^{\infty} w^v f(w;(r+1)\gamma,\lambda,\alpha)\, dw$ is the vth moment of the GIW distribution with parameters $(r+1)\gamma, \lambda$ and α. Then $vi{:}n$th moment of the ith order statistics from GIWPS distribution is easily obtained by substituting the vth moment of the GIW distribution defined in (3), but with parameters $\gamma(r+1), \lambda$ and α, in (20), thereafter, the previous equation can be reduced to

$$\mu_{v_{i:n}}{}' = \frac{1}{\beta(i,n-i+1)} \sum_{j=0}^{i-1} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \tau_{j,k,m,r}\, \lambda^{v} \left(\gamma(r+1)\right)^{\frac{v}{\alpha}} \Gamma\left(1-\frac{v}{\alpha}\right), \qquad \alpha > v, \qquad v = 1,2, \dots .$$

3.3 Rényi Entropy

The entropy is a measure of uncertainty variation. The concept of entropy plays a vital role in information and communication theory. The R´enyi entropy of a random variable W following GIWPS distribution, for $\rho > 0$ and $\rho \neq 1$, is defined as follows

$$I_R(\rho) = (1-\rho)^{-1} \log\left(\int_{\boldsymbol{R}} \left(g(w;\psi)\right)^{\rho} dw\right).$$

Let, $IP = \int_0^\infty \left(g(w;\psi)\right)^{\rho} dw$, then

$$IP = \int_0^\infty (\theta\alpha\gamma\lambda^\alpha)^\rho w^{-\rho(\alpha+1)} e^{-\gamma\rho\left(\frac{\lambda}{w}\right)^\alpha} \left\{ \frac{\sum_{z=1}^\infty a_z z \left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^\alpha}\right)\right]^{z-1}}{c(\theta)} \right\}^\rho dw.$$

But,

$$\left\{\sum_{z=1}^\infty a_z z \left[\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^\alpha}\right)\right]^{z-1}\right\}^\rho = {a_1}^\rho \left(\sum_{m=0}^\infty \delta_m \left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^\alpha}\right)\right)^m\right)^\rho, \delta_m = \frac{a_{m+1}}{a_1}(m+1). \quad (21)$$

By applying relation (16) in (21), then it takes the following form

$$\left(\sum_{z=1}^\infty a_z z \left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^\alpha}\right)\right)^{z-1}\right)^\rho = {a_1}^\rho \sum_{m=0}^\infty d_{\rho,m} \left(\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^\alpha}\right)\right)^m ,$$

$$= {a_1}^\rho \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{m}{k} (-1)^k d_{\rho,m} \theta^m e^{-\gamma k\left(\frac{\lambda}{w}\right)^\alpha}. \quad (22)$$

The coefficients for $t > 1$ are computed from the recurrence equation $d_{\rho,t} = t^{-1} \sum_{m=1}^{t} [m(\rho+1) - t\delta m d\rho, t-m$ and $d\rho,0=1$. Then IP can be written as

$$IP = (\theta\alpha\gamma\lambda^\alpha)^\rho {a_1}^\rho \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{m}{k} (-1)^k d_{\rho,m} \theta^m \int_0^\infty w^{-\rho(\alpha+1)}\, e^{-\gamma(k+\rho)\left(\frac{\lambda}{w}\right)^\alpha} dw . \quad (23)$$

The Rényi entropy can be reduced to the following formula

$$I_R(\rho) = (1-\rho)^{-1} \log\left(\frac{(\theta\alpha\gamma\lambda^\alpha)^\rho {a_1}^\rho}{c(\theta)^\rho} \sum_{m=0}^\infty \sum_{k=0}^\infty \binom{m}{k} (-1)^k \frac{d_{\rho,m}\theta^m \lambda^{-(\rho(\alpha+1)+1)} \Gamma\left(\frac{\rho(\alpha+1)-1}{\alpha}\right)}{(\gamma(k+\rho))^{\frac{\rho(\alpha+1)-1}{\alpha}}}\right).$$

4. ESTIMATION OF THE MODEL PARAMETERS

In this section; the maximum likelihood estimators (MLEs) of the model parameters of the GIWPS distribution are determined from complete samples.

Let W_1, $W_2 \dots, W_n$ be a simple random sample from the GIWPS distribution with parameters $\psi \equiv (\theta, \alpha, \gamma, \lambda)$. The likelihood function based on the observed random sample of size n is given by

$$L(w;\psi) = \theta^n \alpha^n \gamma^n \lambda^{n\alpha} \left(c(\theta)\right)^{-n} \prod_{i=1}^{n} w_i^{-(\alpha+1)} \prod_{i=1}^{n} e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}} \prod_{i=1}^{n} c'\left(\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right).$$

The natural logarithm of the likelihood function, $\ell^* \equiv Ln\, L(w;\psi)$, is given by

$$\ell^* = nln\,\theta + n\ln\alpha + n\ln\gamma + n\alpha\ln\lambda - n\ln c(\theta) - (\alpha+1)\sum_{i=1}^{n}\ln(w_i) - \gamma\sum_{i=1}^{n}\left(\frac{\lambda}{w_i}\right)^{\alpha} + \sum_{i=1}^{n}\ln c'\left(\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right).$$

The maximum likelihood estimators of θ, α, γ and λ, say $\hat{\theta}$, $\hat{\alpha}, \hat{\gamma}$ and $\hat{\lambda}$, are obtained by setting the first partial derivatives of ℓ^* to be zero. The first partial derivatives for log-likelihood function with respect to $\theta, \alpha, \lambda\ and\ \gamma$ are given respectively as follows:

$$\frac{\partial\ell^*}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)c''\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]}{c'\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]} - \frac{nc'(\theta)}{c(\theta)},$$

$$\frac{\partial\ell^*}{\partial\alpha} = \frac{n}{\alpha} + nln\lambda - \gamma\sum_{i=1}^{n}\left(\frac{\lambda}{w_i}\right)^{\alpha}\ln\left(\frac{\lambda}{w_i}\right) - \sum_{i=1}^{n} ln(w_i) + \theta\gamma\lambda^{\alpha}\sum_{i=1}^{n}\frac{\ln\left(\frac{\lambda}{w_i}\right)e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}c''\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]}{w_i{}^{\alpha}\,c'\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]},$$

$$\frac{\partial\ell^*}{\partial\lambda} = \frac{n\alpha}{\lambda} - \frac{\alpha\gamma}{\lambda}\sum_{i=1}^{n}\left(\frac{\lambda}{w_i}\right)^{\alpha} + \theta\gamma\alpha\lambda^{\alpha-1}\sum_{i=1}^{n}\frac{e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}c''\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]}{w_i{}^{\alpha}c'\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]},$$

and

$$\frac{\partial\ell^*}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}\left(\frac{\lambda}{w_i}\right)^{\alpha} + \theta\lambda^{\alpha}\sum_{i=1}^{n}\frac{e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}c''\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]}{w_i{}^{\alpha}c'\left[\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w_i}\right)^{\alpha}}\right)\right]}.$$

The MLEs of the model parameters are obtained after setting the non-linear equations to be zero, i.e, $\frac{\partial\ell^*}{\partial\theta} = 0$, $\frac{\partial\ell^*}{\partial\alpha} = 0$, $\frac{\partial\ell^*}{\partial\lambda} = 0$, and $\frac{\partial\ell^*}{\partial\gamma} = 0$. It is clear that, there is no closed solution for the above non-linear equations, so an extensive numerical solution will be applied via iterative technique**.**

For the interval estimation of the model parameters, the 4×4 observed information matrix $I(\psi)$, whose elements are the second derivatives of the total log likelihood function,

$$I(\psi) = -\left(\frac{\partial^2 \ell^*}{\partial\psi_i \partial\psi_j}\right)_{4\times4} = -\begin{pmatrix} \frac{\partial^2 \ell^*}{\partial\theta^2} & \frac{\partial^2 \ell^*}{\partial\theta\partial\alpha} & \frac{\partial^2 \ell^*}{\partial\theta\partial\lambda} & \frac{\partial^2 \ell^*}{\partial\theta\partial\gamma} \\ \frac{\partial^2 \ell^*}{\partial\alpha\partial\theta} & \frac{\partial^2 \ell^*}{\partial\alpha^2} & \frac{\partial^2 \ell^*}{\partial\alpha\partial\lambda} & \frac{\partial^2 \ell^*}{\partial\alpha\partial\gamma} \\ \frac{\partial^2 \ell^*}{\partial\lambda\partial\theta} & \frac{\partial^2 \ell^*}{\partial\lambda\partial\alpha} & \frac{\partial^2 \ell^*}{\partial\lambda^2} & \frac{\partial^2 \ell^*}{\partial\lambda\partial\gamma} \\ \frac{\partial^2 \ell^*}{\partial\gamma\partial\theta} & \frac{\partial^2 \ell^*}{\partial\gamma\partial\alpha} & \frac{\partial^2 \ell^*}{\partial\gamma\partial\lambda} & \frac{\partial^2 \ell^*}{\partial\gamma^2} \end{pmatrix}_{4\times4}, i,j = 1,2,3,4$$

Under the regularity conditions, the known asymptotic properties of the maximum likelihood method ensure that: $\sqrt{n}(\hat{\psi} - \psi) \xrightarrow{d} N_4(0, I^{-1}(\psi))$ as $n \to \infty$, where $\xrightarrow{d}$ means the convergence in distribution, with mean $0 = (0,0,0,0)^T$ and 4×4 variance covariance matrix $I^{-1}(\psi)$ then, the $100(1-\tau)\%$ confidence interval for $\psi \equiv (\theta, \alpha, \lambda\, \gamma)$ is given, as follows

$$\hat{\psi} \pm Z_{\tau/2}\sqrt{var\ \ (\hat{\psi})}\ ,$$

where $Z_{\tau/2}$ is the standard normal at $\tau/2$, $\tau/2$ is significance level and var (.)'s denote the diagonal elements of $I^{-1}(\psi)$ corresponding to the model parameters.

5. GENERALIZED INVERSE WEIBULL POISSON DISTRIBUTION

As mentioned in Section 2 that the generalized inverse Weibull Poisson distribution is obtained from GIWPS distribution as a special case. As seen from Table 2, the cdf of the GIWP takes the form

$$G(w;\psi) = (e^{\theta} - 1)^{-1}\,[e^{\theta} - \exp\left\{\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\}], w > 0,\ \alpha, \theta, \lambda, \gamma > 0. \qquad (24)$$

The pdf of the GIWP distribution corresponding to (24) is

$$g(w;\psi) = \theta\alpha\gamma\lambda^{\alpha} w^{-\alpha-1} e^{-\gamma(\frac{\lambda}{w})^{\alpha}} (e^{\theta} - 1)^{-1} \exp\left\{\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\}.\ \ w > 0.$$

The reliability and hazard rate functions are obtained directly from (8) and (9) using the quantity $c(\theta) = e^{\theta} - 1, \theta > 0, c'(\theta) = e^{\theta}$, that corresponds to the zero truncated Poisson distribution as the following

$$R(w;\psi) = \frac{\exp\left\{\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\} - 1}{e^{\theta} - 1},$$

and

$$h(w;\psi) = \frac{\theta\alpha\gamma\lambda^{\alpha} w^{-\alpha-1} e^{-\gamma(\frac{\lambda}{w})^{\alpha}} \exp\left\{\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\}}{\exp\left\{\theta\left(1 - e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right\} - 1}.$$

Figs. 1 and 2 represent the probability density and the hazard rate functions plots for the GIWP distribution for some selected values of parameters.

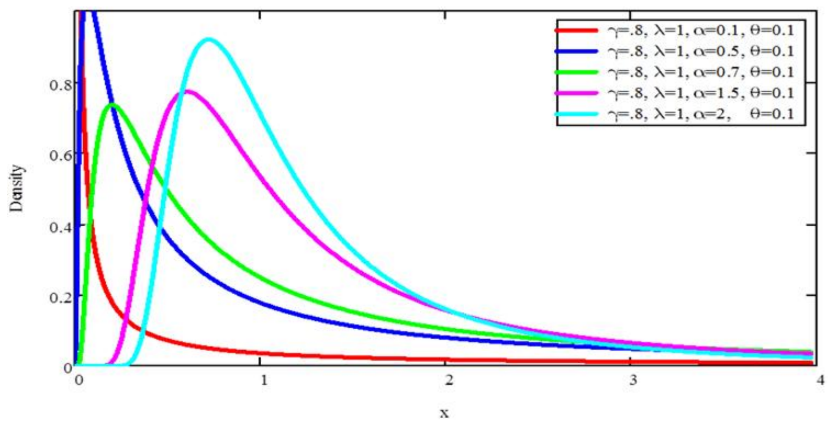

Fig. 1. Plots of the GIWP densities for some values of the parameters

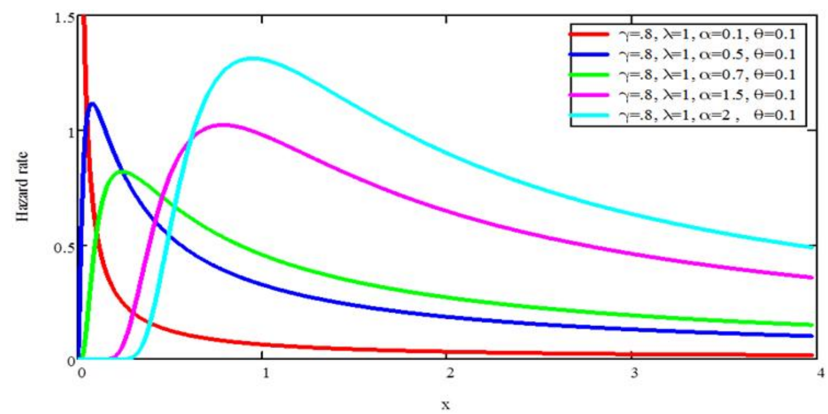

Fig. 2. Hazard rate function plots of the GIWP distribution for some values of the parameters

The GIWP distribution has increasing, decreasing and constant failure rates as shown in Fig. 2.

The quantile function for the GIWP distribution is obtained directly from expression (11) with $c(\theta) = e^{\theta} - 1$ and $c^{-1}(\theta) = \ln(1+\theta)$ as follows

$$w = \lambda \left\{ \frac{-1}{\gamma} \ln\left(1 - \frac{\ln\left(1 + \left((e^{\theta} - 1)(1-u)\right)\right)}{\theta} \right) \right\}^{\frac{-1}{\alpha}} .$$

The rth moment about zero of the GIWP distribution is obtained from (12) with $P_Z(z;\theta) = \frac{\theta^z e^{-\theta}}{\Gamma(z+1)(1-e^{-\theta})}$, $z = 1,2,\dots$ as follows

$$\mu_r{}' = \sum_{z=1}^{\infty} \sum_{j=1}^{\infty} \binom{z}{j} \frac{\theta^z}{\Gamma(z+1)(e^{\theta}-1)} (-1)^{j-1} (\gamma j)^{\frac{r}{\alpha}} \lambda^r \Gamma\left(1 - \frac{r}{\alpha}\right), \ \alpha > r, \ r = 1,2,\dots .$$

6. GENERALIZED INVERSE WEIBULL GEOMETRIC DISTRIBUTION

The second special case of the GIWPS family of distributions will be discussed in some details in this section. As mentioned in Section 2 the distribution function of the generalized inverse Weibull geometric (see Table 2) takes the following form

$$G(w;\psi) = 1 - \frac{(1-\theta)\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}{1-\theta\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}, \ \alpha,\gamma,\lambda, \ w>0, \quad 0<\theta<1. \tag{25}$$

The pdf of the GIWG distribution corresponding to (25) takes the following form

$$g(w;\psi) = (1-\theta)\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}\left[1-\left(\theta(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]^{-2} e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}, \ w>0.$$

In addition, the hazard rate function takes the following form

$$h(w;\psi) = \frac{\alpha\gamma\lambda^{\alpha}w^{-\alpha-1}e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}}{\left[1-\left(\theta(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)\right]\left(1-e^{-\gamma\left(\frac{\lambda}{w}\right)^{\alpha}}\right)}.$$

Figs. 3 and 4 represent the probability density and the hazard rate functions plots for the GIWG distribution for some selected values of parameters.

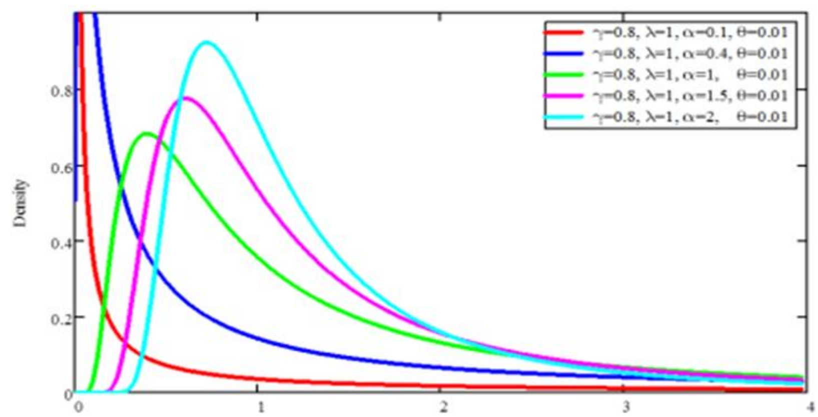

Fig. 3. The pdf plots of the GIWG distribution for some values of parameters

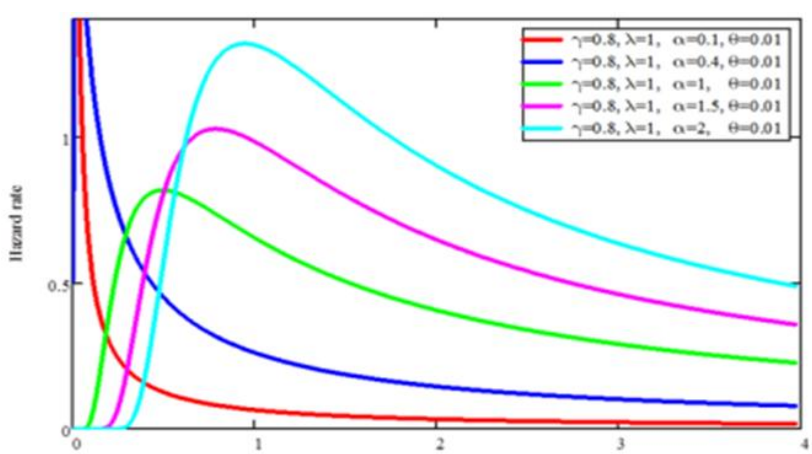

Fig. 4. Hazard rate function plots of the GIWG distribution for some values of parameters

It is clear from Fig. 4 that the GIWG distribution has increasing, decreasing and constant failure rates.

Furthermore, the quantile function and the rth moment formula for the GIWG distribution are obtained from (11), (12) by using $P_Z(z;\theta)=(1-\theta)\theta^{z-1},\ z=1,2\ldots, c(\theta)=\theta(1-\theta)^{-1}$ and $c'(\theta)=(1-\theta-2$ (see Table 1) respectively as follows

$$w=\lambda\left\{\frac{-1}{\gamma}\ln\left(\frac{u-\theta u}{1-\theta u}\right)\right\}^{\frac{-1}{\alpha}},$$

and

$$\mu_r'=\sum_{z=1}^{\infty}\sum_{j=1}^{\infty}\binom{z}{j}(1-\theta)\theta^{z-1}(-1)^{j-1}(\gamma j)^{\frac{r}{\alpha}}\lambda^r\Gamma\left(1-\frac{r}{\alpha}\right),\ \alpha>r,\ r=1,2,\ldots$$

7. APPLICATIONS

In this section, two real data sets will be provided to compare the fit of some special models of the GIWPS distribution and to illustrate the flexibility of the new family.

The first data set is taken from [16], where the vinyl chloride data is obtained from clean upgradient ground –water monitoring wells in mg/L; the data are as follows:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2.

We have fitted the GIWP, GIWG, GIW and IW distributions to this real data set. The pdf of the GIWP is as mentioned in Section 5, the pdf of the GIWG distribution is as mentioned in Section 6.

In addition, the pdf of the GIW is as mentioned in Section 1, and the pdf of the inverse Weibull distribution is as follows

$$f(w;\lambda,\alpha)=\alpha\lambda^\alpha w^{-\alpha-1}e^{-(\frac{\lambda}{w})^\alpha},\ w>0,\ \alpha,\lambda>0,$$

where α is the shape parameter and λ is the scale parameter.

The models parameters of the GIWP, GIWG, GIW and IW distributions are estimated by the maximum likelihood method. The values of Kolmogorov-Smirnov (K-S) statistics (with the corresponding standard errors), maximum likelihood estimates of the parameters (with standard errors), Akike information criterion (AIC) and Bayesian information criterion (BIC) are calculated. The results for all mentioned models are reported in Table 3.

In general the best model corresponds to the smallest values of AIC, BIC, K-S. It clear from Table 3 that the GIWP model fits the data set better than the others competing models.

Plots of the pdfs and cdfs of the fitted GIWP, GIWG, GIW and IW models to the data, displayed in Figs. 5 and 6, indicated the superiority of the GIWP model than the other three models.

As a second example, the data set from [17] will be considered. It consists of 40 observations of the active repair times (in hours) for airborne communication transceiver. The data are:

0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

We have fitted the GIWP, GIWG, Weibull Poisson (WP) and Weibull (W) distributions to this real data set. The pdf of Weibull Poisson (WP) distribution with shape parameter α and scale parameters λ and θ is as follows

$$f(w;\lambda,\alpha,\theta)=\theta\alpha\lambda^\alpha w^{\alpha-1}\left(e^\theta-1\right)^{-1}e^{-\theta(\lambda w)^\alpha}exp\left(\theta\left(1-e^{-(\lambda w)^\alpha}\right)\right),\qquad w>0,\qquad \alpha,\lambda,\theta>0.$$

The Weibull (W) distribution has the following density function

$$f(w;\beta,\alpha,\lambda)=\alpha\lambda^\alpha w^{\alpha-1}e^{-(\lambda w)^\alpha},\qquad w>0,$$

where $\alpha>0$ is a shape parameter and λ is a scale parameter.

Table 3. Estimates ([a] denotes standard errors), K-S ([b] denotes p values), AIC and BIC for the first data set

Models	MLES				Measures		
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$	K-S	AIC	BIC
GIWP	0.168 (0.156)[a]	4.173 (26.01)[a]	5.027 (0.155)[a]	340.545 (1.915*1 0^3)[a]	0.078 (0.985)[b]	117.678	123.783
GIWG	0.148 (0.025)[a]	47.468 (33.55)[a]	4.333 (1.114)[a]	0.999 (7.098*10^-4)[a]	0.131 (0.602)[b]	125.573	131.678
GIW	0.88 (0.109)[a]	0.799 (282.62)[a]	0.797 (247.98)[a]	-	0.113 (0.775)[b]	123.253	127.832
IW	0.88 (0.109)[a]	0.617 (0.128)[a]	-	-	0.113 (0.775)[b]	121.253	124.306

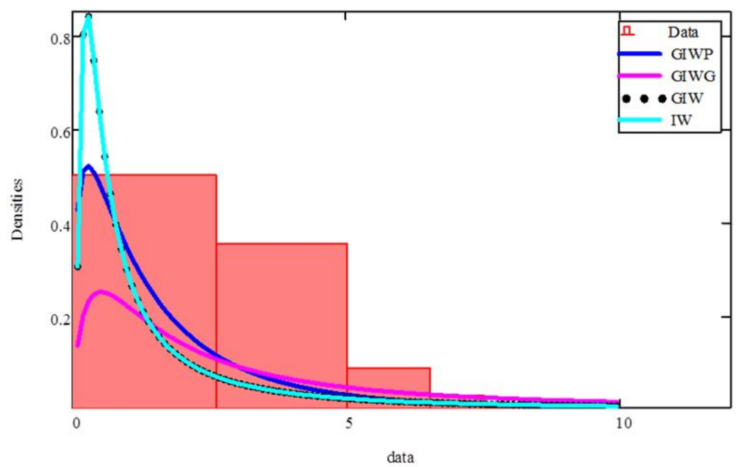

Fig. 5. Estimated densities of models for the first data set

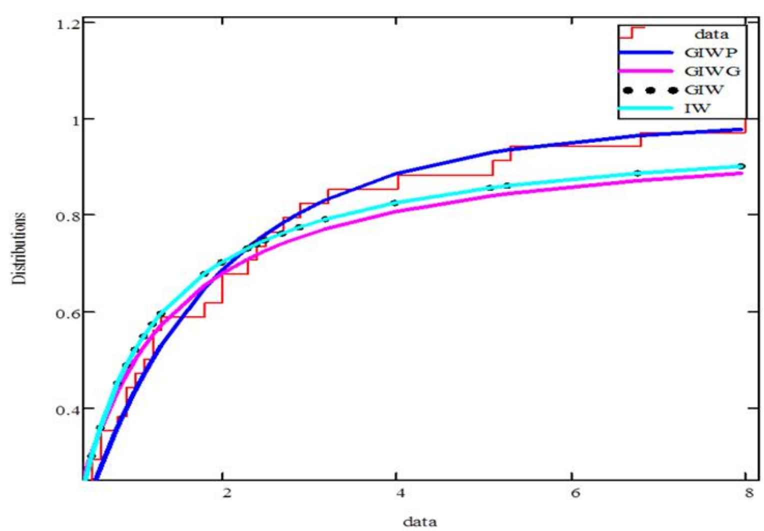

Fig. 6. Estimated cumulative densities for the first data set

Table 4 shows the values of AIC, BIC and K-S statistics. Figs. 7 and 8 provide some plots of the estimated cumulative distribution functions as well as the estimated probability densities of the fitted GIWP, GIWG, WP and W models to the this data set.

Table 4. Estimates ([a] denotes standard errors), K-S ([b] denotes p values), AIC and BIC for the second data set

Models	MLES				Measures		
	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$	K-S	AIC	BIC
GIWP	0.528 (0.242)[a]	1.127 (0.738)[a]	3.256 (0.261)[a]	6.957 (6.66)[a]	0.105 (0.77)[b]	188.772	195.527
GIWG	1.174 (0.328)[a]	0.973 (3.986)[a]	1.606 (7.744)[a]	-0.016 (1.209)[a]	0.159 (0.26)[b]	186.948	193.704
WP	1.187 (0.138)[a]	0.106 (0.046)[a]	-	3.572 (1.894)[a]	0.124 (0.57)[b]	192.944	198.011
W	0.96 (0.109)[a]	0.255 (0.045)[a]	-	-	0.129 (0.51)[b]	195.023	198.4

It is clear from Table 4 that the proposed GIWP distribution fits to this data is better than the other models according to the AIC, BIC, and the statistic K-S.

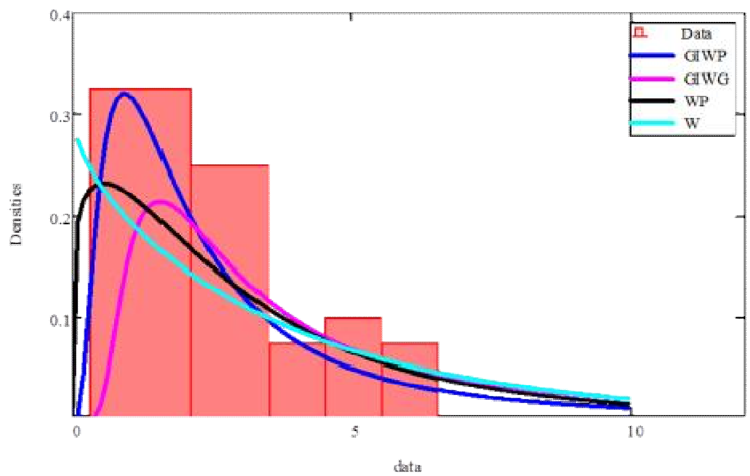

Fig. 7. Estimated densities of models for the second data set

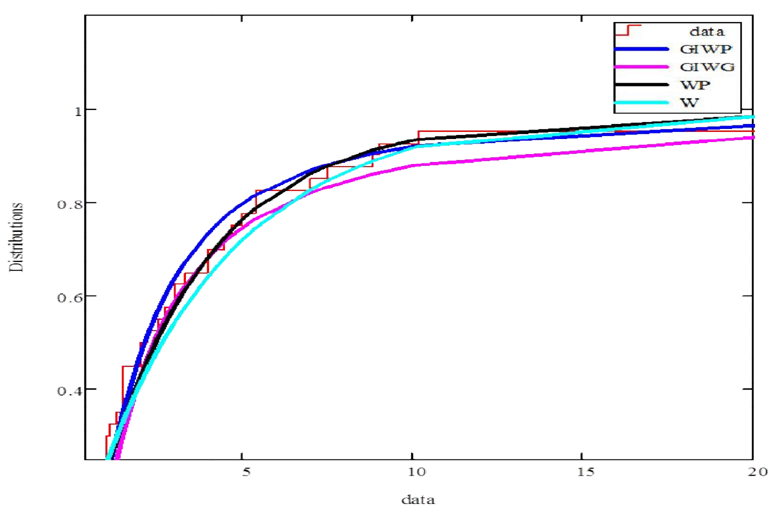

Fig. 8. Estimated cumulative densities for the second data set

It is clear from plots of Fig. 7 that the fitted density of the GIWP model is closer to the empirical histogram than the corresponding densities of the GIWG, WP and W models, also, Fig. 8 confirm this conclusion.

8. CONCLUSION

In this paper, a new family of lifetime distributions called the generalized inverse Weibull power series distributions with increasing, decreasing and constant failure rate functions has been introduced. The GIWPS density function can be expressed as a mixture of GIW density functions. Furthermore, the GIWPS distribution has been extended several new sub-models which have not been studied in the literature. Mathematical properties of the new family, including expressions for density function, moments, moment generating function and quantile function are provided. Further, explicit expressions for the order statistics and Rényi entropy are derived. Maximum likelihood is implemented for estimating the model parameters. The generalized inverse Weibull Poisson and the generalized inverse Weibull geometric distributions have been provided. Some statistical properties of the GIWP and the GIWG models have been discussed. In order to show the usefulness of the suggested family, we fit the GIWP and the GIWG to two real life data sets as two sub-models examples from this family. The GIWP model provides consistently a better fit than the other models. We hope that this generalization may attract wider applications in areas such as engineering, survival and lifetime data, hydrology, economics (income inequality) and others.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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