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Applications to Physical Data Using Four-Parameter Inverted Topp-Leone Model

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Abstract

The Kumaraswamy Marshall-Olkin inverted Topp-Leone (KMOITL) distribution is a new four-parameter generalized version of the inverted Topp-Leone (ITL) distribution proposed in this research. The Marshall-Olkin ITL distribution is a novel model, while the Kumaraswamy ITL and ITL distributions are existing sub-models in the proposed distribution. Different shapes of the density and hazard rate functions are provided by the KMOITL distribution, which has three shape parameters and one scale parameter. The KMOITL's density function can be written as a linear combination of the inverted Topp-Leone density. We construct several statistical expressions for the proposed KMOITL model. The KMOITL distribution parameters are estimated using maximum likelihood and Bayesian estimation techniques. In light of symmetric and asymmetric loss functions, Bayesian estimators are explored. The performance of the suggested estimating techniques is evaluated using simulation results. Finally, the suggested model is tested based on physical real data, with the findings demonstrating the KMOITL distribution's higher performance over some other models.

Keywords: Inverted Topp-Leone distribution, maximum likelihood, Bayesian method, stress strength model, entropy measures

1. Introduction

Various researchers have lately focused their efforts on the development of new families of continuous distributions by extending current continuous distributions. These new families offer a broader range of applications in modelling data in a variety of fields, including engineering, economics, biological research, and environmental sciences, to name a few. The following are the key goals of generalizing this new families of distributions: construct customized models with diverse forms of hazard rate function; accomplish skewness for symmetrical models; build heavy-tailed distributions that may be used in a variety of real-world data sets; achieve a more flexible kurtosis than the baseline distribution; create skewed, symmetric, J-shaped, or reversed-J-shaped distributions that match better than other generalized distributions with the same parameters as the underpinning model.

Marshall and Olkin (1997) presented a transformation of the baseline cumulative distribution function (cdf) into a family of distributions by introducing a new parameter

$$F_{MO}(z)=G(z;\xi)\left[1-\varphi\left(\bar{G}(z;\xi)\right)\right]^{-1}, \tag{1}$$

where φ is the scale parameter and $G(z;\xi)$ is the baseline cdf. Cordeiro and de Castro (2011) defined the Kumaraswamy-G (K-G) class with the cdf given by:

$$F_K(z)=1-\left[1-(G(z;\xi))^c\right]^d, \tag{2}$$

where c and d are two shape parameters. The K-G distribution provided in (2) (for c and d positive integers) has the following physical explanation. Consider a system made up of d separate components, each of which is composed of c independent subcomponents. Assume that if any of the d components fails, the system fails, and that each component fails if all of the c subcomponents fail. Let $Z_{j1},...,Z_{jc}$ indicate the subcomponent lives inside the j^{th} component, $j=1,...,d$ with a common cdf $G(z)$. For $j=1,...,d$, let Z_1 indicate the lifetime of the j^{th} component, and Z denote the lifetime of the entire system. Then cdf of Z is:

$$\begin{aligned}P(Z\le z)&=1-P(Z_1>z,Z_2>z,...Z_d>z)=1-P(Z_1>z)^d=1-[1-P(Z_1\le z)]^d\\&=1-[1-P(Z_{11}\le z,Z_{12}\le z,...,Z_{1c}\le z)]^d=1-[1-P(Z_{11}\le z)^c]^d\\&=1-(1-G^c(z))^d.\end{aligned}$$

Hence, the K-G distribution deduced from (2) is the time to failure distribution of the complete system. Alizadeh et al. (2015) suggested extending the MO family for a given baseline distribution cdf by putting (1) in (2) and defining the Kumaraswamy Marshall-Olkin-G (KMO-G) cdf and density function as follows:

$$F_{KMO}(z;c,d,\varphi,\xi)=1-\left\{1-\left[\frac{G(z;\xi)}{1-\varphi\bar{G}(z;\xi)}\right]^c\right\}^d, \tag{3}$$

$$f_{KMO}(z;c,d,\varphi,\xi)=\frac{cd\bar{\varphi}g(z;\xi)\left(G(z;\xi)\right)^{c-1}}{\left(1-\varphi\bar{G}(z;\xi)\right)^{c+1}}\left\{1-\left[\frac{G(z;\xi)}{1-\varphi\bar{G}(z;\xi)}\right]^c\right\}^{d-1}. \tag{4}$$

The cdf (3) encompasses a broader range of continuous distributions. According to Alizadeh et al. (2015), it encompasses the K-G family, proportional and inverted hazard rate models, and the MO-G and other sub-models. The density function of KMO-G is symmetrical, left-skewed, right-skewed and reversed-J shaped, and has constant, increasing, decreasing, upside-down bathtub, bathtub and S-shaped hazard rates.

Many notable inverted distributions have recently been proposed to model varied data in many areas. Some of the important inverted distributions are the inverse Weibull (Keller and Kamath 1982), inverse Lindley (Sharma et al. 2015), inverted Kumaraswamy (Abd AL-Fattah et al. 2017), inverse power Lindley (Barco et al. 2017), inverted Nadarajah-Haghighi (Tahir et al., 2018), inverse power Lomax (Hassan and Abd-Allah 2019), inverted exponentiated Lomax (Hassan and Mohamed 2019), inverted modified Lindley (Chesneau et al. 2020), inverted Topp-Leone (Hassan et al. 2020), inverse xgamma (Yadav et al. 2021), inverse power Maxwell (Al-Kzzaz and Abd El-Monsef 2022), inverse power Cauchy (Sapkota and Kumar 2023) and inverse power Ramos-Louzada (Al Mutairi et al. 2023) distributions.

The inverted Topp-Leone (ITL) distribution with shape parameter $\gamma > 0,$ has the following cdf and probability density function (pdf):

$$G(z;\gamma) = 1 - \left\{ \frac{(1+2z)^{\gamma}}{(1+z)^{2\gamma}} \right\}; \quad z \geq 0, \gamma > 0, \tag{5}$$

and

$$g(z;\gamma) = 2\gamma z \ (1+z)^{-2\gamma-1}(1+2z)^{\gamma-1} \ ; \ z, \ \gamma > 0. \tag{6}$$

Different structural properties of the ITL distribution were pioneered by Hassan et al. (2020). Many researchers consider extensions and generalizations of the ITL distribution to improve flexibility in modelling a wide range of data. Abushal et al. (2021) proposed a power ITL distribution with an extra shape parameter. Ibrahim et al. (2021) proposed a new type of ITL distribution with an additional parameter called “alpha power ITL distribution”. With application to COVID-19, Hassan et al. (2021) developed a three-parameter ITL distribution based on the K-G family. The two-parameter half-logistic ITL distribution was introduced by Bantan et al. (2021), and parameter estimators based on ranked set samples were explored. Almetwally (2021) introduced another two-parameter ITL distribution using the odd Weibull-G family. Almetwally et al. (2021) introduced the modified Kies ITL distribution and discussed parameter estimators using different estimation methods. The truncated Cauchy power-ITL distribution was presented and its estimators were explored by Mohamed et al. (2023) under the hybrid censoring scheme. The truncated-ITL distribution was established by Elgarhy et al. (2023) and its parameter was investigated under progressive censoring.

The current article’s contribution can be summarized as follows:

(i) We introduce a new generalization of the ITL distribution based on KMO-G family, called Kumaraswamy Marshall-Olkin inverted Topp-Leone (KMOITL) distribution.

(ii) We provide some new models as well as some existing models as seen in Section 2.

(iii) We discuss several statistical properties as provided in Section 3.

(iv) We investigate the Bayesian and non-Bayesian estimation of the KMOITL model parameters using symmetric (squared error loss function (SELF)) as well as asymmetric loss function (linear exponential (LINEX) and entropy loss (ELS).

(v) We apply this model to actual engineering datasets according to the physical explanation of one sub-model (Kumaraswamy ITL) of the proposed distribution.

The following is the article's content: The description of the KMOITL distribution is found in Section 2. The major statistical features of this model are discussed in Section 3. The study of maximum likelihood (ML) and Bayesian estimation techniques is explored in Section 4. Section 5 contains simulation studies that are used to ensure the consistency of estimates. In Section 6, we use two real data sets to demonstrate the potential of the new distribution. Finally, in Section 7, some closing results were observed.

2. Description of the Model

The cdf of the KMOITL distribution with set of parameters $\Psi \equiv (c, d, \varphi, \gamma)$ is obtained by setting (5) in (3) as follows:

$$F(z;\Psi) = 1 - \left\{ 1 - \left[\frac{1-\Xi(z,\gamma)}{1-\varphi[\Xi(z,\gamma)]} \right]^{c} \right\}^{d}, \quad z, c, d, \gamma, \varphi > 0, \tag{7}$$

where $\Xi(z,\gamma)=\left\{\frac{(1+2z)^{\gamma}}{(1+z)^{2\gamma}}\right\}, c,d,\gamma$ are shape parameters and φ is the scale parameter. Based on (7), some special distributions can be summarized as follows:

1) For $\varphi = 0$, then (7) reduces to Kumaraswamy ITL (KITL) distribution (see Hassen et al. 2021).

2) For $c = d = 1$, then (7) yields the Marshall- Olkin ITL (MOITL) distribution (new).

3) For $c = d = 1$ and $\varphi = 0$, then (7) reduces to ITL distribution (Hassen et al. 2020).

4) For $c = 1$ and $\varphi = 0$, we have an ITL distribution with parameter d,γ.

5) For $d = 1$ and $\varphi = 0$, yields exponentiated ITL distribution with parameters d and γ, (new).

The pdf of KMOITL distribution is obtained by inserting (5) and (6) in (4) as follows

$$f(z;\Psi)=\frac{2cd\bar{\varphi}\gamma z\,(1+z)^{-2\gamma-1}(1+2z)^{\gamma-1}\left(1-\Xi(z,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z,\gamma)\right)^{c+1}}\left\{1-\left[\frac{1-\Xi(z,\gamma)}{1-\varphi[\Xi(z,\gamma)]}\right]^{c}\right\}^{d-1},\quad z>0, \tag{8}$$

where $c,d,\gamma,\bar{\varphi}>0,$ and $\Psi=(\varphi,c,d,\gamma)$ is the set of parameters. A random variable with pdf (8) is represented as $Z\sim(c,d,\varphi,\gamma)$. The hazard rate function (hrf) of the KMOITL distribution is given by

$$h(z;\Psi)=\frac{2cd\bar{\varphi}\gamma z\,(1+z)^{-2\gamma-1}(1+2z)^{\gamma-1}\left(1-\Xi(z,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z,\gamma)\right)^{c+1}}\left\{1-\left[\frac{1-\Xi(z,\gamma)}{1-\varphi[\Xi(z,\gamma)]}\right]^{c}\right\}^{-1}.$$

The graphs of the KMOITL density for various parameter values are shown in Figure 1. The versatility and modality of the new distribution are seen in these graphs. We see in Figure 1 that the KMOITL density is uni-modal or less-bell shaped. It is right-skewed for set of parameters $(0.85, 2, 0.5, 3)$. Note that the red and purple curves have a reversed J-shaped form, corresponding to the pdf defined with same values of c and d. The hrf of Z can be in the shape of a decreasing, up-side down, unimodal, J-shaped or reversed J-shaped, as seen in Figure 2.

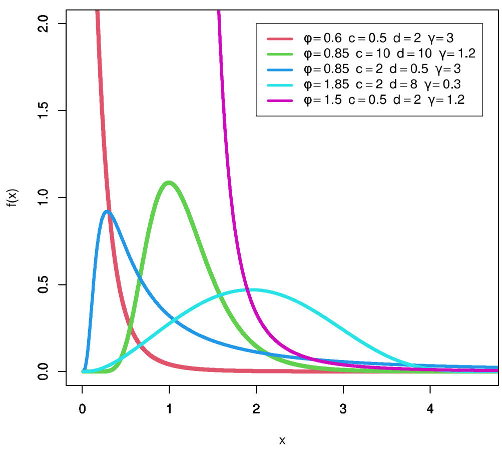

Figure 1 Plots of the KMOITL density function with various values of parameters

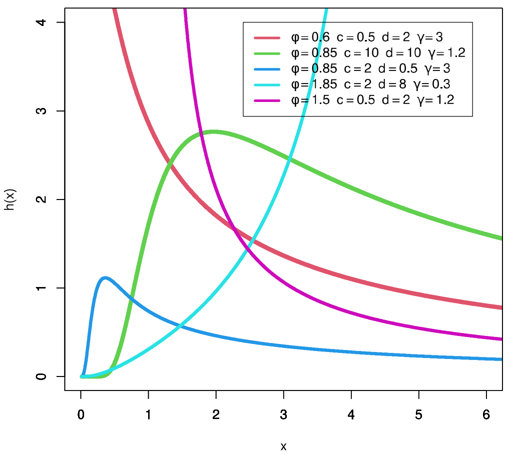

Figure 2 Plots of the KMOITL hazard function with various values of parameters

3. Statistical Properties

In this section, several features of the KMOITL distribution are investigated. To be more specific, we investigate the linear representation of the density function, quantile function, moments, incomplete moments, some entropy measures, stochastic ordering, and stress-strength reliability.

3.1. Linear representation

For the KMOITL density, we give an appropriate linear representation. Using the following binomial expansion

$$\left\{1-\left[\frac{1-\Xi(z,\gamma)}{1-\varphi[\Xi(z,\gamma)]}\right]^{c}\right\}^{d-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{d-1}{j}\left[\frac{1-\Xi(z,\gamma)}{1-\varphi[\Xi(z,\gamma)]}\right]^{cj}, \tag{9}$$

in pdf (8), as follows:

$$f(z;\Psi)=\sum_{j=0}^{\infty}(-1)^{j}\binom{d-1}{j}\frac{2cd\bar{\varphi}\gamma z\,(1+z)^{-2\gamma-1}(1+2z)^{\gamma-1}\left(1-\Xi(z,\gamma)\right)^{c(j+1)-1}}{\left(1-\varphi\Xi(z,\gamma)\right)^{c(j+1)+1}}. \tag{10}$$

Using the following binomial expansion:

$$\left(1-\varphi\Xi(z,\gamma)\right)^{-c(j+1)-1}=\sum_{\ell=0}^{\infty}\frac{(\varphi)^{\ell}\Gamma\left(c(j+1)+1+\ell\right)}{\Gamma\left(c(j+1)+1\right)\ell!}(\Xi(z,\gamma))^{\ell}, \tag{11}$$

in (10) and again using the binomial expansion yields;

$$f(z;\Psi)=\sum_{j,\ell,m=0}^{\infty}(-1)^{j+m}\binom{d-1}{j}\binom{c(j+1)-1}{m}\frac{2cd\bar{\varphi}\gamma\Gamma\left(c(j+1)+1+\ell\right)(\varphi)^{\ell}z}{\Gamma\left(c(j+1)+1\right)\ell!}\left\{\frac{(1+2z)^{\gamma(\ell+m+1)-1}}{(1+z)^{2\gamma(\ell+m+1)+1}}\right\}.$$

Hence, the pdf of the KMOITL distribution can be written as:

$$f(z;\Psi)=\sum_{j,\ell,m=0}^{\infty}\mathrm{A}_{j,\ell,m}\,g(z;\gamma(\ell+m+1)), \tag{12}$$

where $\mathrm{A}_{j,\ell,m}=(-1)^{j+m}\binom{d-1}{j}\binom{c(j+1)-1}{m}\dfrac{cd(\bar{\varphi})\varphi^{\ell}\Gamma\left(c(j+1)+1+\ell\right)}{\Gamma\left(c(j+1)+1\right)\ell!(\ell+m+1)}$, and $g(z;\gamma(\ell+m+1))$ is the pdf of the ITL distribution.

Furthermore, the linear representation of the KMOITL cdf is obtained using the same expansions in (9) and (11) as follows:

$$F(z;\Psi)=1-\sum_{i,u=0}^{\infty}\binom{d}{i}\frac{(\varphi)^{u}(-1)^{i}\Gamma(ci+u)}{\Gamma(ci+1)u!}\frac{[1-\Xi(z,\gamma)]^{ci}[1+2z]^{\gamma u}}{[1+z]^{2\gamma u}}.$$

Again, we use binomial expansion in the previous equation gives:

$$F(z;\Psi)=1-\sum_{i,u,v=0}^{\infty}\mathrm{N}_{i,u,v}G(z;\gamma(u+v)), \tag{13}$$

where $N_{i,u,v}=\binom{d}{i}\binom{ci}{v}\frac{(\varphi)^{u}(-1)^{i+v}\Gamma(ci+u)}{\Gamma(ci+1)u!}$ and $G(z;\gamma(u+v))$ is the cdf of the ITL distribution.

3.2. Quantile function

The quantile function (qf) of a continuous distribution is used in a variety of applications, including Bowley's skewness and Moor's kurtosis. The qf of the KMOITL distribution is obtained by inverting the cdf (7) as follows:

$$z=Q(u)=-L-L\sqrt{L-1},\quad L=1-\left(\frac{\left\{1-(1-u)^{1/d}\right\}^{1/c}-1}{\varphi\left\{1-(1-u)^{1/d}\right\}^{1/c}-1}\right)^{-1/\gamma}.$$

In particular, the KMOITL median of Z is $Q(0.5)$. Based on quantile measures, the effects of shape parameters on skewness (SK) and kurtosis (KR) may be examined. The Bowley skewness and Moors kurtosis are calculated using the following formula:

$$B=\frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)},\quad M=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

3.3. Moments measures

The k^{th} non-central moment of the KMOITL distribution is covered. These moments may be used to derive information about the model's other features, such as spread, skewness, and kurtosis. The desired moments are determined by

$$\begin{aligned}\mu_k' &= \sum_{j,\ell,m=0}^{\infty}\mathrm{A}_{j,\ell,m}\int_0^{\infty}2\gamma(\ell+m+1)z^{k+1}(1+2z)^{\gamma(\ell+m+1)-1}(1+z)^{-2\gamma(\ell+m+1)-1}dz\\ &= \sum_{j,\ell,m,q=0}^{\infty}D_{j,\ell,m,q}\mathrm{B}(k+q+2,\gamma(\ell+m+1)-k),\end{aligned} \tag{14}$$

where $D_{j,\ell,m,q}=2\mathrm{A}_{j,\ell,m}\gamma(\ell+m+1)\binom{\gamma(\ell+m+1)-1}{q}$, and $B(\cdot,\cdot)$ is the beta function. Based on (14), some moments values for KMOITL distribution, such as mean (μ_1'), variance (σ^2), SK and KR for some selected parameter values are listed in Table 1. It can be noticed from Table 1 that the distribution is skewed to the right, as shown by the skewnees values. According to kurtosis values, the distribution is also Leptokurtic.

Table1 Some moments values of the KMOITL distribution

(ϕ,γ,c,d)	μ_1'	σ^2	SK	KR
(0.6,3,0.5,2)	0.197	0.058	4.418	56.138
(0.85,1.2,10,10)	1.199	0.196	1.311	6.605
(0.85,2,2,5)	0.277	0.021	1.724	9.875
(0.5,2.5,3,2)	1.179	0.729	3.666	49.335
(0.6, 0.7,5,8)	2.815	3.040	2.879	27.024
(0.3,3.5,2,3)	0.612	0.122	1.713	9.058

The k^{th} incomplete moment of Z can be obtained from (12) as

$$\psi_k(t)=\sum_{j,\ell,m=0}^{\infty}\mathrm{A}_{j,\ell,m}\int_0^t 2\gamma(\ell+m+1)z^{k+1}(1+2z)^{\gamma(\ell+m+1)-1}(1+z)^{-2\gamma(\ell+m+1)-1}dz$$
$$=\sum_{j,\ell,m,q=0}^{\infty}D_{j,\ell,m,q}\mathrm{B}\left(k+q+2,\gamma(\ell+m+1)-k,t(1+t)^{-1}\right),$$

where $D_{j,\ell,m,q}=2\mathrm{A}_{j,\ell,m}\gamma(\ell+m+1)\binom{\gamma(\ell+m+1)-1}{q}$, and $B(\cdot,\cdot,t)$ is the incomplete beta function. The Bonferroni and Lorenz curves, as well as the mean residual life and mean waiting time, are all notable uses of the first incomplete moment.

3.4. Entropy measures

The entropy of a random variable with a probability density (8) is a measure of the uncertainty's fluctuation. A high entropy number suggests that the data is more unpredictable. The Rényi (RE) entropy, presented by Rényi (1960), is defined by

$$\Lambda(\vartheta)=(1-\vartheta)^{-1}\log\left(\int_0^\infty\left(f(z)\right)^\vartheta dz\right),\quad \vartheta\neq1,\quad \vartheta>0.$$

To obtain $\Lambda(\vartheta)$ of the KMOITL distribution, we must first obtain $\left(f(z;\Psi)\right)^\vartheta$ by using the same expansions as given in (9) and (11), then

$$\left(f(z;\Psi)\right)^\vartheta=\sum_{j,\ell=0}^{\infty}(-1)^j\binom{\vartheta(d-1)}{j}\frac{(2cd\bar{\varphi}\gamma)^\vartheta\varphi^\ell\Gamma\left(\vartheta(c+1)+cj+\ell\right)z^\vartheta\ (1+z)^{-\vartheta(2\gamma+1)}}{\ell!\Gamma\left(\vartheta(c+1)+cj\right)}\times$$
$$(1+2z)^{\vartheta(\gamma-1)}\left(1-\Xi(z,\gamma)\right)^{\vartheta(c-1)+cj+\ell}\left(\Xi(z,\gamma)\right)^\ell.$$

Using the binomial expansion in the previous equation leads to

$$\left(f(z;\Psi)\right)^\vartheta=\mathrm{H}_{j,\ell,m}z^\vartheta(1+2z)^{\gamma(\ell+m)+\vartheta(\gamma-1)}(1+z)^{-2\gamma(\ell+m)-\vartheta(2\gamma+1)},$$
$$\mathrm{H}_{j,\ell,m}=\sum_{j,\ell,m=0}^{\infty}(-1)^{j+m}\binom{\vartheta(d-1)}{j}\binom{\vartheta(c-1)+cj}{m}\frac{(2cd\bar{\varphi}\gamma)^\vartheta\varphi^\ell\Gamma\left(\vartheta(c+1)+cj+\ell\right)}{\ell!\Gamma\left(\vartheta(c+1)+cj\right)}.$$

The Rényi entropy of the KMOITL distribution is obtained as follows:

$$\Lambda(\vartheta)=(1-\vartheta)^{-1}\log\left[\zeta_{j,\ell,m,n}\mathrm{B}\left(n+\vartheta+1,\gamma(\ell+m+\vartheta)+\vartheta-1\right)\right],$$
$$\zeta_{j,\ell,m,n}=\sum_{n=0}^{\infty}\binom{\gamma(\ell+m)+\vartheta(\gamma-1)}{n}\mathrm{H}_{j,\ell,m}.$$

The Arimoto entropy (AO) measure (see Arimoto 1971) is defined by

$$\omega(\vartheta)=\frac{\vartheta}{1-\vartheta}\left[\left(\int_0^\infty (f(z))^\vartheta dz\right)^{\frac{1}{\vartheta}}-1\right],\quad \vartheta\neq 1,\quad \vartheta>0.$$

Hence, the AO of the KMOITL distribution is given by

$$\omega(\vartheta)=\frac{\vartheta}{1-\vartheta}\left[\left(\zeta_{j,\ell,m,n}\mathrm{B}\left(n+\vartheta+1,\gamma(\ell+m+\vartheta)+\vartheta-1\right)\right)^{\frac{1}{\vartheta}}-1\right].$$

The Tsallis (TS) entropy measure (see Tsallis, 1988) is defined by

$$\eta(\vartheta)=\frac{1}{\vartheta-1}\left[1-\int_0^\infty (f(z))^\vartheta dz\right],\quad \vartheta\neq 1,\quad \vartheta>0.$$

Hence, the TS entropy of MOKITL distribution is obtained as follows:

$$\eta(\vartheta)=\frac{1}{\vartheta-1}\left[1-\zeta_{j,\ell,m,n}\mathrm{B}\left(n+\vartheta+1,\gamma(\ell+m+\vartheta)+\vartheta-1\right)\right].$$

Table 2 illustrates the numerical entropy measures for some of the recommended parameter values. We conclude from Table 2 that the values of all entropy measures decrease with the increasing value of ϑ.

3.5. Stochastic ordering

Stochastic ordering is an extensively researched notion in probability distributions and is an essential tool in reliability theory and other domains to examine comparative behaviour across random variables. Assume that Z_i has the KMOITL distribution with a set of the parameters Ψ_i i =1, and 2. Let's $F_i(z,\Psi_i)$ indicate Z_i's cdf and $f_i(z,\Psi_i)$ signify Z_i's pdf. If $f_1(z,\Psi_1)/f_2(z,\Psi_2)$ is a decreasing function for all values of z, then Z_1 is said to be stochastically smaller than Z_2 (denoted by $Z_1\leq_{lr} Z_2$), in terms of likelihood ratio order.

Table 2 Entropy values for the KMOITL model

ϑ	(ϕ,γ,c,d)	TS	RE	AO
0.5	(0.6,3,0.5,2)	11.525	3.823	44.733
	(0.85,1.2,10,10)	0.948	0.776	1.173
	(0.85,2,5,2)	1.777	1.271	2.566
	(0.5,2.5,3,2)	0.243	0.229	0.257
	(0.6,0.7,5,8)	3.714	2.099	7.161
	(0.5,0.5,3,3)	19.418	4.742	113.679
1.5	(0.6,3,2,0.5)	1.096	1.589	1.234
	(0.85,1.2,10,10)	0.341	0.374	0.352
	(0.85,2,5,2)	0.435	0.49	0.452
	(0.5,2.5,3,2)	0.602	0.716	0.637
	(0.6,0.7,5,8)	1.067	1.525	1.195
	(0.5,0.5,3,3)	1.463	2.632	1.752

Let $Z_1\sim$ KMOITL (Ψ_1), $\Psi_1\equiv(c_1,d_1,\gamma_1,\varphi_1)$ and $Z_2\sim$ KMOITL (Ψ_2), $\Psi_2\equiv(c_2,d_2,\gamma_2,\varphi_2)$, then the likelihood ratio ordering is as follows:

$$\frac{f_1(z;\Psi_1)}{f_2(z;\Psi_2)}=\frac{c_1d_1\gamma_1\varphi_1(1+z)^{-2\gamma_1-1}(1+2z)^{\gamma_1-1}\left(1-\Xi(z,\gamma_1)\right)^{c_1-1}\left(1-\varphi_2\Xi(z,\gamma_2)\right)^{c_2+1}\mathfrak{A}_1^{d_1-1}}{c_2d_2\gamma_2\varphi_2(1+z)^{-2\gamma_2-1}(1+2z)^{\gamma_2-1}\left(1-\Xi(z,\gamma_2)\right)^{c_2-1}\left(1-\varphi_1\Xi(z,\gamma_1)\right)^{c_1+1}\mathfrak{A}_2^{d_2-1}},$$

$$\mathfrak{A}_1=\left\{1-\left[\frac{1-\Xi(z,\gamma_1)}{1-\varphi_1\left[\Xi(z,\gamma_1)\right]}\right]^{c_1}\right\},\ \mathfrak{A}_2=\left\{1-\left[\frac{1-\Xi(z,\gamma_2)}{1-\varphi_2\left[\Xi(z,\gamma_2)\right]}\right]^{c_2}\right\}$$

which leads to

$$\frac{d}{dz}\log\left[\frac{f_1(z;\Psi_1)}{f_2(z;\Psi_2)}\right]=\frac{2\left(\gamma_2-\gamma_1\right)}{1+z}+\frac{2\left(\gamma_1-\gamma_2\right)}{1+2z}+\left[\frac{(c_1-1)}{\left(1-\Xi(z,\gamma_1)\right)}-\frac{(c_1+1)\varphi_1}{\left(1-\varphi_1\Xi(z,\gamma_1)\right)}\right]\frac{d\Xi(z,\gamma_1)}{dz}$$
$$-\left[\frac{(c_2-1)}{\left(1-\Xi(z,\gamma_2)\right)}-\frac{(c_2+1)\varphi_2}{\left(1-\varphi_2\Xi(z,\gamma_2)\right)}\right]\frac{d\Xi(z,\gamma_2)}{dz}+\frac{(d_1-1)}{\mathfrak{A}_1}\frac{d\mathfrak{A}_1}{dz}-\frac{(d_2-1)}{\mathfrak{A}_2}\frac{d\mathfrak{A}_2}{dz},$$

$$\frac{d\Xi(z,\gamma_i)}{dz}=\frac{-2\gamma_i z(1+2z)^{\gamma_i-1}}{(1+z)^{2\gamma_i+1}},\ i=1,2,\ \frac{d\mathfrak{A}_i}{dz}=\left[\frac{c_i\left[1-\Xi(z,\gamma_i)\right]^{c_i-1}\left(1-\varphi_i\right)}{\left[1-\varphi_i\Xi(z,\gamma_i)\right]^{c_i+1}}\right]\frac{d\Xi(z,\gamma_i)}{dz},\ i=1,2.$$

For $c_1<c_2, d_1<d_2<\gamma_1<\gamma_2, \varphi_1<\varphi_2,$ we get $\frac{d}{dz}\log\left[\frac{f_1(z;\Psi_1)}{f_2(z;\Psi_2)}\right]<0,$ for all $z\geq 0,$ hence $\frac{d}{dz}\log\left[\frac{f_1(z;\Psi_1)}{f_2(z;\Psi_2)}\right]$ is decreasing in *z* and hence $Z_1\leq_{lr}Z_2$. Moreover, Z_1 is said to be smaller than Z_2 in other different orderings as stochastic order (denoted by $Z_1\leq_{st}Z_2$), hazard rate order (denoted by $Z_1\leq_{hr}Z_2$), and reversed hazard rate order (denoted by $Z_1\leq_{rhr}Z_2$).

3.6. Stress Strength Reliability

In reliability studies, the stress-strength (S-S) model is frequently employed. Strength failure, structures, degradation of rocket motors, and static fatigue of ceramic components are just some of the applications of the S-S model in physics and engineering. In the S-S model, reliability $\Re=P\left(Z_1<Z_2\right)$ refers to the component's capacity to withstand random stress Z_2 when it possesses strength Z_1. If the applied stress exceeds the component's strength, it will fail. Let Z_1 and Z_2, be two independent random variables with distributions of KMOITL$(c_1,d_1,\gamma_1,\varphi)$ and KMOITL$(c_2,d_2,\gamma_2,\varphi)$). The KMOITL distribution's S-S reliability is then calculated using the same expansions in (12) and (13) as follows:

$$\Re=1-A^*N^*\int_0^\infty 2z\left\{\frac{(1+2z)^{\gamma_1(\ell+m+1)-1}}{(1+z)^{2\gamma_1(\ell+m+1)+1}}\right\}\frac{\left[1+2z\right]^{\gamma_2(u+v)}}{\left[1+z\right]^{2\gamma_2(u+v)}}dz$$
$$=1-A^*N^*\int_0^\infty 2z\left\{\frac{(1+2z)^{\gamma_1(\ell+m+1)+\gamma_2(u+v)-1}}{(1+z)^{2\gamma_1(\ell+m+1)+2\gamma_2(u+v)+1}}\right\}dz$$
$$=1-\frac{A^*N^*}{\gamma_1(\ell+m+1)+\gamma_2(u+v)},$$

where

$$A^*=\sum_{j,\ell,m=0}^{\infty}(-1)^{j+m}\binom{d_1-1}{j}\binom{c_1(j+1)-1}{m}\frac{2c_1d_1\bar{\varphi}\gamma_1\Gamma\left(c_1(j+1)+1+\ell\right)(\varphi)^\ell}{\Gamma\left(c_1(j+1)+1\right)\ell!}\ \text{and}$$

$$N^{*}=\sum_{i,u,v=0}^{\infty}\binom{d_2}{i}\binom{c_2 i}{v}\frac{(\varphi)^{u}(-1)^{i+v}\Gamma\left(c_2 i+u\right)}{\Gamma\left(c_2 i+1\right)u!}.$$

4. Parameter Estimation of the KMOITL Model

The parameters estimators of the KMOITL distribution based on ML and Bayesian methods are discussed in this section. The approximate confidence intervals (CIs) and Bayesian credible intervals are given.

4.1. ML estimators

Asume $z_1,...,z_n$ be the obsrved values from the KMOITL distribution with parameters $\Psi=(\varphi,c,d,\gamma)$. The likelihood function, say $L\left(\underline{z}\mid\Psi\right)$ of the KMOITL distribution, is expressed as

$$L\left(\underline{z}\mid\Psi\right)=\left(2cd\bar{\varphi}\gamma\right)^{n}\prod_{i=1}^{n}z_i\,\frac{(1+2z_i)^{\gamma-1}}{(1+z_i)^{2\gamma+1}}\frac{\left(1-\Xi(z_i,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z_i,\gamma)\right)^{c+1}}\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\}^{d-1}, \quad (15)$$

$\wp(z_i,\gamma)=\dfrac{1-\Xi(z_i,\gamma)}{1-\varphi\left[\Xi(z_i,\gamma)\right]}$. Then the log likelihood function, say $\ell\left(\Psi\right)$, of the KMOITL distribution is given as

$$\ell\left(\Psi\right)=n\left[\ln\left(2cd\right)+\ln\left(1-\varphi\right)+\ln\left(\gamma\right)\right]+\sum_{i=1}^{n}\ln\left(z_i\right)+(\gamma-1)\sum_{i=1}^{n}\ln(1+2z_i)-(2\gamma+1)\sum_{i=1}^{n}\ln(1+z_i)$$
$$+\left(c-1\right)\sum_{i=1}^{n}\ln\left(1-\Xi(z_i,\gamma)\right)-\left(c+1\right)\sum_{i=1}^{n}\ln\left(1-\varphi\Xi(z_i,\gamma)\right)+\left(d-1\right)\sum_{i=1}^{n}\ln\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\},$$

where $\wp(z_i,\gamma)=\left[\dfrac{1-\Xi(z,\gamma)}{1-\varphi\left[\Xi(z,\gamma)\right]}\right]$. Therefore, the ML equations are given by

$$\frac{\partial\ell\left(\Psi\right)}{\partial\varphi}=\frac{-n}{1-\varphi}+\left(c+1\right)\sum_{i=1}^{n}\frac{\Xi(z_i,\gamma)}{1-\varphi\Xi(z_i,\gamma)}-\sum_{i=1}^{n}\frac{\left(d-1\right)c\left(\Xi(z_i,\gamma)\right)\left(1-\Xi(z_i,\gamma)\right)\left(\wp(z_i,\gamma)\right)^{c-1}}{\left[1-\left[\wp(z_i,\gamma)\right]^{c}\right]\left[1-\varphi\Xi(z_i,\gamma)\right]^{2}},$$

$$\frac{\partial\ell\left(\Psi\right)}{\partial c}=\frac{n}{c}+\sum_{i=1}^{n}\ln\left(1-\Xi(z_i,\gamma)\right)-\sum_{i=1}^{n}\ln\left(1-\varphi\Xi(z_i,\gamma)\right)-\left(d-1\right)\sum_{i=1}^{n}\frac{\left[\wp(z_i,\gamma)\right]^{c}\ln\left[\wp(z_i,\gamma)\right]}{1-\left[\wp(z_i,\gamma)\right]^{c}},$$

$$\frac{\partial\ell\left(\Psi\right)}{\partial d}=\frac{n}{d}+\sum_{i=1}^{n}\ln\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\},$$

and

$$\frac{\partial\ell\left(\Psi\right)}{\partial\gamma}=\frac{n}{\gamma}-\sum_{i=1}^{n}\frac{\left(c-1\right)}{1-\Xi(z_i,\gamma)}\frac{\partial\Xi(z_i,\gamma)}{\partial\gamma}+\sum_{i=1}^{n}\ln(1+2z_i)-2\sum_{i=1}^{n}\ln(1+z_i)+\sum_{i=1}^{n}\frac{\left(c+1\right)\varphi}{1-\varphi\Xi(z_i,\gamma)}\frac{\partial\Xi(z_i,\gamma)}{\partial\gamma}$$
$$-\sum_{i=1}^{n}\frac{c\left(d-1\right)\left[\wp(z_i,\gamma)\right]^{c-1}}{1-\left[\wp(z_i,\gamma)\right]^{c}}\frac{\partial\wp(z_i,\gamma)}{\partial\gamma},$$

where

$$\frac{\partial\wp(z_i,\gamma)}{\partial\gamma}=\frac{\partial\Xi(z_i,\gamma)}{\partial\gamma}\left[\frac{\varphi-1}{\left(1-\varphi\Xi(z_i,\gamma)\right)^{2}}\right],\quad\frac{\partial\Xi(z_i,\gamma)}{\partial\gamma}=\frac{(1+2z_i)^{\gamma}\left[\ln(1+2z_i)-\ln(1+z_i)\right]}{(1+z_i)^{2\gamma}}.$$

Solving the non-linear equations $\partial\ell(\Psi)/\partial\varphi = 0, \partial\ell(\Psi)/\partial c = 0, \partial\ell(\Psi)/\partial d = 0,$ and $\partial\ell(\Psi)/\partial\gamma = 0$ numerically using optimization algorisms, we determine the ML estimators of $\Psi = (\varphi, c, d, \gamma).$

Furthermore, it is known that under regularity conditions, the asymptotic distribution of ML estimators of a set of parameters $\Psi = (\varphi, c, d, \gamma)$ is given by:

$$(\hat{\varphi}-\varphi), (\hat{c}-c), (\hat{d}-d), (\hat{\gamma}-\gamma) \sim N(0, I^{-1}(\varphi, c, d, \gamma)),$$

where $I^{-1}(\varphi, c, d, \gamma)$ is the variance covariance matrix of set of parameters $\Psi = (\varphi, c, d, \gamma).$ Therefore, the two-sided approximate $\varsigma\, 100$ percent CIs for ML estimates of Ψ can be obtained as follows:

$$L_\Psi = \hat{\Psi} - z_{\varsigma/2}\sqrt{\text{var}(\hat{\Psi})}\ , \quad U_\Psi = \hat{\Psi} + z_{\varsigma/2}\sqrt{\text{var}(\hat{\Psi})}, \ \ \Psi = (\varphi, c, d, \gamma),$$

where $z_{\varsigma/2}$ is the $100(1-\varsigma)\%^{\text{th}}$ standard normal percentile and var(.) denotes the diagonal elements of the variance covariance matrix corresponding to the model parameters.

4.2. Bayesian estimators

Here, we get the Bayesian estimator of the KMOITL parameters. The Bayesian estimator is regarded under SELF, LINEX, and ELF, which are defined, respectively, by:

$$L(\tilde{\Psi}, \Psi) = \left(\tilde{\Psi} - \Psi\right)^2,$$

$$L(\tilde{\Psi}, \Psi) = e^{\delta(\tilde{\Psi}-\Psi)} - \delta(\tilde{\Psi}-\Psi) - 1, \ \delta \neq 0, \ \tilde{\Psi}_L = \frac{-1}{\delta}\ln E_\Psi\left(e^{-\delta\Psi}\right),$$

$$L(\tilde{\Psi}, \Psi) = \left(\frac{\tilde{\Psi}}{\Psi}\right)^\delta - \delta\ln\left(\frac{\tilde{\Psi}}{\Psi}\right) - 1, \ \delta \neq 0, \ \tilde{\Psi}_E = \left[E_\Psi\left(\Psi^{-\delta}\right)\right]^{-1/\delta},$$

where δ is reflects the direction and degree of asymmetry. Assuming that the prior distribution of Ψ denoted by $\pi(\varphi), \pi(\gamma), \pi(c), \pi(d)$ has an independent gamma distribution. The gamma prior was elected because the inverted Topp-Leone, Marshall-Olkin-G, and Kumaraswamy-G distributions all employ a gamma distribution as a prior, and indeed we can recognize the normalized posterior distribution as the kernel of the gamma distribution. The gamma distribution can be a standard choice for non-negative continuous data i.e. 0→∞ because that is the domain of the gamma distribution. It may thus often be used as a prior for the precision of a KMOITL distribution.

The joint gamma prior density of φ, c, d and γ can be written as:

$$\pi(\Psi) \propto \varphi^{a_1-1}e^{-b_1\varphi}c^{a_2-1}e^{-b_2c}d^{a_3-1}e^{-b_3d}\gamma^{a_4-1}e^{-b_4\gamma}; a_j, b_j > 0, \quad j = 1,2,3,4. \tag{16}$$

To elicit the hyper-parameters of the informative priors, the ML estimator for φ, c, d and γ is obtained by equating the estimates and their variances by the inverse of the Fisher information matrix of $\hat{\varphi}, \hat{c}, \hat{d}$ and $\hat{\gamma}.$ For more information, see Dey et al. (2016).

From (15) and the joint prior density (16), the joint posterior of the KMOITL distribution with parameters φ, c, d and γ is

$$\pi(\Psi|\underline{z}) \propto \pi(\Psi)L(\underline{z}\,|\,\Psi).$$

Then the joint posterior can be written as

$$\pi\left(\Psi\left|\underline{z}\right.\right)\propto\bar{\varphi}^{n}\varphi^{a_1-1}e^{-b_1\varphi}c^{n+a_2-1}e^{-b_2c}d^{n+a_3-1}e^{-b_3d}\gamma^{n+a_4-1}e^{-b_4\gamma}$$
$$\times\prod_{i=1}^{n}\frac{(1+2z_i)^{\gamma-1}}{(1+z_i)^{2\gamma+1}}\frac{\left(1-\Xi(z_i,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z_i,\gamma)\right)^{c+1}}\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\}^{d-1}.$$

To obtain the Bayesian estimators, we can use the Markov Chain Monte Carlo (MCMC) approach. A useful sub-class of the MCMC techniques is Gibbs sampling and the more general Metropolis within Gibbs samplers. The Metropolis-Hastings (MH) algorithm and Gibbs sampling are two of the most common MCMC methods. We use the MH within Gibbs sampling steps to generate random samples from conditional posterior densities of (Ψ) as follows:

$$\pi\left(\varphi\left|c,d,\gamma,\underline{z}\right.\right)\propto\bar{\varphi}^{n}\varphi^{a_1-1}e^{-b_1\varphi}\prod_{i=1}^{n}\frac{\left(1-\Xi(z_i,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z_i,\gamma)\right)^{c+1}}\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\}^{d-1},$$

$$\pi\left(c\left|\varphi,d,\gamma,\underline{z}\right.\right)\propto c^{n+a_2-1}e^{-b_2c}\prod_{i=1}^{n}\frac{\left(1-\Xi(z_i,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z_i,\gamma)\right)^{c+1}}\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\}^{d-1},$$

$$\pi\left(d\left|\varphi,c,\gamma,\underline{z}\right.\right)\propto d^{n+a_3-1}e^{-d\left(b_4+\sum_{i=1}^{n}\ln\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\}\right)},$$

$$\pi\left(\gamma\left|\varphi,c,d,\underline{z}\right.\right)\propto\gamma^{n+a_4-1}e^{-b_4\gamma}\prod_{i=1}^{n}\frac{(1+2z_i)^{\gamma-1}}{(1+z_i)^{2\gamma+1}}\frac{\left(1-\Xi(z_i,\gamma)\right)^{c-1}}{\left(1-\varphi\Xi(z_i,\gamma)\right)^{c+1}}\left\{1-\left[\wp(z_i,\gamma)\right]^{c}\right\}^{d-1}.$$

The Bayesian estimators are obtained via SELF, LINEX, and ELF. The 95% two-sided highest posterior density (HPD) credible interval for the unknown parameters $\left[\Psi_{0.025N:N},\Psi_{0.975N:N}\right]$ or any function of them is given by using the method proposed by Chen and Shao (1999), where N is a length of MCMC result.

5. Performance Analysis by Monte-Carlo Simulation

A Monte-Carlo simulation experiment is carried out to analyze the performance of point estimates in terms of mean squared errors (MSE), as well as the performance of interval estimates in terms of accuracy and confidence interval length (L.CI). With various parameter values and sample sizes in mind, the simulation study is carried out. This section is broken into two sub-sections, the first of which is a simulation study, and the second in which the findings of the simulation are outlined as follows.

5.1. Simulation study

First, we select the true values of parameters for a KMOITL distribution as:

$\varphi = 0.5$, $c = 0.7$, $d = 0.8$ and γ is changed from 1.2 to 3 in Table 3.

$\varphi = 0.5$, $d = 2$, $\gamma = 3$ and c is changed from 0.7 to 2 in Table 4.

$\varphi = 0.85$, $c = 2$, $\gamma = 3$ and d is changed from 0.8 to 2 in Table 5.

Altogether, nine sets of simulations of the KMOITL data with different sample sizes as n = 30, 75, and 150 are generated. To avoid the starting bias, 5,000 points are generated for each sample simulation. The generated data of KMOITL distribution are obtained by using quantile function in Subsection 3.2. The simulated data are fitted into the KMOITL model. The estimates of ML and Bayesian methods based on the different loss functions are obtained by using the Newton-Raphson algorithm of numerical analysis and the Metropolis-Hastings algorithm, respectively. The iterative algorithms have been used to obtain $N = 5{,}000$ estimates for each parameter when the first initial is the actual parameter. In the confidence interval, we used the 5% level of significant. This simulation study was implemented via R packages.

5.2. Simulation results

For each estimated item parameter, the MSE and L.CI were calculated. Two summary measures of item parameter recovery are considered. MSE values near to zero indicate situations with best-unbiased estimator item parameter estimates. Tables 3-5 show the results of different strategies for estimating point and interval parameters. The results are shown in Tables 3-5, which include some intriguing data.

- The estimates get more accurate as the sample size grows larger, suggesting that they are asymptotically unbiased.
- In all cases, the MSE reduces as the sample size grows, suggesting that the various estimates are consistent.
- When comparing the various estimates, we can observe that in the majority of cases, the Bayes estimates have the lowest MSE.
- ELF, LINEX, SELF are good alternative losses in Bayesian estimation compared to the ML estimate (MLE).
- The Bayesian estimates via LINEX possess the best performance measures compared to other losses.
- As n grows larger, the L-CI for estimates decreases, suggesting that the CI is the shortest.

Table 3 ML and Bayesian estimation methods based on different loss function with different values of γ

$\varphi = 0.5,\ c = 0.7,\ d = 0.8$								
γ	n		MLE		SELF		LINEX ($\delta = -1.5$)	
			MSE	L.CI	MSE	L.CI	MSE	L.CI
1.2	30	φ	1.2586	4.1337	0.0271	0.5939	0.0253	0.5856
		c	0.2990	1.9238	0.0240	0.6018	0.0258	0.6114
		d	0.2127	1.7823	0.0309	0.6543	0.0335	0.6638
		γ	0.7085	2.9585	0.0564	0.9295	0.0580	0.9578
	75	φ	0.3276	2.1915	0.0175	0.5000	0.0166	0.4864
		c	0.0838	1.0502	0.0112	0.3917	0.0115	0.3949
		d	0.1357	1.4256	0.0153	0.4597	0.0157	0.4728
		γ	0.4788	2.5774	0.0261	0.6068	0.0270	0.6125
	150	φ	0.0855	1.1442	0.0080	0.3412	0.0079	0.3356
		c	0.0246	0.5839	0.0047	0.2606	0.0048	0.2613
		d	0.0797	1.1008	0.0061	0.2984	0.0062	0.2999
		γ	0.2528	1.9257	0.0102	0.3918	0.0101	0.3894
3	30	φ	1.5428	4.5247	0.0308	0.6160	0.0286	0.6090
		c	0.2792	1.8737	0.0266	0.6107	0.0300	0.6159
		d	0.3000	2.0012	0.0296	0.6551	0.0319	0.6764
		γ	0.5794	2.8945	0.0658	0.9568	0.0657	0.9504
	75	φ	0.3341	2.1983	0.0173	0.4931	0.0165	0.4852
		c	0.0611	0.9058	0.0114	0.4001	0.0119	0.4073
		d	0.0864	1.1220	0.0140	0.4593	0.0148	0.4605
		γ	0.3472	2.2896	0.0328	0.7052	0.0331	0.7124
	150	φ	0.1187	1.3171	0.0088	0.3542	0.0085	0.3500
		c	0.0187	0.5225	0.0042	0.2530	0.0043	0.2548
		d	0.0374	0.7420	0.0061	0.2888	0.0061	0.2891
		γ	0.2457	1.9359	0.0115	0.4151	0.0116	0.4172

Table 3 (Continued)

$\varphi = 0.5,\ c = 0.7,\ d = 0.8$								
γ	n		LINEX $(\delta = 1.5)$		ELF $(\delta = -1.5)$		ELF $(\delta = 1.5)$	
			MSE	L.CI	MSE	L.CI	MSE	L.CI
1.2	30	φ	0.0292	0.5977	0.0238	0.5644	0.0694	0.6820
		c	0.0227	0.5909	0.0241	0.5973	0.0252	0.6150
		d	0.0291	0.6351	0.0312	0.6520	0.0312	0.6567
		γ	0.0560	0.9142	0.0556	0.9268	0.0625	0.9719
	75	φ	0.0186	0.5096	0.0159	0.4723	0.0340	0.6559
		c	0.0109	0.3853	0.0111	0.3895	0.0122	0.3935
		d	0.0149	0.4514	0.0153	0.4652	0.0155	0.4575
		γ	0.0256	0.6087	0.0261	0.6029	0.0266	0.6260
	150	φ	0.0082	0.3443	0.0078	0.3333	0.0103	0.3665
		c	0.0047	0.2588	0.0047	0.2601	0.0052	0.2592
		d	0.0061	0.2969	0.0061	0.2983	0.0062	0.3009
		γ	0.0103	0.3904	0.0101	0.3902	0.0105	0.3950
3	30	φ	0.0334	0.6186	0.0267	0.5810	0.0758	0.6786
		c	0.0240	0.5979	0.0270	0.5999	0.0262	0.6152
		d	0.0279	0.6320	0.0299	0.6598	0.0298	0.6644
		γ	0.0675	0.9731	0.0655	0.9596	0.0677	0.9721
	75	φ	0.0183	0.5044	0.0158	0.4747	0.0319	0.6584
		c	0.0110	0.3894	0.0114	0.4016	0.0121	0.3904
		d	0.0134	0.4542	0.0141	0.4555	0.0138	0.4665
		γ	0.0328	0.7038	0.0327	0.7058	0.0330	0.7093
	150	φ	0.0092	0.3571	0.0083	0.3475	0.0124	0.4095
		c	0.0041	0.2512	0.0042	0.2528	0.0042	0.2521
		d	0.0060	0.2851	0.0061	0.2883	0.0061	0.2862
		γ	0.0115	0.4142	0.0115	0.4156	0.0115	0.4157

Table 4 ML and Bayesian estimation methods based on different loss function with different values of c

$\varphi = 0.5,\ d = 2,\ \gamma = 3$								
c	n		MLE		SELF		LINEX $(\delta = -1.5)$	
			MSE	L.CI	MSE	L.CI	MSE	L.CI
1.2	30	φ	1.1156	3.7705	0.0299	0.5749	0.0273	0.5702
		c	0.0821	1.0393	0.0163	0.4340	0.0180	0.4443
		d	1.2634	4.1623	0.0652	0.9914	0.0652	0.9861
		γ	0.9649	3.3794	0.0674	1.0223	0.0686	1.0077
	75	φ	0.4097	2.3125	0.0161	0.4681	0.0152	0.4615
		c	0.0219	0.5565	0.0061	0.2579	0.0064	0.2623
		d	0.6496	3.0060	0.0267	0.6483	0.0271	0.6482
		γ	0.3074	1.9233	0.0321	0.7057	0.0323	0.7089
	150	φ	0.2236	1.7285	0.0064	0.3045	0.0062	0.2997
		c	0.0090	0.3657	0.0024	0.1910	0.0024	0.1931
		d	0.3476	2.2165	0.0100	0.3824	0.0100	0.3825
		γ	0.1623	1.4370	0.0105	0.4034	0.0106	0.4046
3	30	φ	1.5509	4.5062	0.0230	0.5153	0.0206	0.5058
		c	1.1329	3.9114	0.0595	0.9100	0.0582	0.9044
		d	0.9995	3.5797	0.0650	0.9950	0.0668	1.0018
		γ	0.5480	2.6594	0.0658	1.0093	0.0671	0.9896
	75	φ	0.5374	2.7378	0.0107	0.3736	0.0100	0.3652
		c	0.5286	2.7434	0.0241	0.5819	0.0243	0.5944
		d	0.4392	2.4704	0.0273	0.6267	0.0273	0.6251
		γ	0.2066	1.7057	0.0306	0.6633	0.0311	0.6577
	150	φ	0.1414	1.4324	0.0039	0.2406	0.0037	0.2357
		c	0.2605	1.9403	0.0102	0.3933	0.0101	0.3873
		d	0.1772	1.6213	0.0104	0.3892	0.0104	0.3923
		γ	0.1291	1.3610	0.0115	0.4150	0.0115	0.4146

Table 4 (Continued)

$\varphi = 0.5,\ d = 2,\ \gamma = 3$								
c	n		LINEX $(\delta = 1.5)$		ELF $(\delta = -1.5)$		ELF $(\delta = 1.5)$	
			MSE	L.CI	MSE	L.CI	MSE	L.CI
1.2	30	φ	0.0327	0.5750	0.0257	0.5484	0.0826	0.6630
		c	0.0146	0.4276	0.0166	0.4367	0.0152	0.4271
		d	0.0667	0.9862	0.0646	0.9906	0.0689	1.0101
		γ	0.0680	0.9948	0.0673	1.0185	0.0686	1.0130
	75	φ	0.0172	0.4769	0.0147	0.4509	0.0325	0.6373
		c	0.0059	0.2543	0.0060	0.2588	0.0068	0.2537
		d	0.0268	0.6524	0.0267	0.6466	0.0272	0.6591
		γ	0.0323	0.7102	0.0320	0.7059	0.0324	0.7117
	150	φ	0.0066	0.3061	0.0062	0.2984	0.0086	0.3230
		c	0.0024	0.1894	0.0024	0.1904	0.0025	0.1879
		d	0.0100	0.3829	0.0100	0.3828	0.0101	0.3836
		γ	0.0105	0.4026	0.0105	0.4037	0.0105	0.4027
3	30	φ	0.0258	0.5218	0.0196	0.4884	0.0689	0.6249
		c	0.0621	0.9200	0.0588	0.9033	0.0641	0.9368
		d	0.0645	0.9825	0.0648	0.9954	0.0667	0.9974
		γ	0.0659	0.9808	0.0657	1.0077	0.0667	1.0054
	75	φ	0.0115	0.3831	0.0097	0.3600	0.0213	0.4857
		c	0.0242	0.5800	0.0240	0.5833	0.0245	0.5861
		d	0.0276	0.6343	0.0272	0.6219	0.0280	0.6419
		γ	0.0304	0.6592	0.0306	0.6638	0.0307	0.6604
	150	φ	0.0040	0.2427	0.0037	0.2350	0.0051	0.2548
		c	0.0103	0.3986	0.0101	0.3926	0.0104	0.3999
		d	0.0105	0.3900	0.0104	0.3890	0.0106	0.3932
		γ	0.0115	0.4165	0.0115	0.4144	0.0115	0.4137

Table 5 ML and Bayesian estimation methods based on different loss functions with different values of d

$\varphi = 0.85,\ c = 2,\ \gamma = 3$								
d	n		MLE		SELF		LINEX $(\delta = -1.5)$	
			MSE	L.CI	MSE	L.CI	MSE	L.CI
1.2	30	φ	0.2628	1.9016	0.0054	0.2210	0.0047	0.2129
		c	0.4089	2.4675	0.0600	0.9251	0.0571	0.9210
		d	1.2339	3.9194	0.0670	0.9885	0.0694	0.9923
		γ	0.4697	2.4488	0.0641	0.9649	0.0678	0.9832
	75	φ	0.0404	0.7545	0.0017	0.1333	0.0016	0.1299
		c	0.1933	1.7136	0.0277	0.6643	0.0271	0.6466
		d	0.4214	2.4054	0.0281	0.6640	0.0283	0.6665
		γ	0.1016	1.1720	0.0293	0.6483	0.0297	0.6514
	150	φ	0.0089	0.3590	0.0005	0.0799	0.0005	0.0796
		c	0.1048	1.2657	0.0104	0.3912	0.0103	0.3895
		d	0.1442	1.4324	0.0100	0.3861	0.0101	0.3840
		γ	0.0305	0.6608	0.0106	0.3888	0.0108	0.3851
3	30	φ	0.3314	2.1509	0.0107	0.2843	0.0092	0.2719
		c	0.7210	3.2486	0.0612	0.9665	0.0574	0.9257
		d	0.2314	1.7733	0.0307	0.6500	0.0345	0.6838
		γ	0.3978	2.4312	0.0672	0.9674	0.0700	0.9831
	75	φ	0.0385	0.7436	0.0025	0.1654	0.0023	0.1634
		c	0.3203	2.2017	0.0292	0.6722	0.0283	0.6648
		d	0.0448	0.7918	0.0155	0.4705	0.0164	0.4810
		γ	0.2370	1.9019	0.0306	0.6945	0.0307	0.6839
	150	φ	0.0093	0.3661	0.0009	0.1040	0.0008	0.1031
		c	0.1772	1.6416	0.0104	0.3886	0.0103	0.3866
		d	0.0190	0.5304	0.0064	0.3027	0.0065	0.3052
		γ	0.1549	1.5307	0.0112	0.4028	0.0114	0.4042

Table 5 (Continued)

$\varphi = 0.85$, $c = 2$, $\gamma = 3$								
d	n		LINEX $(\delta = 1.5)$		ELF $(\delta = -1.5)$		ELF $(\delta = 1.5)$	
			MSE	L.CI	MSE	L.CI	MSE	L.CI
1.2	30	φ	0.0061	0.2260	0.0050	0.2167	0.0077	0.2392
		c	0.0642	0.9300	0.0590	0.9202	0.0657	0.9486
		d	0.0658	0.9824	0.0668	0.9859	0.0683	1.0064
		γ	0.0621	0.9492	0.0642	0.9635	0.0638	0.9613
	75	φ	0.0018	0.1358	0.0016	0.1321	0.0020	0.1377
		c	0.0286	0.6777	0.0275	0.6619	0.0289	0.6845
		d	0.0281	0.6595	0.0281	0.6646	0.0286	0.6628
		γ	0.0292	0.6382	0.0293	0.6493	0.0294	0.6463
	150	φ	0.0005	0.0802	0.0005	0.0797	0.0005	0.0807
		c	0.0106	0.3918	0.0104	0.3903	0.0106	0.3926
		d	0.0099	0.3852	0.0100	0.3857	0.0100	0.3853
		γ	0.0106	0.3883	0.0106	0.3892	0.0106	0.3872
3	30	φ	0.0123	0.3000	0.0097	0.2769	0.0184	0.3264
		c	0.0668	1.0017	0.0596	0.9526	0.0722	1.0302
		d	0.0278	0.6264	0.0314	0.6564	0.0289	0.6383
		γ	0.0662	0.9469	0.0672	0.9721	0.0675	0.9642
	75	φ	0.0026	0.1690	0.0024	0.1640	0.0029	0.1741
		c	0.0303	0.6768	0.0289	0.6708	0.0306	0.6842
		d	0.0147	0.4639	0.0157	0.4724	0.0149	0.4722
		γ	0.0309	0.6979	0.0306	0.6921	0.0310	0.7000
	150	φ	0.0009	0.1052	0.0008	0.1035	0.0009	0.1060
		c	0.0105	0.3929	0.0104	0.3879	0.0106	0.3931
		d	0.0063	0.3017	0.0064	0.3038	0.0063	0.3050
		γ	0.0112	0.4022	0.0112	0.4031	0.0112	0.4027

6. Applications to Physical Data

In this section, we fit the KMOITL distribution into two distinct real data sets and we compare the performance with that of Topp Leone inverted Kumaraswamy (TLIK) (Behariy et al. 2020), Weibull-Lomax (WL) (Tahir et al. 2015), new exponential ITL (NEITL) (Metwally et al. 2021), modified Kies ITL (MKITL) (Almetwally et al., 2021), Kumaraswamy Weibull (KW) (Cordeiro et al., 2010), Marshall-Olkin alpha power inverse Weibull (MOAPIW) (Basheer et al. 2021), and odds exponential-Pareto IV (OEPIV) (Baharith et al., 2020) distributions. We focus on the physical data set because of its many applications and has been used in many fields such as engineering, agriculture, medicine, and other different sciences. So, the first data collected by Birnbaum and Saunders (1969) describes the fatigue 101 times of 6061-T6 aluminum coupons with a maximum stress per cycle of 26,000 psi. The second data sat was discussed in Ristić and Kundu (2015), which represents the strength data measured in GPA, the single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at a gauge length 1 mm.

The MLE and their accompanying standard error (SE) of the model parameters are calculated for each physical data set. The effectiveness of the models is evaluated using the Kolmogorov-Smirnov statistic (KSS) with P-value (PV), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), Anderson-Darling (A*) statistic, and the Cramér-von Mises (W*) statistic. The model with the lowest value of these measures represents the data set better than the others.

Table 6 lists the MLEs and their accompanying SEs for the model parameters for the first data. Table 7 includes the above-mentioned statistical measures for all models. As seen in Table 7, the KMOITL distribution fits the first data better than other fitted models. As a result, the KMOITL model might be the best option. Figure 3 gives the PP-plots of the fitted models. Figure 4 provides the estimated cdf with empirical for different models, while Figure 5 shows the estimated pdf with a histogram of probability. Figure 6 confirmed the estimates of KMOITL distribution parameters are maximum point of likelihood value.

Table 6 MLE with SE for different models of first data

	KMOITL		KITL		MKITL	
	Estimate	SE	Estimate	SE	Estimate	SE
φ	0.9733	0.0194				
c	4807.4947	15.1594	603.5653	840.6244		
d	5253.6846	19.9050	173.9450	320.4342	18.8251	1.4623
γ	0.4175	0.0925	0.7193	0.2620	0.1043	0.0004
	OEPIV		**WL**		**MOAPIW**	
	Estimate	SE	Estimate	SE	Estimate	SE
c	298.1848	681.5192	2.2714	13.7151	873.8797	1.1494
d	0.2307	0.0334	14.6647	1.2930	4.2440	0.0222
θ	0.0205	0.0320	0.1305	0.0986	1489357.3062	556.6520
γ	2281.0608	977.6489	9.4390	9.8743	1846464.1491	48.6719

Table 7 Different measures for different models of first data

	KSS	PV	AIC	BIC	W*	A*
KMOITL	0.0416	0.9948	1499.6943	1507.1548	0.0199	0.1523
WL	0.0498	0.9639	1499.7509	1509.9114	0.0295	0.2050
KITL	0.0499	0.9627	1499.7663	1507.5116	0.0296	0.2184
MKITL	0.0510	0.9551	1500.5630	1507.7932	0.0315	0.2172
MOAPIW	0.1109	0.1663	1520.7786	1531.2391	0.0852	0.5565
OEPIV	0.0495	0.9657	1499.7850	1509.9454	0.0285	0.1959

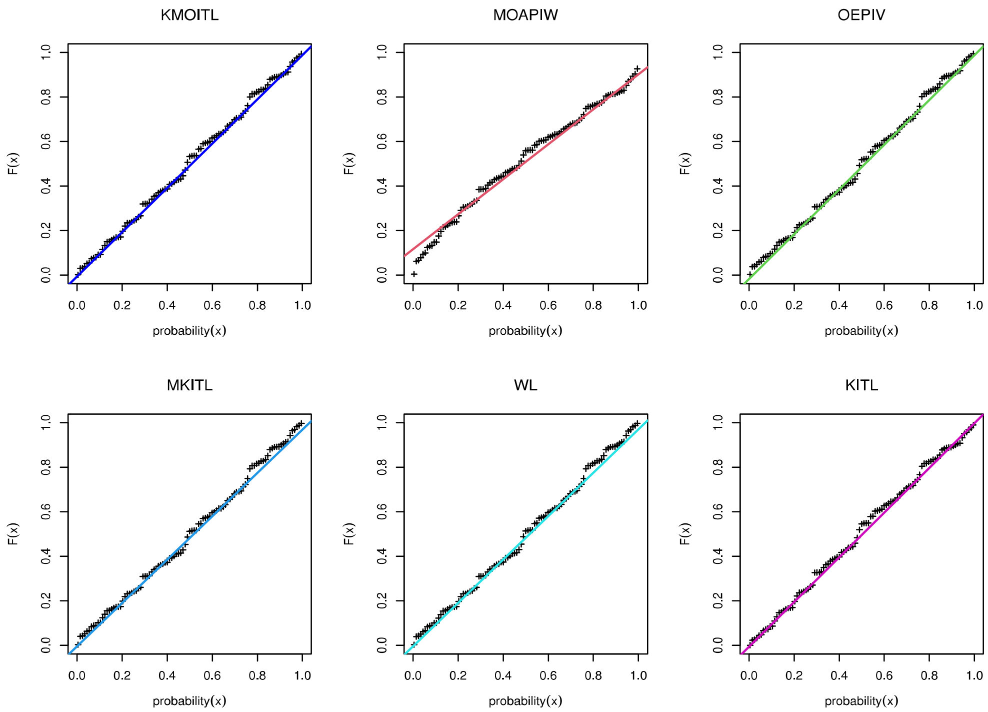

Figure 3: The PP plots for different models

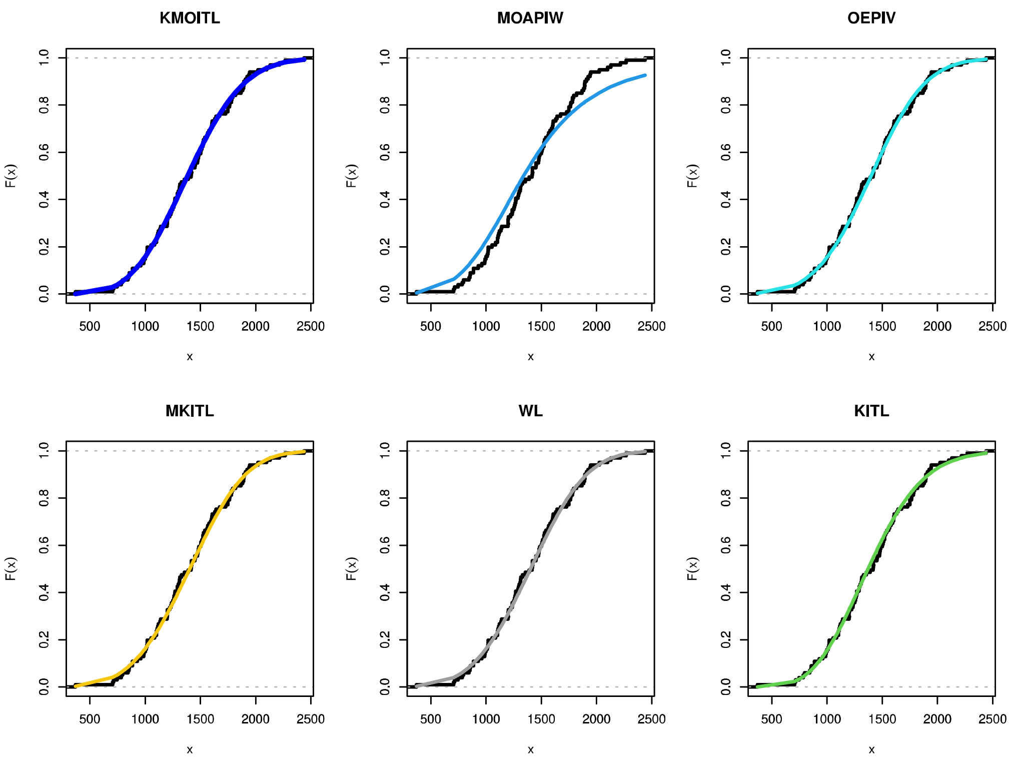

Figure 4 Estimated and empirical cdf for different models

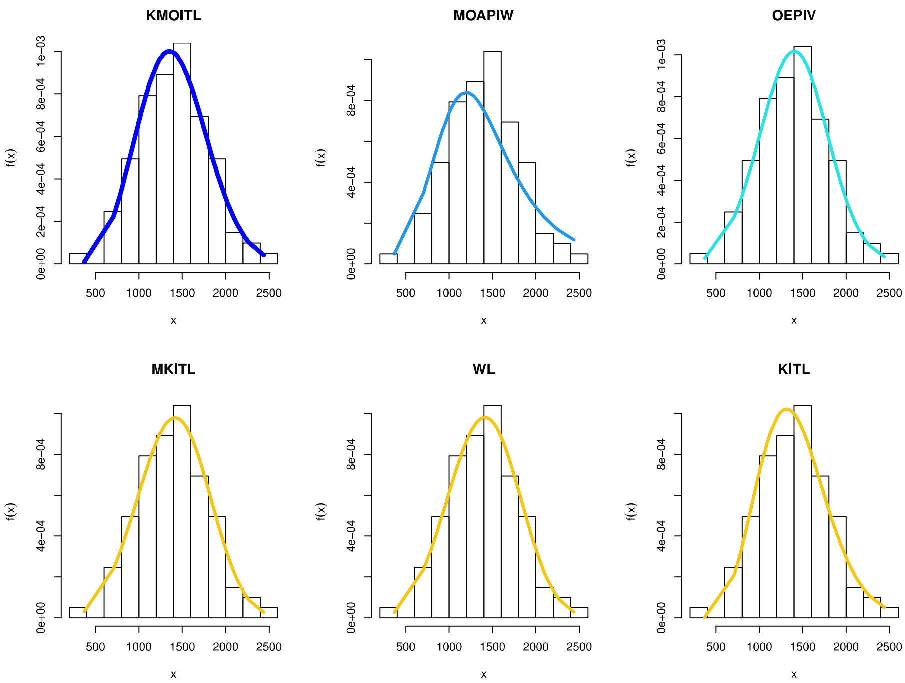

Figure 5 Estimated pdf with histogram of probability for different models

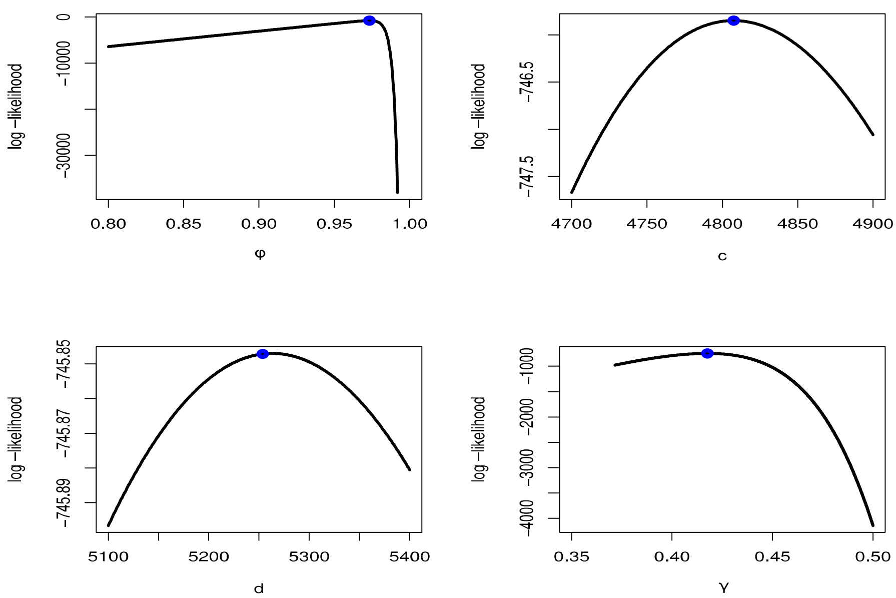

Figure 6 Profile likelihood for parameters for the KMOITL model of the first data

Table 8 shows the MLEs for the model parameters for the second data set, as well as the SEs that go with them. The above-mentioned statistical indicators are included in Table 9 for all models. As seen in Table 9, the KMOITL distribution matches the second data better than other fitted models. As a result, the KMOITL model may be the most appropriate choice. Figure 7 depicts the fitted models' PP-plots. Figure 8 displays the estimated cdf with their empirical for various models, on the other hand Figure 9 depicts the estimated pdf with a probability histogram. Figure 10 confirmed the estimates of KMOIT distribution parameters are maximum point of likelihood value.

Table 8 MLE with SE for different models of strength data

	KMOITL		KITL		MKITL	
	Estimate	SE	Estimate	SE	Estimate	SE
φ	0.9456	0.7564				
c	112.5643	13.8934	13.06934	2.545155		
d	537.6359	35.1579	200.0008	145.7951	6.2059	0.6391
γ	0.6001	0.9641	0.991621	0.201765	0.6186	0.0101
	KW		**WL**		**TLIK**	
	Estimate	SE	Estimate	SE	Estimate	SE
c	0.0080	0.0044	47.4593	6.5699	0.678367	0.185589
d	4.1936	0.9938	8.1616	1.9922	188.9215	73.99957
θ	2.8883	1.0412	0.3652	0.1335	5.82707	0.587424
γ	0.2909	0.4782	1.6541	1.5890		
	MOAPIW		**OEPIV**		**NEITL**	
	Estimate	SE	Estimate	SE	Estimate	SE
α	10.5695	19.2738	40.7601	52.7389		
β	7.9752	0.7340	0.1777	0.0185		
θ	353.0412	361.6230	54.1619	91.1551	50.7331	31.5002
γ	100.1504	99.2120	18.1516	7.9363	0.0186	0.0117

Table 9 Different measures for different models of strength data

	KSS	PV	AIC	BIC	W*	A*
KMOITL	0.0595	0.9820	143.9724	152.0738	0.0260	0.1797
KW	0.0636	0.9667	143.9780	152.9714	0.0280	0.1883
MOAPIW	0.0660	0.9543	146.7408	154.8422	0.0308	0.2753
OEPIV	0.0966	0.6381	146.0936	154.1950	0.0721	0.4695
MKITL	0.0848	0.7836	144.0877	152.1384	0.0551	0.3631
WL	0.0824	0.8120	145.1803	153.2817	0.0566	0.3733
TLIK	0.1415	0.1929	165.0296	171.1057	0.2157	0.5361
KITL	0.0993	0.6035	147.2683	154.3444	0.0330	0.4817

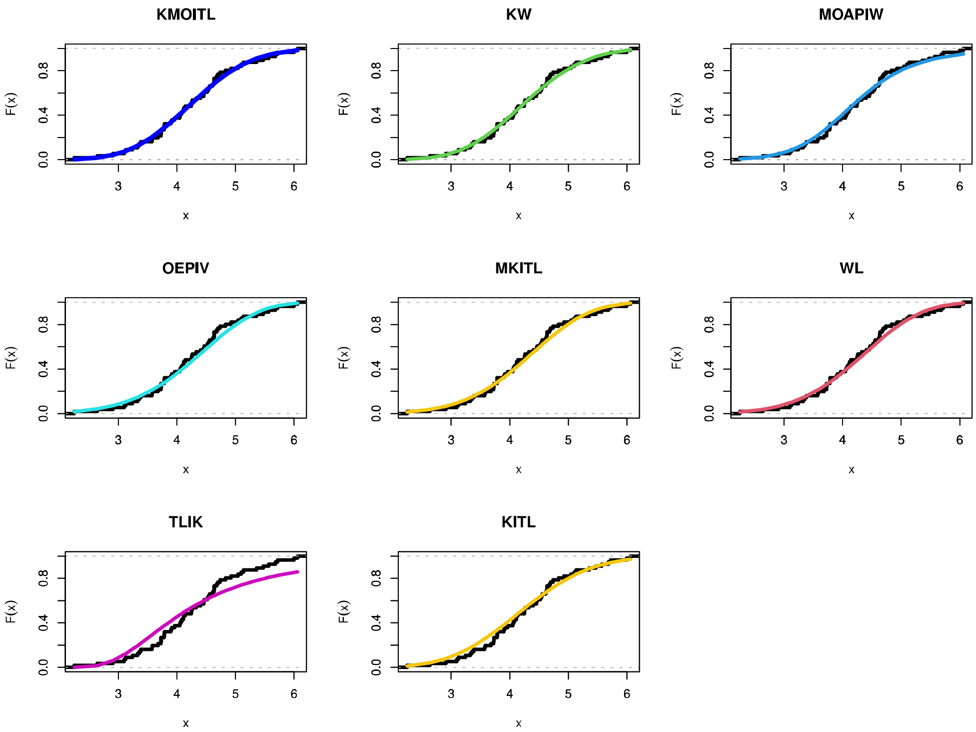

Figure 7 Estimated and empirical cdf for different models of strength data

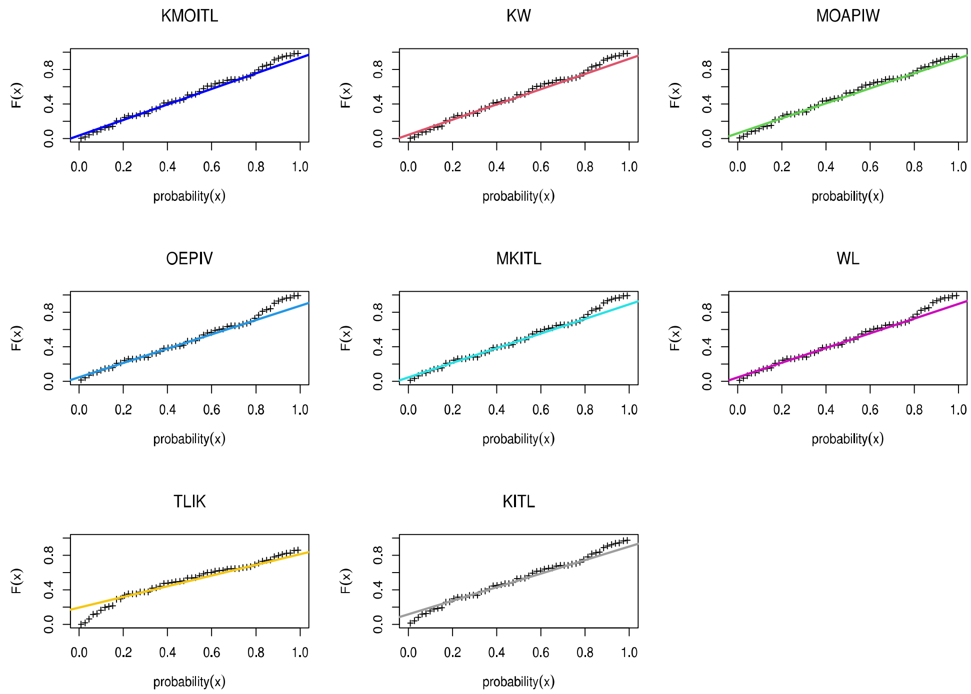

Figure 8 PP plots for different models for strength data

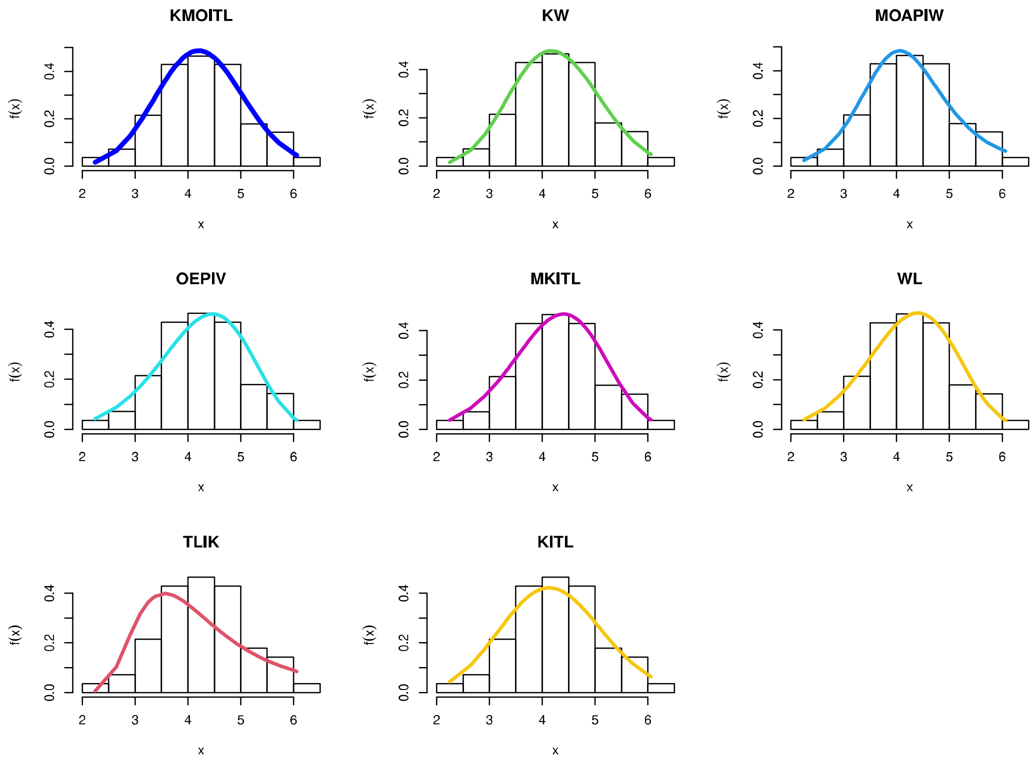

Figure 9 Histogram and estimated pdf for different models of strength data

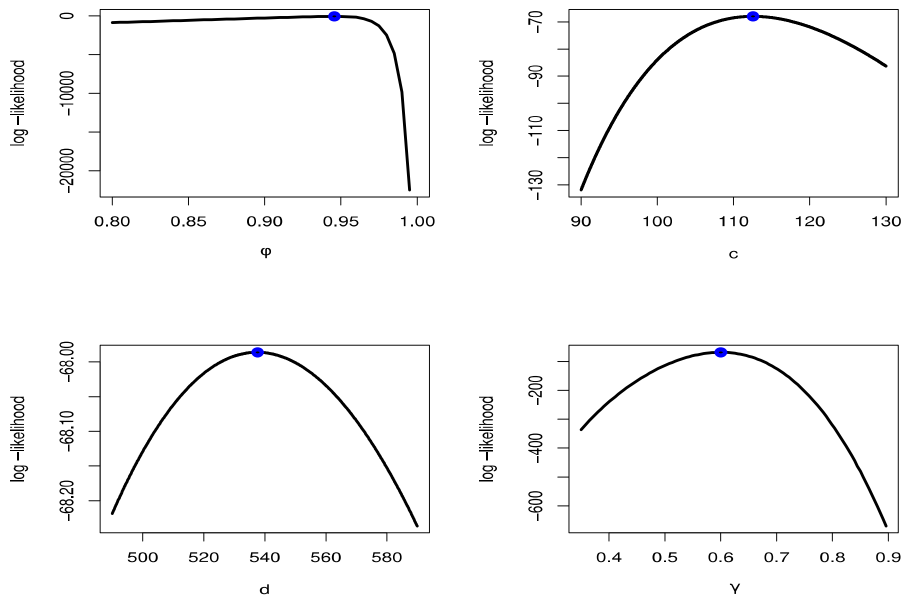

Figure 10 Profile likelihood for parameters for KMOITL model of strength data

7. Concluding Remarks

We propose the Kumaraswamy Marshall-Olkin inverted Topp-Leone distribution, with a four-parameter as a particular model from the Kumaraswamy Marshall-Olkin-G family. The KMOITL includes some sub-models such as the Kumaraswamy ITL, Marshall-Olkin ITL, and ITL distributions. The KMOITL distribution offers several forms of the density and hazard rate functions. The density function of the KMOITL model can be represented as a linear combination of the inverted Topp-Leone densities. Several statistical properties, including moments, incomplete moments, quantile function, some entropy measures, stochastic ordering, and stress-strength reliability, are derived. Simulation studies are used to assess maximum likelihood estimators and approximate confidence intervals. The MCMC method is used to construct Bayesian estimators and Bayesian credible intervals under different loss functions. The overall results revealed that as the sample size becomes greater; the estimators become more accurate, implying that they are asymptotically unbiased. Bayesian estimators via LINEX are preferred over other Bayesian estimators. Finally, two physical real data sets are used to assess the new model's flexibility. Based on the specified criteria, it is discovered that the KMOITL model offers closer matches than certain other models.

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