Best Linear Unbiased Estimators of the Three Parameter Gamma Distribution using doubly Type-II censoring

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Abstract

Recently order statistics and their moments have assumed considerable interest in many applications involving data relating to life testing. In this article, the single and double moments of order statistics from the three parameter gamma distribution are studied. The best linear unbiased estimators (BLUE's) of the location and scale parameters based on the moments of order statistics for doubly type II censored samples are obtained. In particular, the BLUE's based on type II right censored and complete samples are obtained as special cases. Furthermore, the BLUE's are used to construct the confidence intervals for the location and scale parameters. In addition, the sampling distribution of the pivotal quantities are obtained for different sample sizes and censorning types using Pearson's system technique. Monte Carlo simulation is performed to obtain these estimators and the sampling distribution numerically using Mathcad statistical package.

Key words: *Best Linear Unbiased Estimator; Order Statistics; Moments; Gamma Distribution; Doubly Type- II Censored Samples; Approximate Confidence Intervals; Pivotal Quantities; Sampling Distribution ;Monte Carlo Simulation.*

1. Introduction

Gamma distribution is the most popular model for analyzing lifetime data, so it has wide applications in the area of life testing and reliability. It has quite a bit of flexibility for analyzing any positive real data set. Shine and Bain (1983) have mentioned that the gamma distribution has received considerable attention in the area of weather analysis. The gamma distribution has a long history and it has several desirable properties. Johnson et al. (1994) have presented some properties of the three-parameter gamma distribution. It is a generalization of the chi-square and exponential distributions.

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The probability density function of the three-parameter gamma distribution is given by

$$f(x;\alpha,\theta,\lambda)=\frac{(x-\theta)^{\alpha-1}}{\Gamma(\alpha)\lambda^{\alpha}}\exp(-(\frac{x-\theta}{\lambda})), \qquad \alpha,\lambda>0,\ x>\theta \tag{1.1}$$

where α, θ and λ are the shape, location and scale parameters, respectively, [see Johnson et al (1994)]. The corresponding distribution function is given by

$$F(x;\alpha,\theta,\lambda)=\frac{\Gamma_x(\alpha)}{\Gamma(\alpha)}, \qquad x>\theta, \quad \alpha,\lambda>0, \tag{1.2}$$

where $\Gamma_x(\alpha)=\frac{1}{\lambda^{\alpha}}\int_0^x (z-\theta)^{\alpha-1}e^{-\left(\frac{z-\theta}{\lambda}\right)}dz.$

The standard form of the three parameter gamma probability density function (1.1) and distribution function (1.2) are obtained by setting $\theta=0$ and $\lambda=1$ in equations (1.1) and (1.2) as follows:

$$f(x;\alpha)=\frac{1}{\Gamma(\alpha)}(x)^{\alpha-1}\exp(-x), \qquad x\geq 0,\alpha>0, \tag{1.3}$$

and

$$F(x;\alpha)=\frac{\Gamma_x(\alpha)}{\Gamma(\alpha)}. \tag{1.4}$$

In many practical applications, such as life testing, it is quite common not to observe complete data but only observe some forms of censored data. This may be based on cost or time considerations. The doubly type II censored samples arise if the first and the last few failure observations are unknown. In particular, if the last few failure observations of a given sample are unknown, the censored sample is referred to as type II right censored samples.

Gupta (1960) has obtained the first four moments of the order statistics from gamma distribution for sample sizes $n=1(1)10$ and shape parameter $\alpha=1(1)5$. Prescott (1974) has computed the variances and covariance of the gamma order statistics for $n=2(1)10$ and $\alpha=2$ and 5. Mehrotra and Nanda (1974) have obtained approximate maximum likelihood estimators for the ordered sample from normal and gamma distributions. In addition, they obtained the BLUE's and made comparison between the two methods to study the efficiencies between them.

For an overview of the applications for BLUE's, Mahmoud et al. (2003) have derived the exact expressions for the single moments of order statistics from the inverse Weibull distribution based on type II right censored sample. The variance and covariance matrix of ordered sample are also calculated. They used these moments to obtain the BLUE's of the location and scale parameters of the inverse Weibull distribution. Furthermore, they applied their results to draw inference of this distribution. Mahmoud et al. (2005) have derived the exact explicit expressions for the single, double, triple, and quadruple moments of order statistics from the generalized Pareto distribution. Also, they obtained the BLUE's of the parameters of this model. Approximate confidence intervals of the parameters from the generalized Pareto distribution are obtained using Edgeworth approximation and comparing them with those based on Monte Carlo simulation. Sajeevkumar and Thomas (2005) have obtained the BLUE's of the scale and location parameters for the normal and double exponential distributions.

This article focuses on the moments of order statistics from the three parameter gamma distribution. These moments will be used to estimate the location and scale parameters based on doubly and right type II censored samples. Monte Carlo simulation is performed to obtain the point and interval estimates of the parameters using Mathcad statistical package.

This article is organized as follows, the single and double moments of order statistics from the three parameter gamma distribution are discussed in Section (2). In Section (3), BLUE's of the parameters are calculated using the moments of order statistics. In Section (4), the confidence intervals of the parameters are constructed through the percentage points of some pivotal quantities; in addition the sampling distribution of these pivotal quantities is obtained. A numerical study is carried out in Section (5). Tables for the numerical study are displayed at the appendix.

2. Moments of Order Statistics

Suppose that $X_{(1:n)} \leq X_{(2:n)} \leq ... \leq X_{(n:n)}$ are the order statistics corresponding to n independent random variables each having standard gamma distribution (1.3), and the probability density function of the rth order statistics $X_{(r:n)}$ is given by:

$$f_n(x_{(r:n)}) = \frac{1}{\beta(\mathrm{r}, \mathrm{n-r+1})} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j x_{(r)}{}^{\alpha-1} e^{-x_{(r)}} (\Gamma_{x_{(r)}}(\alpha))^{r+j-1} (\Gamma(\alpha))^{-(r+j)}, \tag{2.1}$$

$$\mathrm{r} = 1,2...\mathrm{n}, \quad \alpha > 0 \quad \text{and} \quad 0 < x_{(r:n)} < \infty,$$

where $\beta(.,.)$ stands for beta function.

The kth moments of the rth order statistics denoted by $\mu_{r:n}{}^k$, is given by

$$\mu_{r:n}{}^k = E\left(X^k{}_{r:n}\right) = \frac{1}{\beta(\mathrm{r}, \mathrm{n-r+1})} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j (\Gamma(\alpha))^{-(r+j)} g_r(\alpha+k), \tag{2.2}$$

where,

$$g_r(\alpha+\mathrm{k}) = \int_0^\infty \mathrm{x}_{(r)}{}^{\alpha+\mathrm{k}-1} \mathrm{e}^{-\mathrm{x}_{(\mathrm{r})}} (\Gamma_{x_{(r)}}(\alpha))^{\mathrm{r+j-1}} \mathrm{dx}_{(\mathrm{r})}, \qquad r = 1,2,....$$

Let $X_{(r:n)}$ and $X_{(s:n)}$ be any two order statistics from the set $X_{(1:n)} \le X_{(2:n)} \le ... \le X_{(n:n)}$ corresponding to the n independent random variables having the standard gamma distribution (1.3), then the joint density of $X_{(r:n)}$ and $X_{(s:n)}$ is given by

$$f_n(x,y) = \frac{\mathrm{x}^{\alpha-1} y^{\alpha-1} e^{-(x+y)}}{\beta(\mathrm{r}, \mathrm{s-r})\beta(\mathrm{s}, \mathrm{n-s+1})} \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^j (\Gamma_x(\alpha))^{r+j-1} (\Gamma_y(\alpha))^{s-r-1-j}$$
$$\sum_{l=0}^{n-s} \binom{n-s}{l} (-1)^l (\Gamma_y(\alpha))^l \, (\Gamma(\alpha))^{-(l+s)} \qquad 0 < x < y < \infty, \tag{2.3}$$

The joint moment of $X_{(r:n)}$ and $X_{(s:n)}$, $1 \le \mathrm{r} < \mathrm{s} \le \mathrm{n}$ denoted by $\mu_{r,s:n}{}^{k_1,k_2}$ is given by

$$\mu_{r,s:n}{}^{k_1,k_2} = E\left(X_{r:n}{}^{k_1}, Y_{s:n}{}^{k_2}\right) = \frac{1}{\beta(\mathrm{r}, \mathrm{s-r})\beta(\mathrm{s}, \mathrm{n-s+1})} \int_0^\infty \int_x^\infty \mathrm{x}^{\mathrm{k}_1+\alpha-1} y^{k_2+\alpha-1} e^{-(x+y)} \sum_{j=0}^{s-r-1} \binom{s-r-1}{j} (-1)^j$$
$$\left(\Gamma_x(\alpha)\right)^{r+j-1} \left(\Gamma_y(\alpha)\right)^{s-r-1-j} \sum_{l=0}^{n-s} \binom{n-s}{l} (-1)^l (\Gamma_y(\alpha))^l \, (\Gamma(\alpha))^{-(l+s)} \mathrm{dy}\,\mathrm{dx} \tag{2.4}$$

In view of equations (2.2) and (2.4), it is difficult to derive exact forms for the single and joint moments for the three parameter gamma distribution. So, numerical techinque is applied to obtain these moments. In addition, the variance and covariance matrix of order statistics are computed.

3. Best Linear Unbiased Estimation

This Section discusses the BLUE's of the scale and location parameters of the three parameter gamma probability density function (1.1), based on doubly type II censored samples. The BLUE's of the parameters can be determined using single and double moments of order statistics presented in Section (2)

Suppose that $X_{(r+1:n)} \le X_{(r+2:n)} \le \ldots \le X_{(n-s:n)}$ be the available doubly type II censored samples from the proposed density function (1.1) and let

$$z_{(i)} = \left(\frac{x_{(i)} - \theta}{\lambda}\right), \qquad i = r+1, r+2, \ldots, n-s \tag{3.1}$$

be the corresponding order statistics for the standard gamma distribution. Let us denote $E(Z_{(i)})$ by $\mu_{(i)}$, $\mathrm{var}(Z_{(i)})$ by $\sigma_{(i,i)}$ and $cov(Z_{(i)}, Z_{(j)})$ by $\sigma_{(i,j)}$; further let

$$X = (X_{(r+1)}, X_{(r+2)}, \ldots X_{(n-s)})^T,$$

$$\mu = (\mu_{(r+1)}, \mu_{(r+2)}, \ldots \mu_{(n-s)})^T,$$

$$1 = \underbrace{(1,1,\ldots,1)^T}_{n-r-s}$$

and,

$$\sum = (\sigma_{(i,j)}),\ r+1 \le i, j \le n-s$$

According to Balakrishnan and Cohen (1991), the BLUE's of the location parameter θ and, the scale parameter λ are given by

$$\theta^* = \left(\frac{\mu^T \sum^{-1} \mu 1^T \sum^{-1} - \mu^T \sum^{-1} 1 \mu^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(1^T \sum^{-1} 1) - (\mu^T \sum^{-1} 1)^2} \right) X = \sum_{i=r+1}^{n-s} a_{(i)} x_{(i)}, \tag{3.2}$$

and

$$\lambda^* = \left(\frac{1^T \sum^{-1} 1 \mu^T \sum^{-1} - 1^T \sum^{-1} \mu 1^T \sum^{-1}}{(\mu^T \sum^{-1} \mu)(1^T \sum^{-1} 1) - (\mu^T \sum^{-1} 1)^2} \right) X = \sum_{i=r+1}^{n-s} b_{(i)} x_{(i)}. \tag{3.3}$$

In addition, the variance and covariance of θ^* and λ^* are given by

$$Var(\theta^*) = \sigma^2 \left(\frac{\mu^T \sum^{-1} \mu}{(\mu^T \sum^{-1} \mu)(1^T \sum^{-1} 1) - (\mu^T \sum^{-1} 1)^2} \right) = \sigma^2 V_1, \tag{3.4}$$

$$Var(\lambda^*) = \sigma^2 \left(\frac{1^T \sum^{-1} 1}{(\mu^T \sum^{-1} \mu)(1^T \sum^{-1} 1) - (\mu^T \sum^{-1} 1)^2} \right) = \sigma^2 V_2, \tag{3.5}$$

and

$$Cov(\theta^*, \lambda^*) = \sigma^2 \left(\frac{-\mu^T \sum^{-1} 1}{(\mu^T \sum^{-1} \mu)(1^T \sum^{-1} 1) - (\mu^T \sum^{-1} 1)^2} \right) = \sigma^2 V_3. \quad (3.6)$$

For more details, refer to Lloyd (1952), David (1981), Balakrishnan and Cohen (1991), and Arnold et al. (1992). Using equations (3.2) and (3.3) the coefficients of BLUE's based on doubly and right type II censored samples are calculated. The coefficients of BLUE's based on doubly type-II censored samples are displayed in Tables (1) and (2), while Tables (3) and (4) represent the coefficients of BLUE's based on type-II right censored samples.

To check the accuracy of the coefficients in Tables (1) to (4), the following conditions must be verified

$$\sum_{i=r+1}^{n-s} a_i = 1 \quad \text{and} \quad \sum_{i=r+1}^{n-s} b_i = 0\,. \quad (3.7)$$

According to equations (3.4), (3.5) and (3.6), the variance, covariance and mean square errors (MSEs) of the estimators based on doubly and right type II censored samples are calculated and displayed in Tables (7) and (8).

4. Approximate Confidence Intervals

According to Balakrishnan and Cohen (1991) the confidence intervals for the parameters for the proposed distribution were given through the pivotal quantities

$$R_1 = \frac{\theta^* - \theta}{\lambda \sqrt{V_1}}, \quad R_2 = \frac{\theta^* - \theta}{\lambda^* \sqrt{V_1}} \quad \text{and} \quad R_3 = \frac{\lambda^* - \lambda}{\lambda \sqrt{V_2}} \quad (4.1)$$

where, θ^* and λ^* are the BLUE's of θ and λ with variances $\sigma^2 V_1$ and $\sigma^2 V_2$, respectively as shown in equations (3.4),(3.5). R_1 can be used to draw inference for θ when λ is known, while R_2 can be used to draw inference for θ when λ is unknown. Also, R_3 can be used to draw inference for λ.

The confidence intervals for θ are constructed when λ is known through the formula

$$P(\theta^* - \lambda \sqrt{V_1} (R_1)_{1-\delta/2} \le \theta \le \theta^* - \lambda \sqrt{V_1} (R_1)_{\delta/2}) = 1 - \delta \quad (4.2)$$

when λ is unknown, the confidence intervals for θ is given by

$$P(\theta^* - \lambda^* \sqrt{V_1} (R_2)_{1-\delta/2} \le \theta \le \theta^* - \lambda^* \sqrt{V_1} (R_2)_{\delta/2}) = 1 - \delta \quad (4.3)$$

where V_1 is given in equation (3.4).

and the confidence intervals for λ is constructed through the formula

$$P\left(\frac{\lambda^*}{1+\sqrt{V_2}\,(R_3)_{1-\delta/2}} \le \lambda \le \frac{\lambda^*}{1+\sqrt{V_2}\,(R_3)_{\delta/2}}\right) = 1-\delta \tag{4.4}$$

where V_2 is given in equation (3.5).

Since the sampling distributions for the pivotal quantities R_1, R_2 and R_3 are unknown, so the sampling distribution of these pivotal quantities are derived, using Pearson's system approach. The Pearson's system approach is based on computing the following criterion for fixing the distribution family

$$K = \frac{\beta_1(\beta_2+3)^2}{4(4\beta_2-3\beta_1)(2\beta_2-3\beta_1-6)} \tag{4.5}$$

where the two moment ratios $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ denote the skewness and kurtosis measures, respectively and μ_r is the *r*th central moment. So for different values of K, there exist different types of distributions [see Elderton and Johnson (1969)]. The sampling distribution of the pivotal quantities R_1, R_2 and R_3 for different sample sizes using Pearson's system technique are illustrated in Table (9). It is clear from Table (9) that the sampling distributions of the pivotal quantities R_1, R_2 and R_3 follow Pearson's type I, VI, III or IV distributions for some sample sizes. It is found that most of the sampling distributions of the pivotal quantities R_1 and R_2 are close to the normal distribution for different censored samples when the sample sizes n=20, 25.

5. Numerical Illustration

It is obvious that, the single and double moments of order statistics from the three parameter gamma distribution has no exact forms. So the single and double integration involved in the computations of single and double moments were performed using Mathcad statistical package. The method of BLUE's is applied to estimate the parameters of this model. The accuracy of the estimates of the parameters is studied through the properties of the estimates. The performances of the obtained estimators are investigated in terms of their mean square errors (MSEs). The percentage points of the estimates based on the pivotal quantities R_1, R_2 and R_3 are used to construct confidence intervals with confidence level at 0.99. In addition, the sampling distribution of the pivotal quantities is also determined. The simulation procedures are described through the following steps:

Step (1): Let $X_{(r+1:n)} \le X_{(r+2:n)} \le ... \le X_{(n-s:n)}$ be the available doubly type II censored samples of sizes $n = 5(5)25$ from the three parameter gamma distribution with shape parameter $\alpha = 2$.

Step (2): Compute the single and double moments from the doubly type II censored samples with $r = 1,2$ and $s = 1,2.$

Step (3): As a special case, the single and double moments from the type II right censored will be obtained with $r = 0$ and $s = 1,2,3,4.$

Step (4): Compute the variance covariance matrix $\sum$ using steps (2) and (3) and then compute the inverse of variance covariance matrix $\sum^{-1}$.

Step (5): Generate ordered samples of size n from the standard gamma distribution, according to Mathcad package "the commant *pgamma* (x, α) is used to generate random samples from standard gamma distribution".

Step (6): From equations (3.2) and (3.3), the estimates of the scale parameter λ and, location parameter θ are calculated under doubly and right type II censored samples.

Step (7): Compute the pivotal quantities R_1, R_2 and R_3 using equation (4.1).

Step (8): Construct the confidence intervals of the parameters at Confidence level(99%).

Step (9): Repeat the steps from (5) to (9) for specified 5,0000 times .

Step (10): The resulting 5,0000 pivotal quantities R_1, R_2 and R_3 will be used to calculate the first four central moment for different sample sizes and censoring types.

Step (11): Use the central moment of the pivotal quantities R_1, R_2 and R_3 to compute β_1 and β_2.

Step (12): Use β_1 and β_2 to calculate K using equation (4.5).

Numerical results are summarized in Tables (1) to (9). Tables (1) to (4) represent the coefficients of BLUE's based on doubly and right type II censored samples. The variances , covariances and the MSE of the estimators are appeared in Tables (5) and (6). Tables (7) and (8) contain the approximate confidence intervals for the estimators.

Finally the sampling distributions of R_1, R_2 and R_3 under each level of censoring and for different sample sizes are displayed in Table (9). From these Tables, we have:

1- It is clear from Tables (1) to (4) that, for different sample sizes and censored levels, the coefficients of the BLUE's of the parameters based on doubly and right type II censored sample verify the conditions (3.7).

2- Under doubly and right type II censored samples, the estimates of the parameters approach the true values. The covariances of the estimates, $Cov(\theta^*, \lambda^*)$, decrease as the double censoring level increases and as the sample size decreases. The MSE of estimators decreases as the sample size increases, while increases as the censorning levels increase [see Tables (5) and (6)].

3- Regarding to the confidence intervals, as the censoring levels increase, the average width of the confidence intervals increase. Also, as the sample size increases the average width of the confidence intervals decreases for the two mentioned cases for the censored samples [see Tables (7) and (8)].

4- The sampling distributions of the pivotal quantities for different sample sizes and censored samples take the following forms:

a- For small sample sizes (n=5,10) and different censored samples, the sampling distribution of the pivotal quantities R_1, R_2 and R_3 follow Pearson's type IV, III and I respectively.

b- For moderate sample sizes (n=15,20) the sampling distribution of the pivotal quantities R_1, R_2 and R_3 follows Pearson's type VI, IV, III or I for most different censored samples. While in the case of doubly censored samples the sampling distribution of the pivotal quantities R_1 has a normal distribution for n=20.

c- For n=25 the sampling distribution of the pivotal quantities R_1 and R_2 follows normal distribution for most different censored samples, while a few cases has Pearson's type VI, IV, III or I. The sampling distribution of the pivotal quantities R_3 follows Pearson's type IV, I for different censored samples.

d. It is clear from Table (9) that the the sampling distributions of the pivotal quantities R_1 and R_2 are close to the normal distribution for different censored samples at sample sizes n=20, 25.

Appendix

Table(1): Coefficients of the BLUE's for the Location Parameter θ Based on the Doubly Type II Censored Samples with $\alpha = 2$

n	r	s	a_i							
5	0	0	1.293	0.029	-0.066	-0.113	0.142			
	0	0	1.046	0.109	0.039	0.007	-0.012	-0.024	-0.033	-0.039
			-0.044	-0.049						
10	1	1	1.234	0.315	-0.198	-0.063	0.044	-0.041	-0.133	-0.157
	1	2	1.305	0.328	-0.217	-0.075	0.027	-0.046	-0.320	
	2	1	1.555	0.037	-0.037	-0.075	-0.095	-0.117	-0.270	
	0	0	0.937	0.118	0.061	0.036	-0.007	0.030	-0.003	-0.037
			0.009	-0.026	-0.025	-0.019	-0.023	-0.025	-0.027	
15	1	1	1.113	0.089	0.067	-0.016	0.031	0.021	-0.110	0.070
			-0.095	-0.009	-0.041	-0.034	-0.087			
	1	2	1.123	0.282	-0.360	0.452	-0.183	0.082	-0.102	-0.070
			-0.134	0.092	-0.077	-0.105				
	2	1	1.269	0.065	0.019	0.031	0.110	-0.087	0.027	-0.088
			-0.009	-0.071	-0.045	-0.122				
	0	0	0.885	0.132	0.048	0.000	0.060	-0.000	0.002	0.043
			-0.051	0.010	-0.000	-0.000	-0.039	0.019	0.000	-0.054
			0.003	-0.023	-0.016	-0.018				
	1	1	0.994	0.125	0.006	0.061	0.047	0.002	-0.011	-0.000
			-0.021	-0.004	0.004	-0.045	-0.002	0.005	-0.071	-0.000
			-0.031	-0.054						
	1	2	1.017	0.115	0.011	0.062	0.043	0.004	-0.013	-0.007
20			-0.021	-0.004	0.003	-0.050	-0.000	0.002	-0.074	-0.000
			-0.089							
	2	1	1.105	0.118	0.003	0.037	0.010	0.042	-0.006	-0.039
			0.002	-0.034	0.030	-0.022	0.033	-0.111	-0.006	-0.026
			-0.078							
	2	2	1. 124	0.130	0.001	0.031	0.014	0.042	-0.011	-0.040
			0.002	-0.036	0.036	-0.020	0.029	-0.116	-0.006	-0.110
	0	0	0.403	0.215	0.143	0.105	0.079	0.062	0.049	0.038
			0.030	0.023	0.017	0.012	0.007	-0.013	-0.000	-0.004
			-0.007	-0.009	-0.011	-0.016	-0.02	-0.017	-0.023	-0.026
			-0.039							
	1	1	0.521	0.797	-0.201	-0.083	0.039	0.015	0.056	0.011
			-0.050	0.119	-0.130	-0.017	0.040	0.122	-0.115	-0.07
			0.002	0.425	-0.351	-0.177	0.275	-0.192	-0.035	
25	1	2	0.536	0.793	-0.198	-0.088	0.046	0.005	0.063	0.011
			-0.057	0.124	-0.134	-0.018	0.042	0.121	-0.116	-0.070
			-0.002	0.423	-0.355	-0.172	0.272	-0.228		
	2	1	1.104	-0.080	0.112	-0.008	0.123	0.005	-0.039	0.047
			0.003	-0.113	0.060	0.087	-0.058	-0.151	-0.003	0.672
			-0.558	-0.151	0.085	-0.077	-0.060	0.000		
	2	2	1.091	0.019	0.148	-0.003	-0.023	0.054	0.024	-0.047
			-0.056	0.071	0.086	-0.085	-0.153	0.008	0.708	-0.602
			-0.176	0.103	-0.093	-0.073	0.000			

Note that $r = 0$ and $s = 0$ stand for complete samples

Table(2): Coefficients of the BLUE's for the Scale Parameter λ Based on the Doubly Type II Censored Samples with $\alpha = 2$.

n	r	s	b_i							
5	0	0	-0.456	0.085	0.133	0.157	0.171			
10	0	0	-0.473	-0.005	0.031	0.047	0.056	0.062	0.067	0.069
			0.072	0.074						
	1	1	-0.554	-0.084	0.132	0.089	0.033	0.078	0.117	0.189
	1	2	-0.638	-0.100	0.155	0.103	0.054	0.085	0.342	
	2	1	-0.646	0.033	0.074	0.090	0.100	0.111	0.237	
15	0	0	-0.430	-0.034	0.006	0.013	0.040	0.017	-0.030	0.172
			-0.078	0.139	0.012	0.033	0.048	0.046	0.048	
	1	1	-0.512	-0.008	-0.000	0.047	0.015	-0.041	0.213	-0.108
			0.176	0.005	0.047	0.055	0.112			
	1	2	-0.558	-0.049	0.130	-0.096	0.052	0.013	0.100	0.067
			0.110	0.011	0.038	0.181				
	2	1	-0.551	0.000	0.031	0.015	-0.036	0.203	-0.088	0.173
			0.005	0.061	0.060	0.128				
20	0	0	-0.417	-0.047	0.007	0.023	-0.002	0.023	0.024	0.006
			0.051	0.032	-0.004	0.050	0.034	-0.004	0.078	0.035
			-0.004	0.051	0.033	0.034				
	1	1	-0.468	-0.031	0.019	-0.003	0.000	0.025	0.034	0.028
			0.050	-0.003	0.049	0.038	0.006	0.079	0.044	-0.002
			0.056	0.078						
	1	2	-0.051	-0.016	0.011	-0.004	0.005	0.022	0.036	0.033
			0.050	-0.002	0.049	0.045	0.003	0.083	0.048	-0.002
			0.140							
	2	1	-0.492	-0.034	0.025	0.005	0.021	0.009	0.029	0.058
			-0.005	0.067	0.031	0.015	0.066	0.063	0.000	0.054
			0.089							
	2	2	-0.513	-0.047	0.026	0.011	0.017	0.009	0.035	0.059
			-0.006	0.069	0.038	0.013	0.070	0.069	0.000	0.150
25	0	0	-0.410	-0.099	0.073	-0.035	0.009	0.032	0.035	-0.048
			0.001	0.117	-0.025	0.037	0.049	-0.049	0.046	0.101
			-0.018	-0.058	0.138	0.150	-0.179	0.046	0.082	0.028
			-0.000							
	1	1	-0.470	0.032	-0.082	0.054	-0.011	0.051	-0.035	-0.024
			0.103	-0.025	0.062	0.022	-0.019	0.028	0.093	0.007
			-0.054	0.073	0.195	-0.154	0.025	0.097	0.034	
	1	2	-0.485	0.036	-0.085	0.059	-0.018	0.060	-0.041	-0.024
			0.109	-0.029	0.066	0.023	-0.022	0.029	0.094	0.008
			-0.055	0.072	0.193	-0.157	0.023	0.095	0.037	
	2	1	-0.478	0.043	-0.090	0.054	-0.019	0.058	-0.041	-0.026
			0.119	-0.040	0.070	0.025	-0.035	0.022	0.102	0.005
			-0.088	0.095	0.245	-0.140	0.021	0.078	0.030	
	2	2	-0.508	0.036	-0.036	-0.014	0.029	0.009	0.037	0.062
			-0.070	0.023	0.130	0.021	-0.055	-0.063	-0.063	0.309
			-0.108	-0.009	0.115	0.054	-0.000			

Table(3): Coefficients of the BLUE's for the Location Parameter θ Based on the Type II Right Censored Samples with $\alpha = 2$.

n	s	a_i								
5	1	1.427	0.006	-0.102	0.331					
10	1	1.076	0.112	0.049	-0.015	-0.014	-0.023	-0.039	-0.045	-0.101
	2	1.117	0.111	0.048	-0.022	-0.019	-0.028	-0.046	-0.161	
15	1	0.956	0.124	0.057	0.037	-0.007	0.023	-0.005	-0.03	0.000
		0.022	-0.026	0.087	-0.191	-0.003				
	2	0.972	0.122	0.060	0.033	-0.008	0.025	-0.000	-0.048	0.012
		-0.034	-0.026	-0.024	-0.084					
	3	1.021	0.065	-0.023	0.279	-0.230	-0.138	0.555	-0.402	0.194
		-0.325	0.149	-0.144						
20	1	0.895	0.132	0.047	0.001	0.060	-0.001	0.002	-0.042	0.052
		0.008	0.000	-0.002	-0.039	0.019	-0.002	-0.054	0.003	-0.024
		-0.035								
	2	1.085	0.115	0.004	0.037	0.011	0.044	-0.007	-0.034	0.002
		-0.031	-0.028	-0.021	0.037	-0.109	-0.005	-0.023	-0.039	-0.036
	3	1.227	0.066	-0.028	0.074	-0.021	0.054	-0.066	0.000	0.013
		-0.093	0.002	0.030	-0.124	-0.003	-0.045	-0.042	-0.044	
25	1	0.425	0.225	0.148	0.108	0.081	0.062	0.048	0.037	0.028
		0.020	0.014	0.008	0.003	-0.001	-0.005	-0.009	-0.013	-0.015
		-0.017	-0.022	-0.027	-0.026	-0.032	-0.040			
	2	0.444	0.233	0.153	0.110	0.082	0.062	0.047	0.035	0.026
		0.018	0.011	0.005	-0.000	0.005	-0.009	-0.014	-0.018	-0.020
		-0.023	-0.028	-0.033	-0.034	-0.042				
	3	0.463	0.241	0.157	0.112	0.082	0.062	0.046	0.033	0.024
		0.015	0.007	0.001	-0.004	-0.009	-0.014	-0.019	-0.023	0.026
		-0.029	-0.035	-0.041	-0.045					
	4	0.481	0.249	0.162	0.114	0.082	0.061	0.045	0.031	0.021
		0.012	0.004	-0.003	-0.009	-0.014	-0.019	-0.024	-0.029	-0.033
		-0.037	-0.043	-0.050						

Table(4): Coefficients of the BLUE's for the scale Parameter λ Based on the Type II Right Censored Samples with $\alpha = 2$

n	s	b_i								
5	1	-0.707	0.112	0.112	0.177					
10	1	-0.520	-0.011	0.033	0.058	0.063	0.066	0.075	0.078	0.159
	2	-0.586	-0.010	0.035	0.070	0.071	0.073	0.086	0.261	
15	1	-0.450	-0.030	0.006	0.017	0.046	0.015	-0.032	0.189	-0.084
		0.150	0.012	0.053	0.027	0.107				
	2	-0.487	-0.040	0.009	0.017	0.040	0.025	-0.034	0.192	-0.083
		0.154	0.014	0.041	0.155					
	3	-0.576	0.031	0.07	-0.165	0.215	0.102	-0.318	0.313	-0.094
		0.254	-0.058	0.226						
20	1	-0.437	-0.046	0.007	0.021	-0.002	0.024	0.025	0.008	0.052
		0.035	-0.004	0.052	0.035	-0.005	0.082	0.036	-0.004	0.053
		0.069								
	2	-0.460	-0.054	0.022	0.015	-0.003	0.029	0.022	0.008	0.057
		0.035	-0.005	0.052	0.041	-0.007	0.086	0.039	-0.004	0.126
	3	-0.509	-0.003	0.033	-0.007	0.034	0.004	0.067	-0.005	0.044
		0.057	0.004	0.064	0.067	-0.001	0.060	0.045	0.046	
25	1	-0.202	-0.097	-0.056	-0.036	-0.021	-0.012	-0.004	0.002	0.006
		0.011	0.014	0.017	0.019	0.022	0.024	0.026	0.028	0.029
		0.032	0.034	0.034	0.038	0.043	0.048			

Continued Table 4

n	s	b_i								
25	2	-0.225	-0.107	-0.061	-0.039	-0.022	-0.012	-0.003	0.004	0.009
		0.013	0.017	0.021	0.024	0.026	0.029	0.032	0.034	0.035
		0.039	0.041	0.042	0.048	0.055				
	3	-0.248	-0.117	-0.066	-0.041	-0.023	-0.011	-0.002	0.006	0.012
		0.017	0.021	0.025	0.029	0.032	0.035	0.038	0.041	0.043
		0.047	0.050	0.052	0.061					
	4	-0.273	-0.128	-0.074	-0.043	-0.023	-0.010	0.000	0.009	0.016
		0.022	0.026	0.031	0.035	0.039	0.042	0.046	0.049	0.053
		0.058	0.061	0.065						

Table (5): BLUE's of θ and λ, Variances, Covariances and MSEs Based on the Doubly Type II Censored Samples with $\alpha = 2$

n	r	s	θ^*	λ^*	$\text{var}(\theta^*)$	$\text{var}(\lambda^*)$	$\text{cov}(\theta^*, \lambda^*)$	$\text{MSE}(\theta^*)$	$\text{MSE}(\lambda^\bullet)$
5	0	0	-0.005	1.003	0.809	0.453	-0.404	0.809	0.453
10	0	0	-0.004	0.998	0.234	0.169	-0.117	0.234	0.169
	1	1	-0.002	1.001	0.644	0.343	-0.331	0.644	0.343
	1	2	-0.003	1.002	0.966	0.533	-0.533	0.966	0.562
	2	1	-0.007	0.998	1.477	0.600	-0.722	1.477	0.600
15	0	0	-0.002	1.004	0.125	0.102	-0.062	0.125	0.102
	1	1	-0.003	0.999	0.250	0.155	-0.127	0.250	0.155
	1	2	-0.000	0.998	0.283	0.196	-0.154	0.283	0.196
	2	1	-0.001	1.001	0.422	0.209	-0.208	0.442	0.209
20	0	0	0.001	1.005	0.080	0.073	-0.040	0.080	0.073
	1	1	-0.000	0.999	0.136	0.094	-0.069	0.136	0.094
	1	2	-0.003	1.001	0.152	0.109	-0.080	0.152	0.109
	2	1	-0.006	1.002	0.204	0.115	-0.101	0.204	0.115
	2	2	-0.005	1.003	0.250	0.147	-0.131	0.250	0.147
25	0	0	0.008	0.998	0.054	0.053	-0.032	0.054	0.053
	1	1	-0.012	0.999	0.088	0.067	-0.050	0.088	0.067
	1	2	-0.009	0.998	0.095	0.072	-0.055	0.095	0.072
	2	1	-0.012	1.005	0.100	0.076	-0.074	0.100	0.076
	2	2	-0.004	1.002	0.195	0.130	-0.098	0.195	0.130

Table (6): BLUE's of θ and λ, Variances, Covariances and MSEs Based on the Type II Right Censored Samples with $\alpha = 2$

n	s	θ^*	λ^*	$\text{var}(\theta^*)$	$\text{var}(\lambda^*)$	$\text{cov}(\theta^*, \lambda^*)$	$\text{MSE}(\theta^*)$	$\text{MSE}(\lambda^\bullet)$
5	1	-0.008	1.003	1.365	0.900	-0.790	1.365	0.900
10	1	-0.002	1.037	0.139	0.121	-0.074	0.139	0.121
	2	-0.004	1.018	0.365	0.310	-0.211	0.365	0.310
15	1	-0.030	1.031	0.138	0.108	-0.073	0.138	0.108
	2	-0.005	1.003	0.159	0.144	-0.087	0.159	0.144
	3	-0.000	1.000	0.203	0.187	-0.120	0.203	0.187
20	1	-0.002	1.003	0.086	0.081	-0.045	0.086	0.081
	2	-0.003	1.002	0.094	0.092	-0.050	0.094	0.092
	3	-0.023	1.014	0.138	0.135	-0.081	0.138	0.135
25	1	-0.005	0.995	0.064	0.039	-0.033	0.064	0.039
	2	0.002	0.990	0.069	0.044	-0.017	0.069	0.044
	3	-0.005	0.998	0.095	0.080	-0.020	0.095	0.080
	4	-0.006	1.001	0.121	0.115	-0.081	0.121	0.115

Table(7): Confidence Intervals in the case of Doubly Type II censored sample for θ and λ, Based on the Percentage Point of R_1, R_2 and R_3 with Confidence level 99% with $\alpha = 2$.

n	r	s	$R_1(\theta^*)$	$R_2(\theta^*)$	$R_3(\lambda^*)$
5	0	0	(-0.931, 2.162)	(-2.541, 1.267)	(0.279, 3.172)
10	0	0	(-0.787, 0.928)	(-0.495, 0.421)	(0.223, 0.987)
	1	1	(-0.648, 1.757)	(-0.948, 1.025)	(0.416, 1.207)
	1	2	(0.019, 1.842)	(-1.334,1.454)	(0.420, 1.790)
	2	1	(-0.037, 2.992)	(-1.541, 2.10)	(0.547, 1.341)
15	0	0	(-0.676, 0.389)	(-0.243, 0.241)	(0.193, 0.629)
	1	1	(-0.653, 0.966)	(-0.345, 0.433)	(0.296, 0.729)
	1	2	(-0.610, 1.032)	(-1.62, 0.671)	(0.427, 1.302)
	2	1	(-0.610, 1.330)	(-0.425, 0.692)	(0.215, 0.667)
20	0	0	(-0.656, 0.376)	(-0.168, 0.264)	(0.252, 0.624)
	1	1	(-0.562, 0.775)	(-0.198, 0.272)	(0.199, 0.580)
	1	2	(-0.561, 0.787)	(-0.209, 0.287)	(0.211, 0.639)
	2	1	(-0.523, 0.570)	(-0.222, 0.385)	(0.340, 0.785)
	2	2	(-0.569, 0.948)	(-0.244, 0.437)	(0.232, 0.753)
25	0	0	(-0.254, 0.529)	(0.019, 0.431)	(0.215, 0.485)
	1	1	(-0.515, 0.434)	(-0.198, 0.272)	(0.199, 0.580)
	1	2	(-0.447, 0.452)	(-0.143, 0.247)	(0.252, 0.624)
	2	1	(-0.476, 0.558)	(-0.197, 0.272)	(0.170, 0.495)
	2	2	(-0.158, 1.557)	(-0.249, 1.177)	(0.382, 1.084)

Table(8): Confidence Intervals in the case of Type II Right censored sample for θ and λ, Based on the Percentage Point of R_1, R_2 and R_3 with Confidence level 99% with $\alpha = 2$.

n	s	$R_1(\theta^*)$	$R_2(\theta^*)$	$R_3(\lambda^*)$
5	1	(-0.430, 2.870)	(-5.371, 2.009)	(0.359, 6.381)
10	1	(-1.025, 0.620)	(-0.509, 0.229)	(0.387, 1.007)
	2	(-0.687, 1.110)	(-0.752, 0.617)	(0.115, 0.902)
15	1	(-0.452, 0.658)	(0.017, 0.552)	(0.283, 0.644)
	2	(-0.629, 0.669)	(-0.300, 0.297)	(0.245, 0.958)
	3	(-0.625, 0.743)	(-0.378, 0.396)	(0.105, 0.854)
20	1	(-0.493, 0.309)	(-0.242, 0.297)	(0.260, 0.627)
	2	(-0.190, 0.799)	(-0.020, 0.524)	(0.308, 0.779)
	3	(-0.440, 0.553)	(-0.327, 0.328)	(0.327, 0.844)
25	1	(-0.447, 0.289)	(-0.163, 0.174)	(0.183, 0.500)
	2	(-0.190, 0.799)	(-0.176, 0.188)	(0.283, 0.644)
	3	(-0.171, 0.818)	(-0.017, 0.552)	(0.244, 0.600)
	4	(-0.093, 0.899)	(-0.040, 0.655)	(0.370, 0.974)

Table(9): Sampling Distribution of R_1, R_2 and R_3 for Complete, Right Type II and Doubly Censored Samples According to Pearson's type of Distributions.

n	Types of censored	r	s	Pearson's type of $R_1(\theta^*)$	Pearson's type of $R_2(\theta^*)$	Pearson's type of $R_3(\lambda^*)$
5	Complete	0	0	IV	III	I
	Right	0	1	IV	III	I
10	Complete	0	0	IV	III	I
	Right	0	1	IV	III	I
	Right	0	2	IV	I	I
	Doubly	1	1	IV	III	I
	Doubly	1	2	IV	III	I
	Doubly	2	1	VII	III	I
15	Complete	0	0	VI	VI	I
	Right	0	1	IV	VI	I
	Right	0	2	VI	VI	I
	Right	0	3	IV	III	I
	Doubly	1	1	IV	VI	I
	Doubly	1	2	IV	III	I
	Doubly	2	1	IV	VI	I
20	Complete	0	0	VI	IV	I
	Right	0	1	III	VI	I
	Right	0	2	III	III	I
	Right	0	3	IV	III	I
	Doubly	1	1	IV	VI	I
	Doubly	1	2	IV	VI	I
	Doubly	2	1	Normal Dist.	VI	I
	Doubly	2	2	Normal Dist.	VI	I
25	Complete	0	0	Normal Dist.	III	I
	Right	0	1	Normal Dist.	Normal Dist.	IV
	Right	0	2	Normal Dist.	Normal Dist.	IV
	Right	0	3	Normal Dist.	Normal Dist.	IV
	Right	0	4	Normal Dist.	Normal Dist.	IV
	Doubly	1	1	Normal Dist.	Normal Dist.	IV
	Doubly	1	2	IV	I	I
	Doubly	2	1	IV	VI	I
	Doubly	2	2	VI	I	I

References

1- Arnold, B. C., Balakrishnan, N. and Nagaraja, H. N. (1992), *A First Course in Order Statistics,* New York, John Wiley & Sons.

2- Balakrishnan, N. and Cohen, A. C. (1991), *Order Statistics and Inference: Estimation Methods,* San Diego, Academic Press.

3- David, H. A. (1981), *Order Statistics,* 2nd ed. New York, John Wiley & Sons.

4- Elderton, W.P. and Johnson, N.L. (1969), Systems of Frequency Curves, Cambridge University Press.

5- Gupta, S. S. (1960), Order statistics from the gamma distribution, *Technometrics*, Vol.**2**, 243-262.

6- Johnson, N.L., Kotz, S., and Balakrishnan, N. (1994), *Continuous Univariate Distributions,* 2nd ed, Vol. **1.** New York , John Wiley& Sons.

7- Lloyd, E. H. (1952), Least-squares estimators of location and scale parameters using order statistics, *Biometrika*, Vol.**39**, 88-95**.**

8- Mahmoud, M. A. W.,Sultan, K. S. and Amer, S. M. (2003), Order statistics from inverese weibull distribution and associated inference, *Computational Statistics & Data Analysis,* Vol. **42**,149-163.

9- Mahmoud, M. A. W.,Sultan, K. S. and Moshref, M. E. (2005), Inference based on order statistics from the generalized pareto distribution and application, *Communications in Statistics-Simulation and Computation*, Vol.**34**, 267-282.

10- Mehrotra, K. G. and Nanda, P. (1974), Unbiased estimation of parameters by order statistics in the case of censored samples, *Biometrika*, Vol.**61**, 601-606**.**

11- Prescott, P. (1974), Variances and covariances of order statistics from the gamma distribuion, *Biometrika*, Vol.**61**, 607-611**.**

12- Sajeevkumar, N. K. and Thomas, P. Y. (2005), Application of order statistics of independent nonidentically distributed random variables in estimation, *Communications in Statistics-Theory and Methods*, Vol.**34**, 775-783.

13- Shine, W. K. and Bain, L. J. (1983), A two-sample test of equal gamma distribution scale parameters with unknown common shape parameter, *Technometrics*, Vol.**25**, 377-381.