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Estimation in step-stress partially accelerated life tests for the Burr type XII distribution using type I censoring

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Abstract

In this paper, step-stress partially accelerated life tests are considered when the lifetime of a product follows a Burr type XII distribution. Based on type I censoring, the maximum likelihood estimates (MLEs) are obtained for the distribution parameters and acceleration factor. In addition, asymptotic variance and covariance matrix of the estimators are given. An iterative procedure is used to obtain the estimators numerically using Mathcad (2001). Furthermore, confidence intervals of the estimators are presented. Simulation results are carried out to study the precision of the MLEs for the parameters involved.

Keywords: Reliability; Step-stress partially accelerated life test; Accelerated factor; Burr type XII distribution; Maximum likelihood method; Fisher information matrix

1. Introduction

Due to the continual improvement in the manufacturing design, it is more difficult to obtain information about the lifetime of products or materials with high reliability at the time of testing under normal conditions. This makes the lifetime testing under these conditions very costly, take a long time. To get the information about the lifetime distribution of these materials, a sample of these materials is subjected to more severe operation conditions than normal ones. These conditions are called stresses which may be in the form of temperature, voltage, pressure, vibration, cycling rate, load, etc.

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This kind of testing is called the accelerated life test (ALT), where products are put under stresses higher than usual to yield more failure data in a short time. The life data from the high stresses are used to estimate the life distribution at design condition. There are mainly three ALT methods. The first method is called the constant stress ALT; the second one is referred to as step-stress ALT and the third is the progressive stress ALT. The first method is used when the stress remains unchanged, so that if the stress is weak, the test has to loose for a long time. The other two methods, can reduce the testing time and save a lot of manpower, material sources and money (see [25]). The major assumption in ALT is that the mathematical model relating the lifetime of the unit and the stress are known or can be assumed. In some cases, such life–stress relationships are not known and cannot be assumed, i.e ALT data cannot be extrapolated to use condition. So, in such cases, partially accelerated life test (PALT) is a more suitable test to be performed for which tested units are subjected to both normal and accelerated conditions.

According to Nelson [23], the stress can be applied in various ways. One way to accelerate failure in step-stress is by increasing the stress applied to the test product in a specified discrete sequence. Step-stress partially accelerated life test (SS-PALT) is used to get quickly the information of the lifetime of the product with high reliability; specially, when the mathematical model related to test conditions of mean lifetime of the product is unknown and cannot be assumed. The step-stress scheme applies stress to test units in such a way that the stress will be changed at pre-specified time. Generally, a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held for a specified time. Further, stress is repeatedly increased until the test unit fails or the censoring time is reached.

For an overview of SS-PALT, see the sufficient amount of literature on designing SS-PALT. Goel [17] considered the estimation problem of the accelerated factor using both maximum likelihood and Bayesian methods for items having exponential and uniform distributions. DeGroot and Goel [13] estimated the parameters of the exponential distribution and acceleration factor in SS-PALT using Bayesian approach, with different loss functions. Also, Bhattacharyya and Soejoeti [10] estimated the parameters of the Weibull distribution and acceleration factor using maximum likelihood method. Bai and Chung [9] estimated the scale parameter and acceleration factor for exponential distribution under type I censored sample using maximum likelihood method.

Attia et al. [8] considered the maximum likelihood method for estimating the acceleration factor and the parameters of Weibull distribution in SS-PALT under type I censoring. Abdel-Ghaly et al. [1] used Bayesian approach for estimating the parameters of Weibull distribution parameters with known shape parameter. They studied the estimation problem in SS-PALT under both type I and type II censored data. Abdel-Ghani [5] considered the maximum likelihood and Bayesian methods to estimate the parameters of Weibull distribution and the acceleration factor for both SS-PALT and constant-stress PALT under type I and type II censored data. Abdel-Ghaly et al. [2] studied the estimation problem of the acceleration factor and the parameters of Weibull distribution in SS-PALT using maximum likelihood method in two types of data, namely type I and type II censoring.

Abdel-Ghaly et al. [3,4] studied both the estimation and optimal design problems for the Pareto distribution under SS-PALT with type I and type II censoring. Abdel-Ghani [6] considered the estimation problem of log-logistic distribution parameters under SS-PALT.

Recently, Ismail [19] used maximum likelihood and Bayesian methods for estimating the acceleration factor and the parameters of Pareto distribution of the second kind. Ismail [20] studied the estimation and optimal design problems for the Gompertz distribution in SS-PALT with type I censored data. For a concise review of step-stress ALT, readers may refer to [18,24,21,26].

This article is to focus on the maximum likelihood method for estimating the acceleration factor and the parameters of Burr type XII distribution. This work was conducted for SS-PALT under type I censored sample. The performance of the obtained estimators is investigated in terms of relative absolute bias, mean square error and the standard error. Moreover, the confidence intervals of the estimators will be obtained.

This paper can be organized as follows. In Section 2 the Burr type XII distribution is introduced as a lifetime model and the test method is also described. Section 3 presents point and interval estimates of parameters and acceleration factor for the Burr type XII under type I censoring using maximum likelihood method. In Section 4 the approximate asymptotic variances and covariances matrix are investigated. Section 5 explains the simulation studies for illustrating the theoretical results. Finally, conclusions are included in Section 6. Tables are displayed in the Appendix.

2. The model and test model

As a member of the Burr [11] family of distributions, this includes twelve types of cumulative distribution functions with a variety of density shapes. The two parameter Burr type XII distribution denoted by Burr (c, k) has a density function of the form

$$f(t, c, k) = ckt^{c-1}(1 + t^c)^{-(k+1)}, \quad t > 0, c > 0 \text{ and } k > 0 \tag{2.1}$$

where, c and k are the shape parameters of the distribution.

The cumulative distribution function is

$$F(t, c, k) = 1 - (1 + t^c)^{-(k)}, \quad t > 0, c > 0 \text{ and } k > 0. \tag{2.2}$$

The reliability function of the Burr type XII distribution is given by:

$$R(t, c, k) = (1 + t^c)^{-(k)}. \tag{2.3}$$

The Burr(c, k) distribution was first proposed as a lifetime model by Dubey [14,15]. Evans and Simons [16] studied further the distribution as a failure model and they also derived the maximum likelihood estimators as well moments of the Burr(c, k) probability density function. Lewis [22] noted that the Weibull and exponential distributions are special limiting cases of the parameter values of the Burr(c, k)distribution. She proposed the use of the Burr(c, k) distribution as a model in accelerated life test data. Ahmad and Islam [7] developed optimal accelerated life test designs for Burr Type XII distributions under periodic inspection and Type I censoring.

In SS-PALT, all of the n units are tested first under normal condition, if the unit does not fail for a pre-specified time τ, then it runs at accelerated condition until failure. This means that if the item has not failed by some pre-specified time τ, the test is switched to the higher level of stress and it is continued until items fail. The effect of this switch is to multiply the remaining lifetime of the item by the inverse of the acceleration factor β. In this case, switching to the higher stress level will shorten the life of the test item. Thus the total lifetime of a test item, denoted by Y, passes through two stages, which are the normal and accelerated conditions. Then the lifetime of the unit in SS-PALT is given as follows:

$$Y = \begin{cases} T & \text{if } T \le \tau \\ \tau + \beta^{-1}(T - \tau) & \text{if } T > \tau, \end{cases} \tag{2.4}$$

where, T is the lifetime of an item at use condition, τ is the stress change time and β is the acceleration factor which is the ratio of mean life at use condition to that at accelerated condition, usually $\beta > 1$. Assume that the lifetime of the test item follows Burr type XII distribution with shape parameters c and k. Therefore, the probability density function of total lifetime Y of an item is given by:

$$f(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ f_1(y) & \text{if } 0 < y \leq \tau \\ f_2(y) & \text{if } y > \tau \end{cases} \tag{2.5}$$

where, $f_1(y) = cky^{c-1}(1+y^c)^{-(k+1)}c, k > 0$, is the equivalent form to Eq. (2.1), and,

$$f_2(y) = \beta ck[\tau + \beta(y-\tau)]^{c-1}[1 + \{\tau + \beta(y-\tau)\}^c]^{-(k+1)}, \quad c, k > 0, \beta > 1,$$

is obtained by the transformation variable technique using Eqs. (2.1) and (2.4).

3. Maximum likelihood estimation

The maximum likelihood is one of the most important and widely used methods in statistics. The idea behind maximum likelihood parameter estimation is to determine the parameters that maximize the probability (likelihood) of the sample data. Furthermore, maximum likelihood estimators are consistent and asymptotically normally distributed. In this section point and interval estimation for the parameters and acceleration factor of Burr type XII distribution based on type I censoring are evaluated using maximum likelihood method.

3.1. Point estimates

In type I censoring the test terminates when the censoring time η is reached. The observed values of the total lifetime Y are $y_{(1)} < \cdots < y_{(n_u)} \leq \tau < y_{(n_u+1)} < \cdots < y_{(n_u+n_a)} \leq \eta$, where n_u and n_a are the number of items failed at normal conditions and accelerated conditions respectively. Let δ_{1i}, δ_{2i} be indicator functions, such that

$$\delta_{1i} = \begin{cases} 1 & y_{(i)} \leq \tau \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2, \ldots, n$$

and,

$$\delta_{2i} = \begin{cases} 1 & \tau < y_{(j)} \leq \eta \\ 0 & \text{otherwise} \end{cases} \quad i = 1, 2, \ldots, n.$$

For simplifying $y_{(i)}$ can be expressed by y_i. Since the lifetimes $y_1, \ldots, y_n$ of n items are independent and identically distributed random variables, then their likelihood function is given by

$$\begin{aligned} L(\underline{y}; \beta, c, k) = \prod_{i=1}^{n} & \{cky_i^{c-1}(1+y_i^c)^{-(k+1)}\}^{\delta_{1i}} \\ & \times \{ck\beta[\tau + \beta(y_i - \tau)]^{c-1}[1 + (\tau + \beta\{y_i - \tau\})^c]^{-(k+1)}\}^{\delta_{2i}} \\ & \times \{1 + [\tau + \beta(\eta - \tau)]^c\}^{-k\overline{\delta}_{1i}\overline{\delta}_{2i}} \end{aligned} \tag{3.1}$$

where, $\overline{\delta}_{1i} = 1 - \delta_{1i}$ and $\overline{\delta}_{2i} = 1 - \delta_{2i}$.

It is usually easier to maximize the natural logarithm of the likelihood function rather than the likelihood function itself. Therefore, the logarithm of the likelihood function is

$$\ln L = n_0 \ln c + n_0 \ln k + (c-1)\left\{\sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln A\right\} + n_a \ln \beta - (k+1)\left\{\sum_{i=1}^{n} \delta_{1i} \ln[1+y_i^c] + \sum_{i=1}^{n} \delta_{2i} \ln[1+A^c]\right\} - k(n-n_0)\ln[1+D^c] \tag{3.2}$$

where,

$$A = [\tau + \beta\{y_i - \tau\}], \qquad D = [\tau + \beta(\eta - \tau)], \qquad \sum_{i=1}^{n} \delta_{1i} = n_u,$$

$$\sum_{i=1}^{n} \delta_{2i} = n_a, \qquad \sum_{i=1}^{n} \bar{\delta}_{1i}\bar{\delta}_{2i} = n - n_u - n_a \quad \text{and} \quad n_0 = n_u + n_a.$$

Maximum likelihood estimators of β, c and k are solutions to the system of equations obtained by letting the first partial derivatives of the total log likelihood to be zero with respect to β, c and k, respectively. Therefore, the system of equations is as follows:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n_a}{\beta} + (c-1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)(A)^{-1} - k(n-n_0)c(D)^{c-1}(\eta-\tau)(1+D^c)^{-1} - (k+1)c\sum_{i=1}^{n} \delta_{2i}(A)^{c-1}(y_i-\tau)(1+A^c)^{-1}, \tag{3.3}$$

$$\frac{\partial \ln L}{\partial c} = \frac{n_0}{c} + \sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln A - k(n-n_0)(D)^c \ln D(1+D^c)^{-1} - (k+1)\left\{\sum_{i=1}^{n} \delta_{1i} y_i^c \ln y_i (1+y_i^c)^{-1} + \sum_{i=1}^{n} \delta_{2i}(A)^c \ln A(1+A^c)^{-1}\right\} \tag{3.4}$$

and

$$\frac{\partial \ln L}{\partial k} = \frac{n_0}{k} - \sum_{i=1}^{n} \delta_{1i} \ln(1+y_i^c) - \sum_{i=1}^{n} \delta_{2i} \ln(1+A^c) - (n-n_0)\ln(1+D^c). \tag{3.5}$$

From Eq. (3.5) the maximum likelihood estimate of k is expressed by

$$\hat{k} = \frac{n_o}{a_1}, \tag{3.6}$$

where,

$$a_1 = \sum_{i=1}^{n} \delta_{1i} \ln(1+y_i^{\hat{c}}) + \sum_{i=1}^{n} \delta_{2i} \ln(1+A^{\hat{c}}) + (n-n_0)\ln(1+D^{\hat{c}}).$$

Consequently, by substituting for $\hat{k}$ into Eqs. (3.3) and (3.4), the system of equations are reduced to two nonlinear equations as follows:

$$\frac{n_0}{\hat{c}} + \sum_{i=1}^{n} \delta_{1i} \ln y_i + \sum_{i=1}^{n} \delta_{2i} \ln A - \left(\frac{n_0}{a_1} + 1\right) a_2 - \left(\frac{n_0}{a_1}\right) a_3 = 0 \tag{3.7}$$

and

$$\frac{n_a}{\hat{\beta}} + (\hat{c} - 1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)A^{-1} - \left(\frac{n_0}{a_1} + 1\right)a_4 - \left(\frac{n_0}{a_1}\right)a_5 = 0 \tag{3.8}$$

where,

$$a_2 = \sum_{i=1}^{n} \delta_{1i} y_i^{\hat{c}} \ln y_i (1 + y_i^{\hat{c}})^{-1} + \sum_{i=1}^{n} \delta_{2i}(A)^{\hat{c}} \ln A (1 + A^{\hat{c}})^{-1}$$

$$a_3 = (n - n_0)(D)^{\hat{c}} \ln D (1 + D^{\hat{c}})^{-1}, \qquad a_4 = \hat{c}\sum_{i=1}^{n} \delta_{2i}(A)^{\hat{c}-1}(y_i - \tau)(1 + A^{\hat{c}})^{-1},$$

$$\text{and} \quad a_5 = (n - n_0)\hat{c}(D)^{\hat{c}-1}(\eta - \tau)(1 + D^{\hat{c}})^{-1}.$$

Since the closed form solutions to nonlinear equations (3.7) and (3.8) are very hard to obtain, an iterative procedure is applied to solve these equations numerically using Mathcad (2001) statistical package. The Newton–Raphson method is applied for simultaneously solving the nonlinear equations to obtain $\hat{\beta}$ and $\hat{c}$. Therefore $\hat{k}$ is calculated easily from Eq. (3.6).

3.2. Interval estimates

If $L_\theta = L_\theta(y_1, \ldots, y_n)$ and $U_\theta = U_\theta(y_1, \ldots, y_n)$ are functions of the sample data $y_1, \ldots, y_n$ then a confidence interval for a population parameter θ is given by

$$p[L_\theta \le \theta \le U_\theta] = \gamma \tag{3.9}$$

where, L_θ and U_θ are the lower and upper confidence limits which enclose θ with probability γ. The interval $[L_\theta, U_\theta]$ is called a $100\gamma\%$ confidence interval for θ.

For large sample size, the maximum likelihood estimates, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the approximate $100\gamma\%$ confidence limits for the maximum likelihood estimate $\hat{\theta}$ of a population parameter θ can be constructed, such that

$$p\left[-z \le \frac{\hat{\theta} - \theta}{\sigma(\hat{\theta})} \le z\right] = \gamma \tag{3.10}$$

where, z is the $[\frac{100(1-\gamma)}{2}]$ standard normal percentile. Therefore, the approximate $100\gamma\%$ confidence limits for a population parameter θ can be obtained, such that

$$p[\theta - z\sigma(\hat{\theta}) \le \hat{\theta} \le \theta + z\sigma(\hat{\theta})] = \gamma. \tag{3.11}$$

Then, the approximate confidence limits for β, c and k will be constructed using Eq. (3.11) with confidence levels 95% and 99%.

4. Asymptotic variances and covariances of estimates

The asymptotic variances of maximum likelihood estimates are given by the elements of the inverse of the Fisher information matrix $I_{ij}(\underline{\theta}) = E\{-\partial^2 \ln L/\partial\theta_i\partial\theta_j\}$. Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, the observed Fisher information matrix is given by $I_{ij}(\underline{\theta}) = \{-\partial^2 \ln L/\partial\theta_i\partial\theta_j\}$, which is obtained by dropping the expectation on operation E [see Cohen [12]]. The approximate (observed) asymptotic variance–covariance matrix F for the maximum likelihood estimates can be written as follows

$$F = [I_{ij}(\underline{\theta})], \quad i, j = 1, 2, 3 \text{ and } (\underline{\theta}) = (c, \beta, k) \tag{4.1}$$

The second partial derivatives of the maximum likelihood function are given as the following:

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \beta^2} &= -\frac{n_a}{\beta^2} - (c-1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)^2 (A)^{-2} \\
&\quad - (k+1)c \sum_{i=1}^{n} \delta_{2i}(y_i - \tau)[(c-1)(y_i - \tau)(1 + A^c)^{-1}(A)^{c-2} \\
&\quad - c(y_i - \tau)(1 + A^c)^{-2}(A)^{2c-2}] \\
&\quad - k(n - n_0)c(\eta - \tau)[(c-1)(\eta - \tau)(1 + D^c)^{-1}(D)^{c-2} \\
&\quad - c(\eta - \tau)(1 + D^c)^{-2}(D)^{2c-2}],
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \beta \partial c} &= \sum_{i=1}^{n} \delta_{2i}(y_i - \tau)(A)^{-1} - k(n - n_0)(\eta - \tau)[(D)^{c-1}(1 + D^c)^{-1} \\
&\quad + c(1 + D^c)^{-1}(D)^{c-1} \ln D - c(D)^{c-1}(1 + D^c)^{-2}(D)^c \ln D] \\
&\quad - (k+1)\sum_{i=1}^{n} \delta_{2i}(y_i - \tau)[c(1 + A^c)^{-1}(A)^{c-1} \ln A + (A)^{c-1}(1 + A^c)^{-1} \\
&\quad - c(1 + A^c)^{-2}(A)^{c-1}(A)^c \ln A],
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial \beta \partial k} &= -c\sum_{i=1}^{n} \delta_{2i}(A)^{c-1}(y_i - \tau)(1 + A^c)^{-1} \\
&\quad - (n - n_0)c(D)^{c-1}(\eta - \tau)(1 + D^c)^{-1},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial c^2} &= -\frac{n_0}{c^2} - k(n - n_0)\ln D[(1 + D^c)^{-1}(D)^c \ln D - (1 + D^c)^{-2}(D)^{2c} \ln D] \\
&\quad - (k+1)\left[\sum_{i=1}^{n} \delta_{1i} \ln y_i \{(1 + y_i^c)^{-1} y_i^c \ln y_i - (y_i^{2c})(1 + y_i^c)^{-2} \ln y_i\}\right] \\
&\quad - (k+1)\left[\sum_{i=1}^{n} \delta_{2i} \ln A\{(1 + A^c)^{-1}(A)^c \ln A - (1 + A^c)^{-2}(A)^{2c} \ln A\}\right],
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
\frac{\partial^2 \ln L}{\partial c \partial k} &= -\left[\sum_{i=1}^{n} \delta_{1i}(y_i^c) \ln y_i (1 + y_i^c)^{-1}\right. \\
&\quad \left. + \sum_{i=1}^{n} \delta_{2i}(A)^c \ln A(1 + A^c)^{-1} + (n - n_0)(D)^c \ln D(1 + D^c)^{-1}\right],
\end{aligned} \tag{4.6}$$

and

$$\frac{\partial^2 \ln L}{\partial k^2} = -\frac{n_0}{k^2}. \tag{4.7}$$

Consequently, the maximum likelihood estimators of β, c and k have an asymptotic variance–covariance matrix defined by inverting the Fisher information matrix F and by substituting $\hat{\beta}$ for β, $\hat{c}$ for c and $\hat{k}$ for k.

5. Simulation studies

Simulation studies have been performed using Mathcad (2001) for illustrating the theoretical results of estimation problem. The performance of the resulting estimators of the acceleration factor and two shape parameters has been considered in terms of their absolute relative bias (RABias), mean square error (MSE), and relative error (RE), where

$$\text{RABias}(\hat{\theta}) = \left|\frac{\hat{\theta}-\theta}{\theta}\right|, \qquad \text{MSE}(\hat{\theta}) = E(\hat{\theta}-\theta)^2 \quad \text{and} \quad \text{RE}(\hat{\theta}) = \frac{\sqrt{\text{MSE}(\hat{\theta})}}{\theta}.$$

Furthermore, the asymptotic variance and covariance matrix and confidence intervals of the acceleration factor and two shape parameters are obtained. The Simulation procedures were described below:

Step 1. 1000 random samples of sizes 100 (50) 400 and 500 were generated from the Burr type XII distribution. The generation of the Burr type XII distribution is very simple, if U has a uniform (0, 1) random number, then $Y = [(1-U)^{(\frac{-1}{k})} - 1]^{(\frac{1}{c})}$ follows a Burr type XII distribution. The true parameter values are selected as $(c = 1, \beta = 1.25, k = 2)$ and $(c = 1.5, \beta = 2, k = 0.5)$.

Step 2. Choosing the censoring time τ at the normal condition to be $\tau = 3$ and censoring time of a PALT to be $\eta = 6$.

Step 3. For each sample and for the two sets of parameters, the acceleration factor and the parameters of distribution were estimated in SS-PALT under type I censored sample.

Step 4. The Newton–Raphson method was used for solving the two nonlinear likelihood for β and c given in Eqs. (3.7) and (3.8), respectively. The estimate of the shape parameter k was easily obtained from Eq. (3.6).

Step 5. The RABias, MSE, and RE of the estimators for acceleration factor and two shape parameters for all sample sizes and for two sets of parameters were tabulated.

Step 6. The asymptotic variance and covariance matrix of the estimators for different sample sizes were obtained.

Step 7. The confidence limit with confidence level $\gamma = 0.95$ and $\gamma = 0.99$ of the acceleration factor and the two shape parameters were constructed.

Simulation results are summarized in Tables 1–3. Table 1 gives the RABias, MSE, and RE of the estimators. The asymptotic variances and covariance matrix of the estimators are displayed in Table 2. The approximated confidence limits at 95% and 99% for the parameters and acceleration factor are presented in Table 3.

From these tables, the following observations can be made on the performance of SS-PALT parameter estimation of the Burr type XII lifetime distribution:

1. For the first set of parameters $(c = 1.5, \beta = 2, k = 0.5)$, the maximum likelihood estimators have good statistical properties than the second set of parameters $(c = 1, \beta = 1.25, k = 2)$ for all sample sizes (see Table 1).

Table 1
The RABias, MSE and RE of the parameters $(c, \beta, k, \tau, \eta)$ under type I censoring

n	Parameters $(c, \beta, k, \tau, \eta)$	(1, 1.25, 2, 3, 6) RABias	MSE	RE	(1.5, 2, 0.5, 3, 6) RABias	MSE	RE
	c	0.027	0.021	0.145	0.033	0.022	0.099
100	β	0.068	0.440	0.531	0.064	0.085	0.146
	k	0.001	0.041	0.101	0.088	0.005	0.135
	c	0.020	0.012	0.110	0.036	0.016	0.084
150	β	0.015	0.160	0.320	0.069	0.064	0.126
	k	0.003	0.026	0.080	0.085	0.003	0.117
	c	0.020	0.010	0.099	0.042	0.013	0.076
200	β	0.018	0.116	0.272	0.074	0.053	0.115
	k	0.005	0.019	0.069	0.090	0.003	0.116
	c	0.019	0.008	0.087	0.038	0.011	0.070
250	β	0.027	0.095	0.247	0.070	0.045	0.106
	k	0.003	0.016	0.063	0.093	0.003	0.112
	c	0.013	0.006	0.078	0.039	0.010	0.067
300	β	0.030	0.072	0.215	0.071	0.042	0.102
	k	0.004	0.013	0.057	0.092	0.003	0.110
	c	0.013	0.005	0.073	0.042	0.009	0.065
350	β	0.046	0.059	0.194	0.072	0.039	0.099
	k	0.005	0.011	0.052	0.088	0.003	0.104
	c	0.009	0.004	0.066	0.042	0.008	0.061
400	β	0.041	0.049	0.177	0.073	0.037	0.096
	k	0.007	0.010	0.050	0.092	0.003	0.104
	c	0.007	0.004	0.064	0.042	0.008	0.061
450	β	0.039	0.046	0.172	0.072	0.034	0.092
	k	0.007	0.009	0.048	0.092	0.003	0.103
	c	0.010	0.004	0.061	0.045	0.008	0.061
500	β	0.049	0.041	0.162	0.073	0.033	0.091
	k	0.005	0.008	0.045	0.091	0.003	0.102

2. As the acceleration factor increases the estimates have smaller MSE, and RE. As the sample size increases the RABiases and MSEs of the estimated parameters decrease. This indicates that the maximum likelihood estimates provide asymptotically normally distributed and consistent estimators for the parameters and acceleration factor.
3. The asymptotic variances of the estimators are decreasing when the sample size is increasing (see Table 2).
4. The interval of the estimators decreases when the sample size is increasing. Also, the interval of the estimator at $\gamma = 0.95$ is smaller than the interval of estimator at $\gamma = 0.99$ (see Table 3).

6. Conclusion

For products having a high reliability, the test of product life under normal use often requires a long period of time. So ALT or PALT is used to facilitate estimating the reliability of the unit in a short period of time. In ALT test items are run only at accelerated conditions, while in PALT they are run at both normal and accelerated conditions. One way to accelerate failure is

Table 2
Asymptotic variances and covariances of estimates under type 1 censoring

n	(1, 1.25, 2, 3, 6)			(1.5, 2, 0.5, 3, 6)		
	$\hat{c}$	$\hat{\beta}$	$\hat{k}$	$\hat{c}$	$\hat{\beta}$	$\hat{k}$
	0.8190	−0.0210	−0.0540	0.7640	−0.0620	−0.0190
100	−0.0210	0.0070	0.0013	−0.0620	0.0340	−0.0047
	−0.0540	0.0013	0.0440	−0.0190	−0.00476	0.0044
	0.445	−0.0120	−0.0300	0.4920	−0.0400	−0.0120
150	−0.0120	0.0046	0.0008	−0.0400	0.0220	−0.0030
	−0.0300	0.0008	0.0290	−0.0120	−0.0031	0.0029
	0.329	−0.00849	−0.0210	0.3550	−0.0290	−0.0089
200	−0.0085	0.0034	0.0006	−0.0290	0.0160	−0.0023
	−0.0210	0.0006	0.0220	−0.0089	−0.0023	0.0021
	0.2580	−0.0067	−0.0170	0.2880	−0.0230	−0.0072
250	−0.0067	0.0027	0.0005	−0.0230	0.0130	−0.0018
	−0.0170	0.0005	0.0170	−0.0072	−0.0018	0.0017
	0.2110	−0.0054	−0.0140	0.2390	−0.0190	−0.0060
300	−0.0054	0.0022	0.0004	−0.019	0.0110	−0.0015
	−0.014	0.0004	0.0140	−0.0060	−0.0015	0.0014
	0.1740	−0.0045	−0.0120	0.2000	−0.0160	−0.0050
350	−0.0045	0.0019	0.0003	−0.0160	0.0092	−0.0013
	−0.0120	0.0003	0.0120	−0.0050	−0.0013	0.0012
	0.1500	−0.0039	−0.0099	0.1760	−0.0140	−0.0044
400	−0.0039	0.0017	0.0003	−0.0140	0.0081	−0.0011
	−0.0099	0.0003	0.0110	−0.0044	−0.0011	0.0010
	0.1330	−0.0034	−0.0087	0.1530	−0.0120	−0.0038
450	−0.0034	0.0015	0.0003	−0.0120	0.0071	−0.0098
	−0.0087	0.0003	0.0095	−0.0039	−0.0009	0.0009
	0.1200	−0.0031	−0.0079	0.1380	−0.0110	−0.0034
500	−0.0031	0.0013	0.0002	−0.0110	0.0064	−0.0009
	−0.0079	0.0002	0.0086	−0.0034	−0.0008	0.0008

the SS-PALT which increases the stresses applied to test product in a specified discrete sequence. The lifetime of the test items is assumed to follow the Burr type XII distribution. Under type I censoring, the test unit is first run at normal use condition, and if it does not fail for a specified time τ, then it is run at accelerated condition until censoring time η is reached. The maximum likelihood method is used for estimating the acceleration factor and the parameters of Burr type XII distribution under type I censoring. Performance of step-stress testing plans and model assumptions are usually evaluated by the properties of the maximum likelihood estimates of model parameters.

In this study, the first set of parameters have good statistical properties than the second set of parameters for all sample sizes. Maximum likelihood estimators are consistent and asymptotically normally distributed. As the sample size increases the asymptotic variance and covariance of estimators decreases. Regarding the interval of estimators, it can be noted that the interval of the estimators at $\gamma = 0.99$ is greater than the corresponding at $\gamma = 0.95$. Also, as the sample size increases the interval of the estimators decreases for the confidence level.

Table 3
Confidence bounds of the estimates at confidence levels 0.95 and 0.99

n	Parameters $(c, \beta, k, \tau, \eta)$	(1, 1.25, 2, 3, 6)			(1.5, 2, 0.5, 3, 6)		
		Standard deviation	Lower bound	Upper bound	Standard deviation	Lower bound	Upper bound
100	c	0.142	0.750	1.305	0.140	1.177	1.724
			0.662	1.392		1.090	1.811
	β	0.658	0.045	2.624	0.262	1.358	2.386
			0.363	3.032		1.195	2.549
	k	0.203	1.605	2.400	0.052	0.355	0.557
			1.480	2.526		0.323	0.589
150	c	0.107	0.810	1.230	0.117	1.218	1.675
			0.744	1.297		1.146	1.747
	β	0.400	0.448	2.016	0.212	1.447	2.277
			0.200	2.264		1.316	2.409
	k	0.620	1.678	2.312	0.040	0.379	0.536
			1.577	2.412		0.354	0.561
200	c	0.097	0.830	1.210	0.098	1.246	1.629
			0.770	1.270		1.186	1.689
	β	0.339	0.563	1.893	0.177	1.505	2.2
			0.352	2.104		1.395	2.31
	k	0.137	1.722	2.258	0.036	0.384	0.526
			1.638	2.342		0.361	0.549
250	c	0.085	0.852	1.186	0.085	1.276	1.609
			0.799	1.239		1.223	1.662
	β	0.306	0.615	1.816	0.159	1.549	2.173
			0.425	2.006		1.45	2.271
	k	0.125	1.750	2.238	0.031	0.393	0.514
			1.673	2.316		0.374	0.533
300	c	0.077	0.863	1.163	0.081	1.282	1.601
			0.815	1.210		1.232	1.651
	β	0.266	0.692	1.734	0.146	1.571	2.144
			0.527	1.899		1.481	2.234
	k	0.112	1.771	2.211	0.03	0.396	0.512
			1.702	2.280		0.378	0.530
350	c	0.072	0.872	1.154	0.074	1.293	1.581
			0.828	1.198		1.247	1.627
	β	0.237	0.728	1.657	0.135	1.591	2.121
			0.581	1.804		1.508	2.205
	k	0.106	1.783	2.198	0.027	0.403	0.509
			1.718	2.264		0.386	0.526
400	c	0.066	0.881	1.138	0.073	1.295	1.580
			0.840	1.178		1.250	1.625
	β	0.215	0.777	1.621	0.126	1.607	2.101
			0.643	1.754		1.528	2.180
	k	0.099	1.792	2.181	0.020	0.405	0.503
			1.731	2.243		0.390	0.518

Table 3 (*continued*)

n	Parameters (c, β, k, τ, η)	(1, 1.25, 2, 3, 6) Standard deviation	Lower bound	Upper bound	(1.5, 2, 0.5, 3, 6) Standard deviation	Lower bound	Upper bound
450	c	0.064	0.882	1.132	0.066	1.307	1.567
			0.842	1.171		1.266	1.608
	β	0.208	0.792	1.609	0.115	1.63	2.080
			0.663	1.738		1.559	2.151
	k	0.095	1.804	2.178	0.024	0.407	0.501
			1.744	2.237		0.392	0.516
500	c	0.06	0.892	1.129	0.062	1.311	1.555
			0.854	1.166		1.272	1.594
	β	0.193	0.812	1.567	0.109	1.641	2.068
			0.692	1.686		1.574	2.136
	k	0.090	1.813	2.166	0.023	0.410	0.499
			1.757	2.222		0.395	0.513

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Appendix

See Tables 1–3.

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