ESTIMATION OF RELIABILITY IN MULTI-COMPONENT STRESS-STRENGTH MODEL FOLLOWING EXPONENTIATED PARETO DISTRIBUTION

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Abstract

This article deals with the Bayesian and non-Bayesian estimation of reliability of an s-out-of-k system with identical component strengths which are subjected to a common stress. Assuming that both stress and strength are assumed to have an exponentiated Pareto distribution with known and unequal shape parameters(λ_1, λ_2). Five non-Bayesian methods of estimation will be used which are maximum likelihood, moments, percentile, least squares and weighted least squares. The Bayesian estimation will be studied under squared error and LINEX loss functions using Lindley's approximation. Based on a Monte Carlo simulation study, comparisons are made between the different estimators of system reliability by obtaining their absolute biases and mean squared errors. Comparison study revealed that the maximum likelihood estimator works the best among the competitors.

Key words: stress–strength model; reliability; exponentiated Pareto; maximum likelihood estimator; moments estimator; percentile estimator; least squares estimator; weighted least squares estimator; Bayes estimator; noninformative type prior; squared error loss function; LINEX loss function; Lindley's approximation.

1. Introduction

The stress-strength model is used in many applications of physics and engineering such as, strength failure and the system collapse. This model is of special importance in reliability literature. In the statistical approach to the stress-strength model, most of the considerations depend on the assumption that the component strengths are independently and identically distributed (iid) and are subjected to a common stress.

Consider a system made up of k identical components. The strengths of these components $Y_1, \ldots, Y_k$ are iid random distributed variables. Assume that these strengths have an exponentiated (EP) distribution that suggested by Gupta et al. (1998) with parameters (θ_1, λ_1). This system is subjected to a common stress X which is independent random variable distributed as EP with parameters (θ_2, λ_2). Let $f(y; \theta_1, \lambda_1)$ be a common probability density function (pdf) of $Y_1, \dots, Y_k$ and $g(x; \theta_2, \lambda_2)$ be pdf of X and written as

$$f(y; \theta_1, \lambda_1) = \theta_1 \lambda_1 [1-(1+y)^{-\lambda_1}]^{\theta_1 - 1}(1+y)^{-(\lambda_1+1)};\ y > 0, \theta_1 > 0, \lambda_1 > 0,$$

and

$$g(x; \theta_2, \lambda_2) = \theta_2 \lambda_2 [1-(1+x)^{-\lambda_2}]^{\theta_2 - 1}(1+x)^{-(\lambda_2+1)};\ x > 0, \theta_2 > 0, \lambda_2 > 0.$$

The corresponding cumulative distribution functions are given, respectively, by

$$\left.\begin{aligned} F(y; \theta_1, \lambda_1) &= [1-(1+y)^{-\lambda_1}]^{\theta_1};\ y > 0, \theta_1 > 0, \lambda_1 > 0, \\ G(x; \theta_2, \lambda_2) &= [1-(1+x)^{-\lambda_2}]^{\theta_2};\ x > 0, \theta_2 > 0, \lambda_2 > 0. \end{aligned}\right\} \qquad (1.1)$$

The system operates satisfactorily if s or more of k components have strengths larger than the stress X. Consequently, the system reliability $R_{(s,k)}$, which is the probability that the system does not fail, developed by Bhattacharyya and Johnson (1974) is given by $R_{(s,k)}$ = P[at least s of the $(Y_1, \dots, Y_k)$ exceed X],

$$R_{(s,k)} = \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [F_Y(x)]^{k-i} [1 - F_Y(x)]^i \, dG_X(x), \qquad (1.2)$$

The particular cases $s = 1$ and $s = k$ correspond, respectively, to parallel and series systems.

The reliability of s-out-of-k system for EP distribution can be computed by substituting equations (1.1) in equation (1.2) and takes the following form:

$$R_{(s,k)} = \theta_2 \delta \sum_{i=s}^{k} \binom{k}{i} \int_0^1 [1-z]^{\theta_1(k-i)} \, [1-[1-z]^{\theta_1}]^i (1-z^{\delta})^{\theta_2 - 1} z^{\delta - 1} dz, \qquad (1.3)$$

where $\delta = \frac{\lambda_2}{\lambda_1}$.

The problem of estimating system reliability was originally viewed as an extension of the stress-strength model to a multi-component system. The estimation of reliability of s-out-of-k stress–strength system has been discussed by many authors such as, Bhattacharya and Johnson (1974), Draper and Guttman (1978), Pandey and Upadhyay (1986), Pandey and Borhan Uddin (1991), Pandey et al. (1993) and Srinivasa Rao and Kantam (2010). They considered the strengths are iid and are subjected to a common stress. Hassan and Basheikh (2012) studied the Bayesian and non-Bayesian estimation of reliability of an s-out-of-k system with non-identical component strengths which are subjected to a common stress. They assumed that both stress and strength have an exponentiated Pareto distribution with common and known shape parameter.

The main aim of this article is estimating the reliability in multi-component stress-strength model of an s-out-of-k system. Assuming both stress and strength are independently distributed as EP with known and unequal shape parameters (λ_1, λ_2). This problem is studied when the strengths of the components are iid. Maximum likelihood estimator (MLE), moment estimator (ME), percentile estimator (PCE), least squares estimator (LSE) and weighted least squares estimator (WLSE) are obtained. Also, the Bayes estimators under squared error and LINEX loss functions are discussed using Lindley's approximation. Monte Carlo simulation is performed for comparing different methods of estimation.

The rest of the article is organized as follows. In Section 2, different methods of estimation of $R_{(s,k)}$ are discussed. In Section 3, numerical illustration is carried out to illustrate theoretical results. In Section 4, simulation results are displayed. Finally, conclusion is presented in Section 5.

2. Different Methods of Estimation of $R_{(s,k)}$

It is well known that the method of maximum likelihood estimation has invariance property. When the method of estimation of unknown parameter is changed from maximum likelihood to any other traditional method, this invariance principle does not hold good to estimate the parametric function. However, such an adoption of invariance property for other optimal estimators of the parameters to estimate a parametric function is attempted in different situations by different authors [see Srinivasa Rao and Kantam (2010)]. In this direction, in the following subsections some methods of estimation for the reliability of an s-out-of-k system in stress–strength model will be proposed by considering the estimators of the parameters of stress, strength distributions.

2.1 Maximum likelihood estimator of $R_{(s,k)}$

Let $Y_1, Y_2, \dots, Y_m$ be a random sample of size m drawn from EP(θ_1, λ_1), then $Y_{(1)} < Y_{(2)} <..< Y_{(m)}$ denote the order statistic of the observed sample. Let $X_1, X_2, .., X_n$ be a random sample of size n drawn from EP(θ_2, λ_2), then $X_{(1)} < X_{(2)} < .. < X_{(n)}$ denote the order statistic of the observed sample. Then the likelihood function is given by

$$\begin{aligned} L(\theta_1, \theta_2, \lambda_1, \lambda_2; \underline{y}, \underline{x}) &= \prod_{i=1}^{n} g(x_i; \theta_2, \lambda_2) \prod_{j=1}^{m} f\left(y_j; \theta_1, \lambda_1\right) \\ &= \theta_2^n \lambda_2^n \theta_1^m \lambda_1^m \prod_{i=1}^{n} [1-(1+x_i)^{-\lambda_2}]^{\theta_2 - 1} \times \\ &\quad \prod_{i=1}^{n} (1+x_i)^{-(\lambda_2+1)} \prod_{j=1}^{m} [1-(1+y_j)^{-\lambda_1}]^{\theta_1 - 1} \times \\ &\quad \prod_{j=1}^{m} (1+y_j)^{-(\lambda_1+1)} \end{aligned} \tag{2.1}$$

Then the logarithm of the likelihood function is given by

$$\begin{aligned} \ln L(\theta_1, \theta_2, \lambda_1, \lambda_2; \underline{y}, \underline{x}) &= n(\ln \theta_2 + \ln \lambda_2) + m\,(\ln \theta_1 + \ln \lambda_1) + (\theta_2 - 1) \times \\ &\quad \sum_{i=1}^{n} \ln[1-(1+x_i)^{-\lambda_2}] + (\theta_1 - 1) \times \\ &\quad \sum_{j=1}^{m} \ln[1-(1+y_j)^{-\lambda_1}] - (\lambda_2+1) \sum_{i=1}^{n} \ln(1+x_i) - \\ &\quad (\lambda_1+1) \sum_{j=1}^{m} \ln\left(1+y_j\right). \end{aligned}$$

For simplicity; write $\ln L(\theta_1, \theta_2, \lambda_1, \lambda_2; \underline{y}, \underline{x})$ to be $\ln L$

The first derivatives of the log-likelihood function with respect to θ_1 and θ_2 are given, respectively, by

$$\left.\begin{aligned}\frac{\partial \ln L}{\partial \theta_1} = \frac{m}{\theta_1} + \textstyle\sum_{j=1}^{m} \ln[1-(1+y_j)^{-\lambda_1}] = 0,\\ \frac{\partial \ln L}{\partial \theta_2} = \frac{n}{\theta_2} + \textstyle\sum_{i=1}^{n} \ln[1-(1+x_i)^{-\lambda_2}] = 0\,.\end{aligned}\right\} \tag{2.2}$$

Then the MLE's of $\theta_1$and θ_2, denoted by $\hat{\theta}_{1(MLE)}$ and $\hat{\theta}_{2(MLE)}$, respectively, can be obtained as the solution of equations (2.2) as

$$\left.\begin{aligned}\hat{\theta}_{1(MLE)} = -\frac{m}{\sum_{j=1}^{m} \ln[1-(1+y_j)^{-\lambda_1}]},\\ \hat{\theta}_{2(MLE)} = -\frac{n}{\sum_{i=1}^{n} \ln[1-(1+x_i)^{-\lambda_2}]}.\end{aligned}\right\} \tag{2.3}$$

The MLE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)MLE}$, is obtained by substitute $\hat{\theta}_{1(MLE)}$ and $\hat{\theta}_{2(MLE)}$ in equation (1.3).

2.2 Moments estimator of $R_{(s,k)}$

Since the strengths of k components $Y_1, \dots, Y_k$ follow EP (θ_1, λ_1) and the stress X follows EP (θ_2, λ_2), then their population means are given by

$$\left.\begin{aligned}\mu_Y = E(Y) = \theta_1 B(\theta_1, 1-\tfrac{1}{\lambda_1}) - 1\,; \lambda_1 > 1,\\ \mu_X = E(X) = \theta_2 B(\theta_2, 1-\tfrac{1}{\lambda_2}) - 1\,; \lambda_2 > 1.\end{aligned}\right\} \tag{2.4}$$

Here $B(.,.)$ denotes the beta function. According to the method of moments, equating the samples means with the corresponding populations means. Then,

$$\left.\begin{aligned}\bar{y} = \theta_1 B(\theta_1, 1-\tfrac{1}{\lambda_1}) - 1\,; \lambda_1 > 1,\\ \bar{x} = \theta_2 B(\theta_2, 1-\tfrac{1}{\lambda_2}) - 1\,; \lambda_2 > 1.\end{aligned}\right\} \tag{2.5}$$

The ME's of $\theta_1$and θ_2 ,denoted by $\hat{\theta}_{1(ME)}$ and $\hat{\theta}_{2(ME)}$, respectively, can be obtained by solving the non-linear equations (2.5) numerically.

The ME of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)ME}$, is obtained by substitute $\hat{\theta}_{1(ME)}$ and $\hat{\theta}_{2(ME)}$ in equation (1.3).

2.3 Percentile estimator of $R_{(s,k)}$

The percentile estimators can be obtained by equating the sample percentile points with the population percentile points. In case of EP distribution it is possible to use the same concept to obtain the estimators based on the percentiles, because of the structure of its distribution function.

According to Kao (1958,1959) several estimators of p_{1j} and p_{2i}, where p_{1j} and p_{2i} are the samples percentile, can be used as estimates for populations percentile $F(y_{(j)}; \theta_1, \lambda_1)$ and $G(x_{(i)}; \theta_2, \lambda_2)$.

In this work, the following formulas can be used

$$p_{1j} = \frac{j}{m+1};\ j = 1,2,..,m, \qquad \text{and} \qquad p_{2i} = \frac{i}{n+1}\ ;\ i = 1,2,..,n,$$

which are the expected values of $F(Y_{(j)})$ and $G(X_{(i)})$ respectively.

Then the PCE's of $\theta_1$and θ_2 can be obtained by minimizing the following equations with respect to $\theta_1$and θ_2, respectively,

$$\left.\begin{array}{l}\sum_{j=1}^{m}[\ln(p_{1j})-\theta_1\ln[1-(1+y_{(j)})^{-\lambda_1}]]^2,\\ \sum_{i=1}^{n}[\ln(p_{2i})-\theta_2\ln[1-(1+x_{(i)})^{-\lambda_2}]]^2.\end{array}\right\} \quad (2.6)$$

Then the PCE's of $\theta_1$and θ_2, denoted by $\hat{\theta}_{1(PCE)}$ and $\hat{\theta}_{2(PCE)}$, respectively, can be obtained as the solution of equations (2.6) as

$$\left.\begin{array}{l}\hat{\theta}_{1(PCE)}=\frac{\sum_{j=1}^{m}\ln(p_{1j})\ln[1-(1+y_{(j)})^{-\lambda_1}]}{\sum_{j=1}^{m}[\ln[1-(1+y_{(j)})^{-\lambda_1}]]^2},\\ \hat{\theta}_{2(PCE)}=\frac{\sum_{i=1}^{n}\ln(p_{2i})\ln[1-(1+x_{(i)})^{-\lambda_2}]}{\sum_{i=1}^{n}[\ln[1-(1+x_{(i)})^{-\lambda_2}]]^2}.\end{array}\right\} \quad (2.\ 7)$$

The PCE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)PCE}$, is obtained by substitute $\hat{\theta}_{1(PCE)}$ and $\hat{\theta}_{2(PCE)}$ in equation (1.3).

2.4 Least square and weighted least square estimators of $R_{(s,k)}$

Least square estimators are obtained by minimizing the sum of squared errors between the value and its expected value. This estimation method is very popular for model fitting, especially in linear and non-linear regression**.**

According to Johnson et al. (1995),

$$E\big(F(Y_{(j)})\big)=\frac{j}{m+1};\ j=1,2,..,m,\quad E\big(G(X_{(i)})\big)=\frac{i}{n+1};\ i=1,2,..,n,$$

and

$$V\big(F(Y_{(j)})\big)=\frac{j(m-j+1)}{(m+1)^2(m+2)},\qquad V\big(G(X_{(i)})\big)=\frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

Using the expectations and the variances of $F(Y_{(j)})$ and $G(X_{(i)})$, two variants of the least squares methods can be used.

The LSE's of $\theta_1$andθ_2, denoted by $\hat{\theta}_{1(LSE)}$ and$\hat{\theta}_{2(LSE)}$, respectively, can be obtained by minimizing the following equations with respect to $\theta_1$and θ_2

$$\left.\begin{array}{l}\sum_{j=1}^{m}[F(Y_{(j)})-E\big(F(Y_{(j)})\big)]^2,\\ \sum_{i=1}^{n}[G(X_{(i)})-E\big(G(X_{(i)})\big)]^2.\end{array}\right\} \quad (2.8)$$

The LSE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)LSE}$, is obtained by substitute $\hat{\theta}_{1(LSE)}$ and $\hat{\theta}_{2(LSE)}$ in equation (1.3).

Also, the WLSE's of $\theta_1$and θ_2,denoted by $\hat{\theta}_{1(WLSE)}$ and $\hat{\theta}_{2(WLSE)}$, respectively, can be obtained by minimizing the following equations with respect to θ_1 and θ_2

$$\left.\begin{array}{l}\sum_{j=1}^{m}w_{1j}[F(Y_{(j)})-E\big(F(Y_{(j)})\big)]^2,\\ \sum_{i=1}^{n}w_{2i}[G(X_{(i)})-E\big(G(X_{(i)})\big)]^2,\end{array}\right\} \quad (2.9)$$

where,

$$w_{1j}=\frac{1}{V\big(F(Y_{(j)})\big)}=\frac{(m+1)^2(m+2)}{j(m-j+1)},\quad w_{2i}=\frac{1}{V\big(G(X_{(i)})\big)}=\frac{(n+1)^2(n+2)}{i(n-i+1)}.$$

The WLSE of $R_{(s,k)}$, denoted by $\hat{R}_{(s,k)WLSE}$, is obtained by substitute $\hat{\theta}_{1(WLSE)}$ and $\hat{\theta}_{2(WLSE)}$ in equation (1.3).

In the following subsection the approximate Bayes estimators (BE's) of $R_{(s,k)}$ are obtained. The approximate Bayes estimators under squared error loss function and LINEX loss function by using Lindley's approximation, denoted by BESL and BELL, respectively, are discussed.

2.5 Bayes estimator of $R_{(s,k)}$

Assume θ_1 and θ_2 are independent random variables. Following Afify (2010) the noninformative type of prior for parameters θ_1 and θ_2 is considered. Therefore, the joint prior density of (θ_1, θ_2) is

$g(\theta_1, \theta_2) \propto \frac{1}{\theta_1}\frac{1}{\theta_2}$; $0 < \theta_i < \infty$, $i = 1,2$.

Combining the joint prior density of (θ_1, θ_2) and the likelihood function given in equation (2.1) to obtain the joint posterior density of (θ_1, θ_2) as

$$\pi(\theta_1, \theta_2|\underline{y}, \underline{x}) = \frac{g(\theta_1,\theta_2)L(\theta_1,\theta_2,\lambda_1,\lambda_2;\underline{y},\underline{x})}{\int_0^\infty\int_0^\infty g(\theta_1,\theta_2)L(\theta_1,\theta_2,\lambda_1,\lambda_2;\underline{y},\underline{x})d\theta_1 d\theta_2},$$

$$\pi(\theta_1, \theta_2|\underline{y}, \underline{x}) = \frac{t}{t_1}; 0 < \theta_i < \infty \ , i = 1,2, \quad (2.10)$$

where,

$$t = \theta_2^{n-1}\theta_1^{m-1}\prod_{i=1}^{n}[1-(1+x_i)^{-\lambda_2}]^{\theta_2-1}\prod_{j=1}^{m}[1-(1+y_j)^{-\lambda_1}]^{\theta_1-1},$$

$$t_1 = \int_0^\infty\int_0^\infty \theta_2^{n-1}\theta_1^{m-1}\prod_{i=1}^{n}[1-(1+x_i)^{-\lambda_2}]^{\theta_2-1} \times \prod_{j=1}^{m}[1-(1+y_j)^{-\lambda_1}]^{\theta_1-1}\, d\theta_1 d\theta_2.$$

Under squared error and LINEX loss functions, the BE's of $R_{(s,k)}$ denoted by $\hat{R}_{(s,k)BES}$ and $\hat{R}_{(s,k)BEL}$, respectively, defined as

$$\hat{R}_{(s,k)BES} = E(R_{(s,k)}|\underline{y}, \underline{x}) = \int_0^\infty\int_0^\infty R_{(s,k)}\pi(\theta_1, \theta_2|\underline{y}, \underline{x})\, d\theta_1 d\theta_2,$$

$$\hat{R}_{(s,k)BEL} = -\frac{1}{a}\ln[E(e^{-aR_{(s,k)}}|\underline{y}, \underline{x})] = -\frac{1}{a}\ln[\int_0^\infty\int_0^\infty e^{-aR_{(s,k)}}\pi(\theta_1, \theta_2|\underline{y}, \underline{x})\, d\theta_1 d\theta_2],$$

this integrals cannot be obtained in a simple closed form. Alternatively, using the approximation of Lindley (1980) to compute the approximate BE of $R_{(s,k)}$.

Using Lindley's approximation, the approximate BE's of $R_{(s,k)}$ under squared error and LINEX loss functions denoted by $\hat{R}_{(s,k)BESL}$ and $\hat{R}_{(s,k)BELL}$, respectively, take the following forms

$$\left.\begin{aligned} \hat{R}_{(s,k)BESL} &= \tilde{R}_{(s,k)} + \frac{1}{2}[U_{11}\tau_{11} + U_{22}\tau_{22} + Q_{30}U_1\tau_{11}^2 + Q_{03}U_2\tau_{22}^2], \\ \hat{R}_{(s,k)BELL} &= -\frac{1}{a}\ln[e^{-a\tilde{R}_{(s,k)}} + \frac{1}{2}[W_{11}\tau_{11} + W_{22}\tau_{22} + \\ &\quad Q_{30}W_1\tau_{11}^2 + Q_{03}W_2\tau_{22}^2]], \end{aligned}\right\} \quad (2.11)$$

where all functions in equations (2.11) defined in appendix A and evaluated at the posterior mode $\tilde{\theta}_1 = -\frac{m-1}{\sum_{j=1}^{m}\ln[1-(1+y_j)^{-\lambda_1}]}$, $\tilde{\theta}_2 = -\frac{n-1}{\sum_{i=1}^{n}\ln[1-(1+x_i)^{-\lambda_2}]}$.

3. Simulation Study

In this Section, Monte Carlo simulation is performed to observe the behavior of the different methods of estimation of $R_{(s,k)}$ for different sample sizes, different parameter values and for different s-out-of-k systems. The performances of the different estimators of $R_{(s,k)}$ are compared in terms of their absolute biases and mean squared errors (MSE's). The absolute biases and MSE's are computed for the different estimators over 5000 replications for different case. The simulation procedures are described through the following steps:

Step (1): A random samples $Y_1, Y_2, \dots, Y_m$ and $X_1, X_2, .., X_n$ of sizes (m, n) = (10,10), (10,30), (10,50), (30,10), (30,30), (30,50), (50,10), (50,30) and (50,50) are generated from EP distributions.

Step (2): The parameters values are selected as $(\theta_1, \theta_2, \lambda_1, \lambda_2)$ = (1.5,0.5,3,5) and (0.5,1.5,5,3). The selected values for s-out-of-k systems are (1,3), (2,3), (3,3) and (1,1). It is evident that, (1,3) reduce to parallel system, (3,3) reduce to series system and (1,1) reduce to single component.

Step (3): The estimation of the parameters $\theta_1$and θ_2 are considered. The MLE's and PCE's of $\theta_1$and θ_2 can be obtained from equations (2.3) and (2.7), respectively. The ME's of $\theta_1$and θ_2 can be obtained by solving the non-linear equations (2.5). Also, The LSE's and WLSE's of $\theta_1$and θ_2 can be obtained by minimizing equations (2.8) and (2.9) with respect to $\theta_1$and θ_2, respectively.

Step (4): The MLE, ME, PCE, LSE and WLSE of $R_{(s,k)}$ are computed by using the estimates of $\theta_1$and θ_2 obtained in step (3).

Step (5): The approximate Bayes estimates of $R_{(s,k)}$ under squared error and LINEX loss functions, at a=1, using Lindley's approximation can be computed from equations (2.11).

Step (6): Repeat the pervious steps from (1) to (5) r times representing r different samples, where $r = 5000$. Then, the absolute average bias and MSE of the estimates of $R_{(s,k)}$ are computed.

4. Simulation Results

All simulated studies presented here are obtained via MathCAD (14). The results are reported in Tables 1 and 2.

From Tables 1 and 2 many conclusions can be made on the performance of all methods of estimation of $R_{(s,k)}$. These conclusions are summarized as follows:

1- The value of $R_{(s,k)}$ increases as the value of θ_1 and λ_2 increase and as the value of θ_2 and λ_1 decrease (see Tables 1 and 2).

2- It is found that the $R_{(s,k)}$ are broadly in following order of descending value $R_{(1,3)} \rightarrow R_{(2,3)} \rightarrow R_{(1,1)} \rightarrow R_{(3,3)}$ when $(\theta_1, \theta_2, \lambda_1, \lambda_2)$ = (1.5,0.5,3,5) (see

Table 1) and $R_{(1,3)} \rightarrow R_{(1,1)} \rightarrow R_{(2,3)} \rightarrow R_{(3,3)}$ at $(\theta_1, \theta_2, \lambda_1, \lambda_2)$ = (0.5,1.5,5,3) (see Table 2).

3- For all the methods it is observed that when $m = n$ and m, n increases the MSE's decrease. For fixed$m$, as n increases the MSE's decrease. For fixed$n$, as m increases the MSE's decrease. Also, the biases decrease in almost all values expects for some few cases.
4- For fixed value of k, as s increases the value of $R_{(s,k)}$ decreases.
5- With respect to MSE's, the MLE performs the best estimators for $R_{(s,k)}$ in almost all of the cases compared to other reliability estimators. The performance of the BESL and BELL are quite close to the MLE.
6- As far as biases are concerned, it is observed that the MLE's have the minimum biases in almost all of the cases expect for few cases, the PCE's have minimum biases.
7- Regarding to series system at (s, k) = (3,3) the BESL performs the best estimators for $R_{(s,k)}$ when $(\theta_1, \theta_2, \lambda_1, \lambda_2)$ = (1.5,0.5,3,5) for most different sample sizes in terms of MSE's (see Table 1)
8- According to parallel system at (s, k) = (1,3) the BELL works the best estimators for $R_{(s,k)}$ when $(\theta_1, \theta_2, \lambda_1, \lambda_2)$ = (0.5,1.5,5,3) for all different sample sizes with respect to MSE's (see Table 2)
9- WLSE works better estimator for $R_{(s,k)}$ than LSE in all cases
10- In almost all of the cases, the performance of ME is the worst estimators for $R_{(s,k)}$ as for as the MSE's are concerned.
11- In the context of computational complexities, MLE, PCE, BESL and BELL are easiest to compute. They do not involve any non-linear equation solving, whereas the ME, LSE and WLSE involve solving non-linear equations and they need to be calculated by some iterative processes.

5. Conclusion

In this article, different methods of estimation of reliability in multi-component stress-strength systems are considered. Both stress and strength are assumed to be independent and have an EP distribution. Estimation of system reliability in an s-out-of-k system is studied when the component strengths are iid and subjected to a common stress. In particular, the reliability estimation of series and parallel systems are also studied. In addition, the estimation of reliability of single component is also considered as special case. Comparison the performance of all estimators, it is observed that the MLE performs the best among the competitors relative to their absolute biases and MSE's.

Table 1: Results of simulation study of absolute bias and MSE of estimates of reliability for θ_1=1.5, θ_2=0.5, λ_1=3 , λ_2=5 , α=1 and 5000 replications

(s,k)	True $R_{3(s,k)}$	(m,n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(1,3)	0.957	(10,10)	0.00458	0.01988	0.00925	0.01757	0.01564	0.01162	0.01202
			0.00066	0.00401	0.00096	0.00954	0.00837	0.00091	0.00094
		(10,30)	0.00204	0.01891	0.01041	0.00312	0.00365	0.00907	0.00939
			0.00044	0.00267	0.00081	0.00086	0.00095	0.00062	0.00064
		(10,50)	0.00145	0.01929	0.01095	0.00242	0.00287	0.00848	0.00877
			0.00039	0.00255	0.00079	0.00053	0.00051	0.00056	0.00057
		(30,10)	0.00431	0.00822	0.00344	0.01620	0.01423	0.00635	0.00652
			0.00037	0.00181	0.00042	0.00854	0.00744	0.00041	0.00042
		(30,30)	0.00140	0.00721	0.00393	0.00212	0.00194	0.00355	0.00365
			0.00017	0.00099	0.00025	0.00025	0.00022	0.00020	0.00020
		(30,50)	0.00082	0.00675	0.00449	0.00165	0.00154	0.00299	0.00307
			0.00015	0.00077	0.00024	0.00020	0.00018	0.00016	0.00017
		(50,10)	0.00445	0.00668	0.00157	0.01813	0.01491	0.00555	0.00568
			0.00033	0.00165	0.00033	0.00983	0.00801	0.00035	0.00036
		(50,30)	0.00160	0.00603	0.00212	0.00222	0.00206	0.00284	0.00291
			0.00014	0.00074	0.00017	0.00032	0.00032	0.00015	0.00015
		(50,50)	0.00101	0.00503	0.00266	0.00161	0.00145	0.00228	0.00233
			0.00011	0.00055	0.00015	0.00015	0.00014	0.00012	0.00012
(2,3)	0.867	(10,10)	0.00982	0.03888	0.02022	0.02502	0.02261	0.02447	0.02707
			0.00456	0.01949	0.00606	0.01842	0.01598	0.00536	0.00575
		(10,30)	0.00339	0.04031	0.02412	0.00481	0.00621	0.01894	0.02099
			0.00314	0.01454	0.00514	0.00455	0.00446	0.00382	0.00404
		(10,50)	0.00188	0.04181	0.02580	0.00368	0.00502	0.01760	0.01954
			0.00285	0.01389	0.00496	0.00385	0.00372	0.00348	0.00368
		(30,10)	0.01067	0.01500	0.00691	0.02481	0.02191	0.01447	0.01568
			0.00272	0.01072	0.00298	0.01490	0.01307	0.00281	0.00292
		(30,30)	0.00310	0.01515	0.00907	0.00466	0.00427	0.00787	0.00859
			0.00132	0.00643	0.00180	0.00183	0.00166	0.00141	0.00145
		(30,50)	0.00155	0.01471	0.01072	0.00350	0.00329	0.00651	0.00713
			0.00110	0.00512	0.00168	0.00148	0.00135	0.00118	0.00120
		(50,10)	0.01126	0.01161	0.00220	0.02740	0.02269	0.01289	0.01389
			0.00246	0.00972	0.00245	0.01648	0.01355	0.00246	0.00254
		(50,30)	0.00387	0.01318	0.00450	0.00498	0.00459	0.00652	0.00703
			0.00104	0.00502	0.00128	0.00155	0.00142	0.00108	0.00110
		(50,50)	0.00228	0.01107	0.00618	0.00369	0.00333	0.00513	0.00555
			0.00082	0.00383	0.00108	0.00113	0.00102	0.00086	0.00087
(3,3)	0.686	(10,10)	0.00916	0.03484	0.02325	0.01790	0.01674	0.02412	0.03175
			0.01405	0.04333	0.01684	0.02746	0.02497	0.01355	0.01461
		(10,30)	0.00073	0.04607	0.03283	0.00165	0.00135	0.01847	0.02444
			0.01032	0.03481	0.01431	0.01467	0.01401	0.01038	0.01101
		(10,50)	0.00314	0.05021	0.03656	0.00275	0.00020	0.01695	0.02257
			0.00954	0.03348	0.01369	0.01362	0.01293	0.00966	0.01020
		(30,10)	0.01508	0.00558	0.00485	0.02502	0.02203	0.01627	0.02040
			0.00872	0.02876	0.00954	0.01893	0.01716	0.00824	0.00863
		(30,30)	0.00306	0.01422	0.01171	0.00463	0.00427	0.00839	0.01082
			0.00457	0.01865	0.00582	0.00618	0.00565	0.00454	0.00467
		(30,50)	0.00047	0.01602	0.01526	0.00313	0.00304	0.00665	0.00873
			0.00381	0.01522	0.00531	0.00503	0.00461	0.00381	0.00390
		(50,10)	0.01671	0.00123	0.00292	0.02829	0.02325	0.01497	0.01846
			0.00802	0.02672	0.00840	0.01905	0.01657	0.00749	0.00780

Continued Table 1

(s,k)	True $R_{3(s,k)}$	(m,n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(3,3)	0.686	(50,30)	0.00514	0.01365	0.00430	0.00605	0.00559	0.00751	0.00931
			0.00365	0.01521	0.00438	0.00490	0.00451	0.00361	0.00369
		(50,50)	0.00251	0.01193	0.00812	0.00431	0.00389	0.00574	0.00719
			0.00287	0.01198	0.00361	0.00387	0.00351	0.00286	0.00291
(1,1)	0.837	(10,10)	0.00785	0.03120	0.01757	0.02016	0.01833	0.02007	0.02284
			0.00495	0.01834	0.00627	0.01636	0.01444	0.00531	0.00564
		(10,30)	0.00157	0.03510	0.02245	0.00209	0.00374	0.01549	0.01767
			0.00352	0.01410	0.00534	0.00510	0.00497	0.00391	0.00411
		(10,50)	0.00006	0.03710	0.02444	0.00112	0.00270	0.01434	0.01639
			0.00323	0.01352	0.00513	0.00449	0.00430	0.00361	0.00377
		(30,10)	0.01002	0.00960	0.00507	0.02201	0.01939	0.01237	0.01376
			0.00301	0.01088	0.00330	0.01286	0.01137	0.00297	0.00308
		(30,30)	0.00252	0.01219	0.00824	0.00380	0.00349	0.00660	0.00742
			0.00152	0.00678	0.00200	0.00208	0.00190	0.00157	0.00160
		(30,50)	0.00095	0.01249	0.01016	0.00276	0.00262	0.00538	0.00609
			0.00127	0.00547	0.00185	0.00169	0.00154	0.00131	0.00134
		(50,10)	0.01081	0.00651	0.00029	0.02460	0.02028	0.01114	0.01230
			0.00274	0.00998	0.00281	0.01397	0.01164	0.00265	0.00273
		(50,30)	0.00353	0.01095	0.00364	0.00442	0.00408	0.00562	0.00622
			0.00121	0.00540	0.00147	0.00173	0.00160	0.00123	0.00124
		(50,50)	0.00194	0.00934	0.00565	0.00320	0.00289	0.00438	0.00486
			0.00095	0.00419	0.00122	0.00130	0.00117	0.00097	0.00098

Note: The first entry is the simulated about absolute biases.
The second entry is the simulated about MSE’s.

Table 2: Results of simulation study of absolute bias and MSE of estimates of reliability for θ_1=0.5, θ_2=1.5, λ_1=5 , λ_2=3 , α=1 and 5000 replications

(s,k)	True $R_{3(s,k)}$	(m,n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(1,3)	0.345	(10,10)	0.00312	0.01913	0.01714	0.00953	0.01024	0.01154	0.00572
			0.01068	0.03303	0.01264	0.02074	0.01943	0.00990	0.00960
		(10,30)	0.00950	0.00196	0.00100	0.01854	0.01631	0.00743	0.00422
			0.00630	0.02179	0.00727	0.01382	0.01270	0.00589	0.00578
		(10,50)	0.01128	0.00473	0.00493	0.01913	0.01658	0.00699	0.00428
			0.00549	0.01934	0.00621	0.01261	0.01158	0.00511	0.00503
		(30,10)	0.00635	0.02863	0.02403	0.00612	0.00387	0.00709	0.00259
			0.00835	0.02689	0.01075	0.01216	0.01153	0.00794	0.00778
		(30,30)	0.00131	0.00793	0.00983	0.00294	0.00279	0.00426	0.00236
			0.00371	0.01379	0.00478	0.00491	0.00450	0.00362	0.00358
		(30,50)	0.00264	0.00337	0.00345	0.00290	0.00259	0.00337	0.00198
			0.00274	0.01077	0.00341	0.00373	0.00340	0.00268	0.00266
		(50,10)	0.00630	0.03327	0.02954	0.00579	0.00284	0.00816	0.00390
			0.00777	0.02524	0.01040	0.01141	0.01065	0.00745	0.00730
		(50,30)	0.00106	0.00813	0.01292	0.00107	0.00130	0.00295	0.00131
			0.00319	0.01204	0.00411	0.00424	0.00389	0.00313	0.00310
		(50,50)	0.00089	0.00475	0.00718	0.00189	0.00178	0.00268	0.00154
			0.00224	0.00904	0.00292	0.00291	0.00265	0.00221	0.00220
(2,3)	0.118	(10,10)	0.01315	0.04526	0.02288	0.02756	0.02665	0.02932	0.02599
			0.00567	0.02319	0.00746	0.01675	0.01565	0.00634	0.00584
		(10,30)	0.01329	0.02058	0.00714	0.02709	0.02438	0.01775	0.01612
			0.00363	0.01326	0.00387	0.01457	0.01319	0.00361	0.00345
		(10,50)	0.01372	0.01646	0.00224	0.02635	0.02350	0.01579	0.01443
			0.00327	0.01153	0.00324	0.01390	0.01257	0.00316	0.00304
		(30,10)	0.00398	0.04501	0.02566	0.00768	0.00856	0.02128	0.01873
			0.00379	0.01709	0.00620	0.00500	0.00485	0.00444	0.00415
		(30,30)	0.00472	0.01878	0.01060	0.00689	0.00638	0.01025	0.00930
			0.00180	0.00778	0.00250	0.00254	0.00233	0.00190	0.00184
		(30,50)	0.00468	0.01289	0.00508	0.00574	0.00520	0.00781	0.00713
			0.00136	0.00591	0.00171	0.00197	0.00180	0.00139	0.00136
		(50,10)	0.00335	0.04639	0.02906	0.00706	0.00830	0.02094	0.01852
			0.00347	0.01634	0.00603	0.00446	0.00432	0.00413	0.00388
		(50,30)	0.00251	0.01704	0.01207	0.00489	0.00469	0.00826	0.00745
			0.00148	0.00632	0.00213	0.00201	0.00184	0.00157	0.00153
		(50,50)	0.00292	0.01196	0.00715	0.00418	0.00385	0.00626	0.00571
			0.00107	0.00467	0.00147	0.00141	0.00128	0.00111	0.00109
(3,3)	0.027	(10,10)	0.01013	0.03671	0.01449	0.02504	0.02369	0.02224	0.02147
			0.00124	0.00936	0.00182	0.01164	0.01077	0.00195	0.00184
		(10,30)	0.00841	0.01985	0.00599	0.02293	0.02071	0.01302	0.01272
			0.00078	0.00449	0.00078	0.01155	0.01042	0.00093	0.00091
		(10,50)	0.00825	0.01698	0.00374	0.02210	0.01987	0.01147	0.01123
			0.00071	0.00371	0.00062	0.01130	0.01020	0.00080	0.00078
		(30,10)	0.00493	0.03073	0.01423	0.00733	0.00748	0.01595	0.01541
			0.00065	0.00532	0.00136	0.00102	0.00099	0.00111	0.00106
		(30,30)	0.00343	0.01378	0.00585	0.00484	0.00446	0.00720	0.00704
			0.00029	0.00193	0.00044	0.00058	0.00054	0.00036	0.00036
		(30,50)	0.00303	0.01013	0.00324	0.00394	0.00358	0.00544	0.00533
			0.00022	0.00137	0.00028	0.00048	0.00044	0.00025	0.00025
		(50,10)	0.00435	0.03037	0.01527	0.00657	0.00686	0.01523	0.01473
			0.00057	0.00502	0.00127	0.00074	0.00073	0.00100	0.00095

Continued Table 2

(s,k)	True $R_{3(s,k)}$	(m,n)	Method of estimations						
			MLE	MME	PCE	LSE	WLSE	BESL	BELL
(3,3)	0.027	(50,30)	0.00230	0.01174	0.00603	0.00361	0.00338	0.00585	0.00573
			0.00022	0.00138	0.00036	0.00032	0.00029	0.00028	0.00027
		(50,50)	0.00209	0.00855	0.00374	0.00283	0.00259	0.00431	0.00423
			0.00016	0.00094	0.00023	0.00022	0.00020	0.00019	0.00019
(1,1)	0.163	(10,10)	0.00880	0.03370	0.01817	0.02071	0.02019	0.02103	0.01820
			0.00492	0.01950	0.00629	0.01480	0.01381	0.00531	0.00500
		(10,30)	0.01040	0.01282	0.00471	0.02285	0.02047	0.01273	0.01131
			0.00305	0.01141	0.00333	0.01254	0.01137	0.00302	0.00292
		(10,50)	0.01108	0.00957	0.00035	0.02253	0.01998	0.01141	0.01023
			0.00272	0.00993	0.00280	0.01194	0.01082	0.00263	0.00255
		(30,10)	0.00085	0.03479	0.02131	0.00296	0.00406	0.01477	0.01261
			0.00344	0.01437	0.00523	0.00477	0.00458	0.00382	0.00364
		(30,30)	0.00316	0.01350	0.00876	0.00489	0.00454	0.00724	0.00641
			0.00159	0.00667	0.00215	0.00223	0.00205	0.00164	0.00161
		(30,50)	0.00345	0.00880	0.00392	0.00419	0.00379	0.00554	0.00494
			0.00119	0.00509	0.00149	0.00172	0.00157	0.00120	0.00118
		(50,10)	0.00047	0.03668	0.02462	0.00261	0.00411	0.01478	0.01273
			0.00317	0.01365	0.00506	0.00431	0.00411	0.00356	0.00340
		(50,30)	0.00125	0.01230	0.01034	0.00319	0.00312	0.00569	0.00498
			0.00132	0.00551	0.00184	0.00179	0.00164	0.00137	0.00135
		(50,50)	0.00197	0.00842	0.00602	0.00297	0.00274	0.00442	0.00393
			0.00095	0.00407	0.00127	0.00124	0.00113	0.00097	0.00096

Note: The first entry is the simulated about absolute biases.
The second entry is the simulated about MSE's.

APPENDIEX A

$$\tau_{11} = \frac{\theta_1^2}{m-1}. \tag{A.1}$$

$$\tau_{22} = \frac{\theta_2^2}{n-1}. \tag{A.2}$$

$$Q_{30} = \frac{2(m-1)}{\theta_1^3}. \tag{A.3}$$

$$Q_{03} = \frac{2(n-1)}{\theta_2^3}. \tag{A.4}$$

$$\left.\begin{aligned}
U_1 = \frac{\partial R_{(s,k)}}{\partial \theta_1} &= \theta_2 \delta \sum_{\mathrm{i=s}}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{i}} \int_0^1 [1-z^{\delta}]^{\theta_2 - 1} z^{\delta-1}\, [1-[1-z]^{\theta_1}]^{\mathrm{i}} \times \\
&\quad [1-z]^{\theta_1(\mathrm{k-i})} \ln[1-z]\, [\mathrm{k} - \mathrm{i} - \\
&\quad i[1-z]^{\theta_1}[1-[1-z]^{\theta_1}]^{-1}] dz, \\
U_2 = \frac{\partial R_{(s,k)}}{\partial \theta_2} &= \delta \sum_{\mathrm{i=s}}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{i}} \int_0^1 z^{\delta-1}\, [1-[1-z]^{\theta_1}]^{\mathrm{i}} [1-z]^{\theta_1(\mathrm{k-i})} \times \\
&\quad [1-z^{\delta}]^{\theta_2 - 1} [\theta_2 \ln[1-z^{\delta}] + 1] dz, \\
U_{11} = \frac{\partial^2 R_{(s,k)}}{\partial \theta_1^2} &= \theta_2 \delta \sum_{\mathrm{i=s}}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{i}} \int_0^1 [1-z^{\delta}]^{\theta_2 - 1} z^{\delta-1}\, [1-[1-z]^{\theta_1}]^{\mathrm{i}} \times \\
&\quad [1-z]^{\theta_1(\mathrm{k-i})} (\ln[1-z])^2 [i(i-1)[1-z]^{2\theta_1} \times \\
&\quad [1-[1-z]^{\theta_1}]^{-2} - i(2\mathrm{k} - 2\mathrm{i} + 1)[1-z]^{\theta_1} \times \\
&\quad [1-[1-z]^{\theta_1}]^{-1} + (\mathrm{k} - \mathrm{i})^2] dz, \\
U_{22} = \frac{\partial^2 R_{(s,k)}}{\partial \theta_2^2} &= \delta \sum_{\mathrm{i=s}}^{\mathrm{k}} \binom{\mathrm{k}}{\mathrm{i}} \int_0^1 z^{\delta-1}\, [1-[1-z]^{\theta_1}]^{\mathrm{i}} [1-z]^{\theta_1(\mathrm{k-i})} \times \\
&\quad [1-z^{\delta}]^{\theta_2 - 1} \ln[1-z^{\delta}]\, [\theta_2 \ln[1-z^{\delta}] + 2] dz.
\end{aligned}\right\} \tag{A.5}$$

$$\left.\begin{aligned}
W_1 &= -a \left(\frac{\partial R_{(s,k)}}{\partial \theta_1}\right) e^{-aR_{(s,k)}}, \\
W_2 &= -a \left(\frac{\partial R_{(s,k)}}{\partial \theta_2}\right) e^{-aR_{(s,k)}}, \\
W_{11} &= a e^{-aR_{(s,k)}} \left[a \left(\frac{\partial R_{(s,k)}}{\partial \theta_1}\right)^2 - \frac{\partial^2 R_{(s,k)}}{\partial \theta_1^2}\right], \\
W_{22} &= a e^{-aR_{(s,k)}} \left[a \left(\frac{\partial R_{(s,k)}}{\partial \theta_2}\right)^2 - \frac{\partial^2 R_{(s,k)}}{\partial \theta_2^2}\right],
\end{aligned}\right\} \tag{A.6}$$

where,

$\frac{\partial R_{(s,k)}}{\partial \theta_1}$, $\frac{\partial R_{(s,k)}}{\partial \theta_2}$, $\frac{\partial^2 R_{(s,k)}}{\partial \theta_1^2}$ and $\frac{\partial^2 R_{(s,k)}}{\partial \theta_2^2}$ are given, respectively, in equations (A.5).

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