Exponentiated Lomax Geometric Distribution: Properties and Applications

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Abstract

In this paper, a new four-parameter lifetime distribution, called the exponentiated Lomax geometric (*ELG*) is introduced. The new lifetime distribution contains the Lomax geometric and exponentiated Pareto geometric as new sub-models. Explicit algebraic formulas of probability density function, survival and hazard functions are derived. Various structural properties of the new model are derived including; quantile function, Re'nyi entropy, moments, probability weighted moments, order statistic, Lorenz and Bonferroni curves. The estimation of the model parameters is performed by maximum likelihood method and inference for a large sample is discussed. The flexibility and potentiality of the new model in comparison with some other distributions are shown via an application to a real data set. We hope that the new model will be an adequate model for applications in various studies.

Keywords: Exponentiated Lomax distribution, Geometric distribution, Maximum likelihood estimation.

1. Introduction

Lomax (1954) introduced an important and widely used lifetime model, the so-called Lomax distribution, and it used for stochastic modeling of decreasing failure rate.It has been applied in studies of income, size of cities, wealth inequality, engineering, queuing theory and biological analysis.

Studies about Lomax distribution have been discussed by several authors. Some properties and moments for the Lomax distribution have been discussed by Balakrishnan and Ahsanullah (1994). The discrete Poisson-Lomax distribution has been provided by Al-Awadhi and Ghitany (2001). The Bayesian and non-Bayesian estimation of the reliability has been studied by Abd-Elfattah et al. (2007). Ghitany et al. (2007) introduced Marshall-Olkin extended Lomax. Hassan and Al-Ghamdi (2009) determined the optimal times of changing stress level for simple stress plans under a cumulative exposure model for the Lomax distribution. Hassan et al. (2016) discussed the optimal times of changing stress level for *k*-level step stress accelerated life tests based on adaptive type-II progressive hybrid censoring with product's life time following Lomax distribution.

Some extensions of the Lomax distribution have been constructed by several authors. Abdul-Moniem and Abdel-Hameed (2012) introduced the exponentiated Lomax (*EL*) by adding a new shape parameter to the Lomax distribution. Lemonte and Cordeiro (2013) investigated beta Lomax, Kumaraswamy Lomax and McDonald Lomax distributions. The gamma-Lomax distribution has been suggested by Cordeiro et al. (2013). Five-parameter beta Lomax distribution has been investigated by Rajab et al. (2013). The Weibull Lomax and Gumbel-Lomax distributions have been introduced by Tahir et al. (2015a) and (2015 b).

The cumulative distribution function (cdf) of Lomax distribution with shape parameter θ and scale parameter λ is given by

$$G(x;\lambda,\theta)=1-(1+\lambda x)^{-\theta},\ x,\lambda,\theta>0. \tag{1}$$

The probability density function (pdf) of Lomax distribution is as follows

$$g(x;\lambda,\theta)=\lambda\theta(1+\lambda x)^{-(\theta+1)},\ x,\lambda,\theta>0. \tag{2}$$

The exponentiated Lomax has been introduced by Abdul-Moniem and Abdel-Hameed (2012) by adding shape α parameter to the cdf (1). The cdf of the *EL* takes the following form

$$G(x;\lambda,\theta,\alpha)=[1-(1+\lambda x)^{-\theta}]^{\alpha},\ x,\lambda,\alpha,\theta>0. \tag{3}$$

The corresponding pdf is as follows:

$$g(x;\lambda,\theta,\alpha)=\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\alpha},\ x,\lambda,\alpha,\theta>0. \tag{4}$$

Also, a discrete random variable, N is a member of a zero-truncated geometric random variable independent of $X\,'s$, with probability mass function (pmf) given by:

$$P(N=n;p)=(1-p)p^{n-1},\ n=1,2,3...,\ p\in(0,1). \tag{5}$$

Recently, various compounding probability distributions have been proposed by several authors for modeling lifetime data in several areas. Adamidis and Loukas (1998) proposed a two-parameter exponential-geometric distribution. In a similar manner, some examples as, the Weibull geometric, exponentiated exponential geometric and Lindley geometric distributions have been suggested by Barreto-Souza et al. (2010), Rezaei et al. (2011) and Zakerzadeh and Mahmoudi (2012) respectively. Recently, exponentiated Lomax Poisson, Lomax-logarithm, and extended Lomax Poisson distributions have been given, respectively by, Ramos et al. (2013), Al-Zahrani and Sagor (2014), and Al-Zahrani (2015).

In this article, a new compound distribution is introduced by mixing *EL* and geometric distributions. We hope that this new model will serve as a suitable model in several areas. The density, cumulative, survival and hazard rate functions of the new model are obtained in Section 2. Section 3 devotes with some mathematical properties such as; quantile, probability weighted moments, entropy and order statistics. Section 4 discusses the estimation of the unknown parameters by maximum likelihood method and inference for large sample is presented. In Section 5, applications to real data sets are given. Finally, concluding remakes are outlined in Section 6.

2. Construction of the Distribution

In this section, following the same approach of Adamidis and Loukas (1998), we introduce and study the exponentiated Lomax geometric distribution. The probability density function, distribution function, reliability and hazard rate function are obtained.

Let $X_1, X_2, ..., X_N$ be a random sample of size N from the Exponentiated Lomax distribution with cdf(3) and N be a zero-truncated geometric random variable independent of X 's, with pmf (5).

Define $X_{(1)} = \min\{X_1, X_2, ..., X_N\}$, then the conditional probability density function of $X_{(1)} \mid N$ is obtained as follows:

$$f_{X_{(1)}|N}(x \mid n;\alpha,\lambda,\theta) = n\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}(1-(1+\lambda x)^{-\theta})^{\alpha-1}[1-(1-(1+\lambda x)^{-\theta})^{\alpha}]^{n-1}.$$

The joint probability density function of X and N is obtained as follows:

$$f_{X_{(1)}N}(xn;\alpha,\lambda,\theta) = n(1-p)p^{n-1}\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}(1-(1+\lambda x)^{-\theta})^{\alpha-1}[1-(1-(1+\lambda x)^{-\theta})^{\alpha}]^{n-1}.$$

The probability density of exponentiated Lomax geometric isdefined as the marginal density of X, i.e.

$$f(x;\phi) = \frac{(1-p)\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}(1-(1+\lambda x)^{-\theta})^{\alpha-1}}{\{1-p[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]\}^2}, x>0, \alpha,\lambda,\theta>0, \quad (6)$$

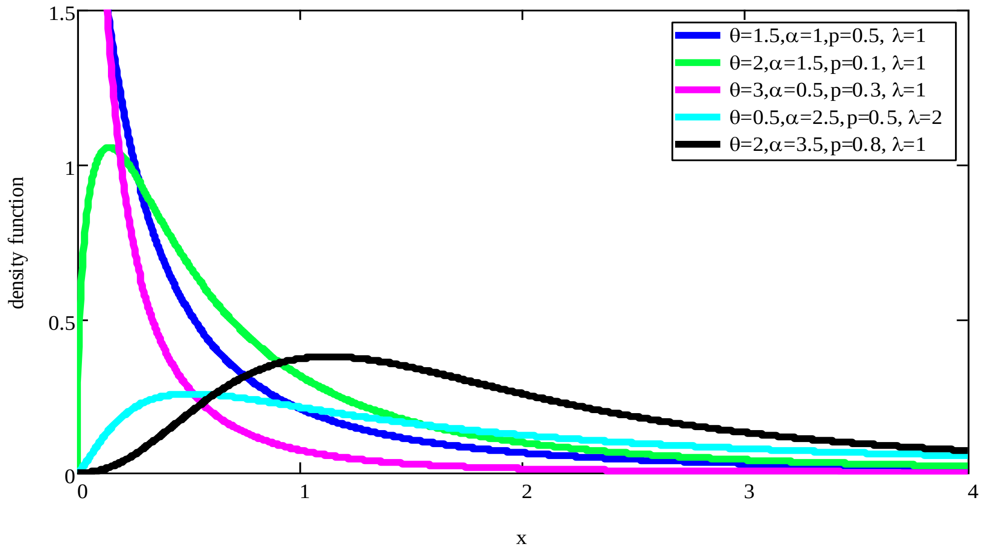
where $0 < p < 1$, and $\phi \equiv (\alpha,\lambda,\theta,p)$, is the set of parameters. A random variable X with density function (6) shall be denoted by $X \sim ELG(x;\phi)$. Furthermore, the cumulative distribution function of ELG corresponding to (6) is derived as follows:

$$F(x;\phi) = \frac{[1-(1+\lambda x)^{-\theta}]^{\alpha}}{1-p[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]}, x>0. \quad (7)$$

Based on cdf (7), some special distributions arisefrom the ELG distribution as follows:

1. As $p \to 0$, the exponentiated Lomax (EL) is a limiting case of the ELG distribution.
2. For $\alpha = 1$ and when $p \to 0$, the exponentiated Lomax geometric reduces to Lomax distribution.
3. When $p \to 0$, and $\lambda = 1$, the exponentiated Lomax geometric reduces to exponentiated Pareto (see Gupta et al. (1998)).
4. The exponentiated Lomax geometric reduces to Lomax geometric when $\alpha = 1$.
5. The exponentiated Lomax geometric reduces to exponentiated Pareto geometric when λ=1.

Figure (1) gives some possible shapes of the density (6) for some selected parameter values.



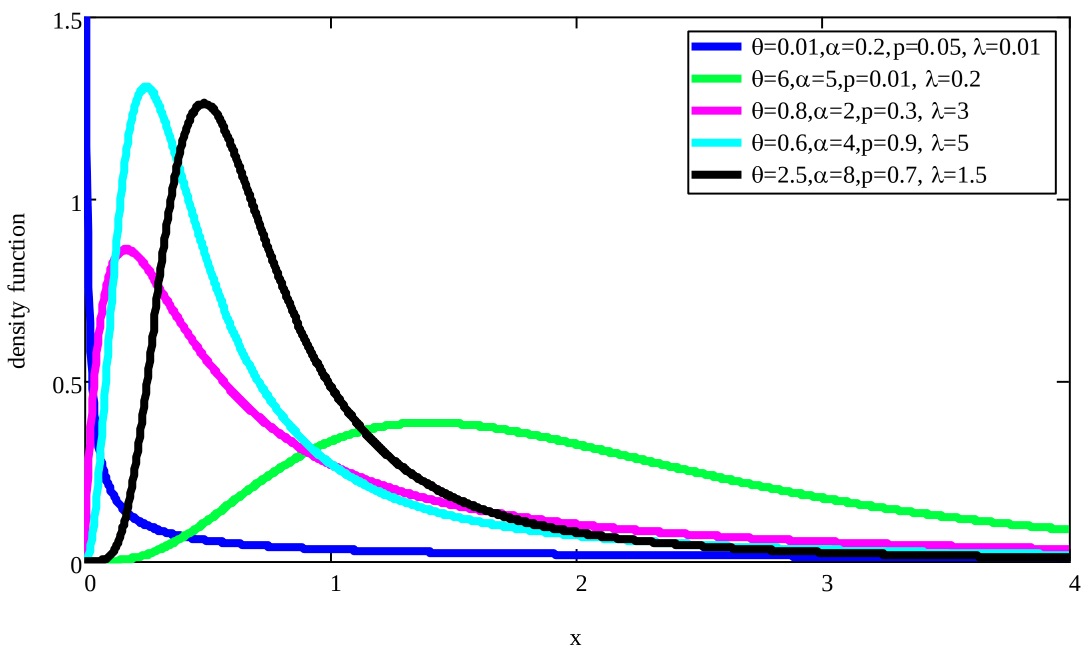


Figure 1: Plots of the *ELG* densities function for some parameter values

In addition, the reliability and hazard rate functions of *ELG* distribution are as follows:

$$R(x;\phi)=\frac{(1-p)\left\{1-[1-(1+\lambda x)^{-\theta}]^{\alpha}\right\}}{1-p[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]},$$

and,

$$\upsilon(x;\phi)=\frac{\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}(1-(1+\lambda x)^{-\theta})^{\alpha-1}}{\{1-p[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]\}[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]},$$

where $\alpha > 0, \lambda > 0, \theta > 0,$ and $0 < p < 1$. Figure (2) illustrates the graphical behavior of hazard rate function for *ELG* for some selected values of parameters.

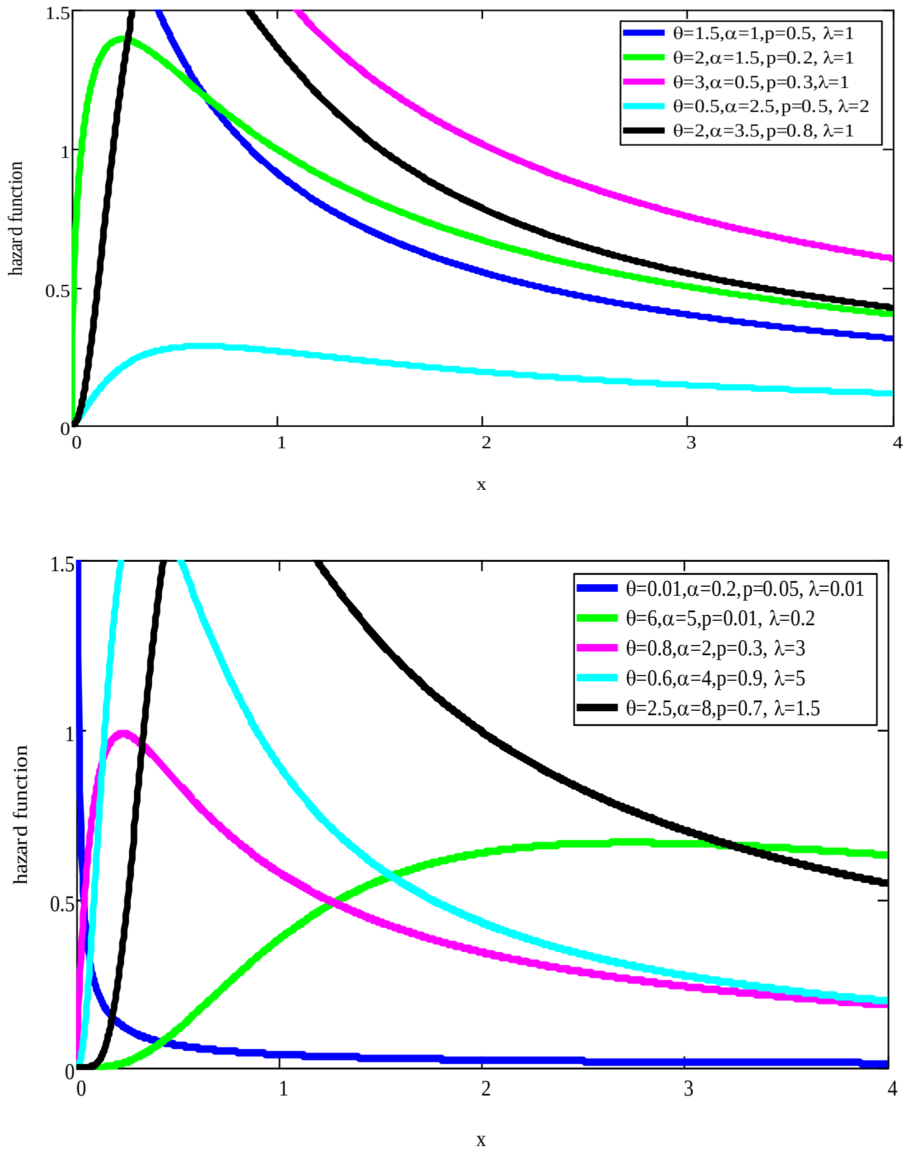


Figure. 2: Plots of the *ELG*hazard function for some parameter values

Figure 2 shows that the shapes of the hazard rate are increasing, decreasing and constant at some selected values of parameters.

2. Some Statistical Properties

In this section, some of statistical properties of the *ELG* distribution including, expansion for pdf (6) and cdf (7), quantile function, rth moment, and probability weighted moments are derived. Furthermore, Re'nyi entropy, distribution of order statistics, Bonferroni and Lorenz curves are provided.

3.1. Usefuel expansions

In this subsection, two useful expansions for the pdf (6) and cdf (7) are derived. We show that the pdf of *ELG* can be expressed as linear combinations of *EL* distributions. Also the two expansions are used to determine some mathematical properties of the *ELG* distribution.

Firstly, the pdf (6) of *ELG* can be expressed as linear combinations of *EL* distributions. Using the following series expansions

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)z^{j}}{\Gamma(k)j!},\ \ |z|<1,k>0. \tag{8}$$

Then, the pdf (6) can be written as follows

$f(x;\phi)=\sum_{j=0}^{\infty}(j+1)p^{j}(1-p)\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\alpha-1}\{1-[1-(1+\lambda x)^{-\theta}]^{\alpha}\}^{j}$. Then, by using the binomial expansion for the previous pdf, then it can be written as

$$f(x;\phi)=\sum_{j,k=0}^{\infty}(-1)^{k}(j+1)\binom{j}{k}p^{j}(1-p)\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\alpha(k+1)-1}, \tag{9}$$

where, pdf (9) leads to the following infinite linear combination

$$f(x;\phi)=\sum_{j,k=0}^{\infty}W_{j,k}h_{\alpha(k+1)}(x;\lambda,\theta), \tag{10}$$

where,

$$W_{j,k}=(-1)^{k}\binom{j+1}{k+1}p^{j}(1-p),\ \sum_{j,k=0}^{\infty}W_{j,k}=1,$$

and $h_{\alpha(k+1)}(x;\lambda,\theta)$ denotes the pdf of *EL* with parameters $\lambda,\alpha(k+1),$ and $\theta.$

Secondly; an expansion for $[F(x;\phi)]^{s}$ is derived from cdf (7) through the expansions defined in (8) as follows

$$[F(x;\phi)]^{s}=\sum_{i,h=0}^{\infty}(-1)^{h}\binom{i}{h}\binom{s+i-1}{s-1}p^{i}[1-(1+\lambda x)^{-\theta}]^{\alpha(i+s)}. \tag{11}$$

3.2 Quantile measures

The quantile function of *ELG* distribution, denoted by, $Q(u)=F^{-1}(u)$ of X is derived as follows:

$$Q(u)=\frac{1}{\lambda}\left\{\left(1-\left[\frac{u(1-p)}{(1-up)}\right]^{\frac{1}{\alpha}}\right)^{\frac{-1}{\theta}}-1\right\}, \tag{12}$$

where u is a uniform random variable on the unit interval $(0,1)$. In particular the median of the *ELG* distribution, denoted by m, is obtained by substituting $u = 0.5$ in (12) as follows

$$m = \frac{1}{\lambda}\left\{\left(1-\left[\frac{0.5(1-p)}{(1-0.5p)}\right]^{\frac{1}{\alpha}}\right)^{\frac{-1}{\theta}} - 1\right\}.$$

The Bowleyskewness (see Kenney and Keeping (1962)) based on quantiles can be calculated by

$$B = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Further, the Moors kurtosis (see Moors (1988)) is defined as

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)},$$

where Q(.) denotes the quantile function. Plots of the skewness and kurtosis for some choices of the parameter α as function of *p*, and for some choices of the parameter *p* as function of α are shown in Figures 3 and 4. We can detect from these figures that the skewness and kurtosis for p decreases as α increases from 0.5 to 2.5. Also, the skewness and kurtosis for α decreases as p increases from 0.1 to 0.7

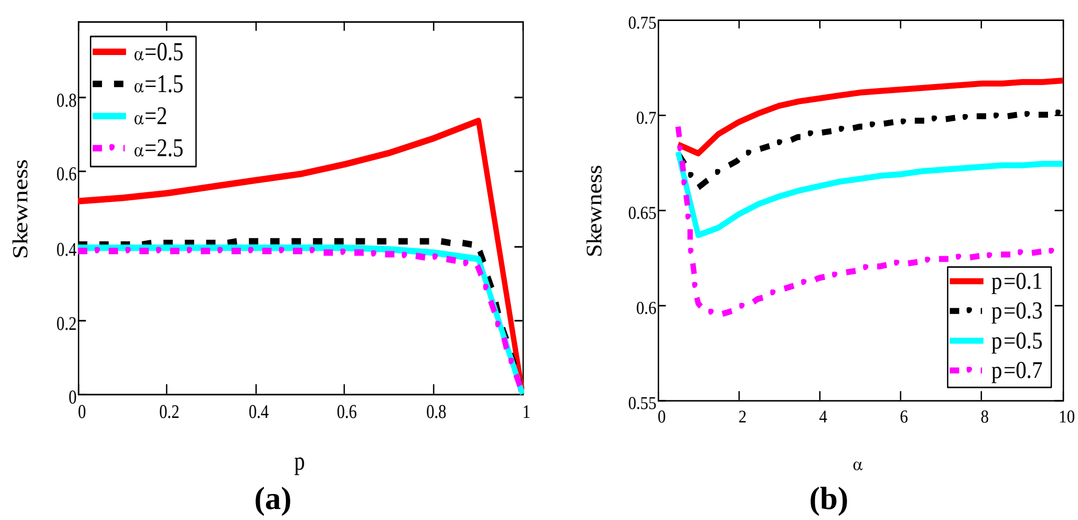
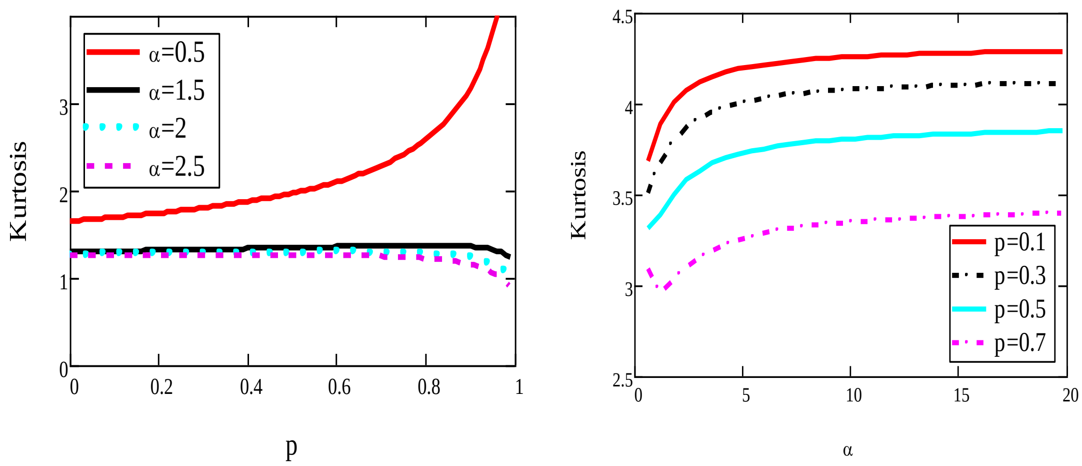


Figure 3: Skewness of the *ELG* distribution. (a) As function of *p* for some values of α with $\theta = 0.5$ and $\lambda = 1$(b) As function of α for some values of *p* with $\theta = 1.5$ and $\lambda = 1$



(a) **(b)**

Figure 4: Kurtosis of the *ELG* distribution.(a) As function of p for some values of α with $\theta = 0.5$ and $\lambda = 1$ (b) As function of α for some values of p with $\theta = 1.5$ and $\lambda = 1$

3.3 Moments

The moments of any probability distribution are necessary and important in any statistical analysis especially in applied work. Some of the most important features and characteristics of a distribution can be studied through moments such as; dispersion, skewness and kurtosis.

An explicit expression for the rth moment of *ELG* distribution about the origin is obtained by using pdf (10) as follows:

$$\mu_r^{'} = \sum_{j,k=0}^{\infty} W_{j,k} \int_0^{\infty} x^r h_{\alpha(k+1)}(x;\lambda,\theta)dx \ ,$$

$$\mu_r^{'} = \sum_{j,k=0}^{\infty} W_{j,k} \int_0^{\infty} \alpha(k+1)\lambda\theta x^r (1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\alpha(k+1)-1}dx \ .$$

Let, $y = (1+\lambda x)^{-\theta}$, and using binomial expansion then the above integral is reduced to

$$\mu_r^{'} = \sum_{j,k=0}^{\infty} W_{j,k} \frac{\alpha(k+1)}{\lambda^r} \int_0^1 \sum_{m=0}^{r} (-1)^m \binom{r}{m} y^{\frac{-1}{\theta}(r-m)} [1-y]^{\alpha(k+1)-1} dy$$

$$\mu_r^{'} = \sum_{j,k=0}^{\infty} \sum_{m=0}^{r} W_{j,k} \frac{(-1)^m \alpha(k+1) \binom{r}{m} \mathrm{B}\left(1-\left(\frac{1}{\theta}(r-m)\right), \alpha(k+1)\right)}{\lambda^r},$$

where, $\mathrm{B}(.,.)$ stands for beta function. Hence the *rth* moment for *ELG* distribution takes the following form:

$$\mu_r^{'} = \sum_{j,k=0}^{\infty}\sum_{m=0}^{r} D_{j,k,m} \frac{\alpha(k+1)\Gamma(1-\frac{1}{\theta}(r-m))\Gamma(\alpha(k+1))}{\lambda^r \Gamma(1-\frac{1}{\theta}(r-m)+\alpha(k+1))}, \qquad (13)$$

where,

$$D_{j,k,m} = (-1)^m \binom{r}{m} W_{k,j}.$$

Furthermore, it is easy to show that, the moment generating $M_X(t)$ function can be written as follows:

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r',$$

where, μ_r' is the *rth* moment. Then by using (13), the moment generating function of *ELG* distribution can be written as follows:

$$M_X(t) = \sum_{j,k,r=0}^{\infty}\sum_{m=0}^{r} D_{j,k,m} \frac{t^r\alpha(k+1)\Gamma(1-\frac{1}{\theta}(r-m))\Gamma(\alpha(k+1))}{r!\lambda^r \Gamma(1-\frac{1}{\theta}(r-m)+\alpha(k+1))}.$$

3.4 The probability weighted moments

Probability weighted moments (PWMs) were devised by Greenwood et al. (1979) primarily as an aid to estimate the parameters of distributions that are analytically expressible only in inverse form. PWMs are the expectations of certain functions of a random variable defined when the ordinary moments of the random variable exist.
The PWMs of a random variable X are formally defined by

$$\tau_{r,s} = E[X^r \mathrm{F}(x)^s] = \int_{-\infty}^{\infty} x^r \mathrm{f}(x)(\mathrm{F}(x))^s dx. \qquad (14)$$

The PWMs of *ELG* distribution is obtained by subsituting pdf (10) and cdf (11) in (14) as follows:

$$\tau_{r,s} = \sum_{j,k,i,h=0}^{\infty} \kappa_{i,j,k,m} \int_0^{\infty} x^r \alpha(k+1)\lambda\theta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\alpha(k+s+h+1)-1}dx,$$

where

$$\kappa_{j,k,i,h} = (-1)^h \binom{i}{h}\binom{s+i-1}{s-1} p^i W_{j,k}.$$

Let $y=(1+\lambda x)^{-\theta}$, and using binomial expansion then $\tau_{r,s}$ takes the following form:

$$\tau_{r,s}=\sum_{i,j,k,h=0}^{\infty}\kappa_{i,j,k,h}\sum_{m=0}^{r}\frac{(-1)^m\alpha(k+1)}{\lambda^r}\int_0^1 z^{-\left(\frac{m-r}{\theta}\right)}[1-z]^{\alpha(k+s+h+1)-1}dz$$

Then

$$\tau_{r,s}=\sum_{i,j,k,h=0}^{\infty}\sum_{m=0}^{r}\frac{(-1)^m\alpha(k+1)}{\lambda^r}\kappa_{i,j,k,h}\mathrm{B}\left(1-\frac{(r-m)}{\theta},\alpha(k+s+h+1)\right)$$

Hence, the PWMs of *ELG* distribution can be expressed as:

$$\tau_{r,s}=\sum_{i,j,k,h}^{\infty}\sum_{m=0}^{r}\eta_{i,j,k,m,h}\frac{\alpha(k+1)\Gamma(1-\frac{(r-m)}{\theta})\Gamma(\alpha(k+s+h+1))}{\lambda^r\Gamma(\alpha(m+s+h+1)+1-\frac{(r-m)}{\theta})},$$

$$\eta_{j,k,m,i,h}=(-1)^{k+m+h}\binom{j+1}{k+1}\binom{i}{h}\binom{s+i-1}{s-1}p^{j+i}(1-p).$$

3.5 Re'nyi entropy

The entropy of a random variable *X*is a measure of uncertainty variation. If *X* is a random variable which distributed as *ELG*, then the Re'nyi entropy, for $\rho>0$ and $\rho\neq 1$ is defined by:

$$I_\rho(x)=\frac{1}{(1-\rho)}\log_b\int_0^\infty\left(f(x;\phi)\right)^\rho dx.$$

Then by using pdf (6), the Re'nyi entropy of *ELG* can be written as follows:

$$I_\rho(x)=\frac{1}{(1-\rho)}\log_b\int_0^\infty\left[\frac{(1-p)\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}(1-(1+\lambda x)^{-\theta})^{\alpha-1}}{\{1-p[1-[1-(1+\lambda x)^{-\theta}]^\alpha]\}^2}\right]^\rho dx.$$

Let

$$IP=\left((1-p)\alpha\lambda\theta\right)^\rho\int_0^\infty\frac{(1+\lambda x)^{-\rho(\theta+1)}(1-(1+\lambda x)^{-\theta})^{\rho(\alpha-1)}}{\{1-p[1-[1-(1+\lambda x)^{-\theta}]^\alpha]\}^{2\rho}}dx.$$

By using the series expansion (8), then the above integration IP can be written as follows

$$IP=\sum_{i,j=0}^{\infty}(-1)^i\binom{j}{i}\binom{2\rho+j-1}{2\rho-1}p^j\left((1-p)\alpha\lambda\theta\right)^\rho\int_0^\infty[1-(1+\lambda x)^{-\theta}]^{\rho(\alpha-1)+\rho i}(1+\lambda x)^{-\rho(\theta+1)}dx$$

Let, $y=(1+\lambda x)^{-\theta}$, then the IP is reduced to

$$IP=\sum_{i,j=0}^{\infty}\binom{2\rho+j-1}{2\rho-1}\binom{j}{i}\frac{(-1)^{i)}p^j\left((1-p)\alpha\right)^\rho}{(\theta\lambda)^{1-\rho}}\mathrm{B}\left(\frac{\rho(\theta+1)-1}{\theta},\alpha i+\rho(\alpha-1)+1\right).$$

Therefore, the Re'nyi entropy of *ELG* distribution is as follows

$$I_R(x) = (1-\rho)^{-1}\log_b \sum_{i,j=0}^{\infty} L_{i,j} \frac{\Gamma\left(\frac{\rho(\theta+1)-1}{\theta}\right)\Gamma\left(\alpha i + \rho(\alpha-1)+1\right)}{\Gamma\left(\frac{\rho(\theta+1)-1}{\theta} + \alpha i + \rho(\alpha-1)+1\right)},$$

where,

$$L_{i,j} = \binom{j}{i}\binom{2\rho+j-1}{2\rho-1}\frac{(-1)^i p^j\left((1-p)\alpha\right)^{\rho}}{(\theta\lambda)^{1-\rho}}.$$

3.6 Order statistics

In this subsection, a closed form expression for the pdf of the *rth* order statistics of the *ELG* distribution will be derived. Let $X_1, X_2, \dots, X_n$ be a simple random sample from *ELG* distribution and let $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ denote the order statistics obtained from this sample. According to David (1981), the pdf of rth order statistics is as follows

$$f_{r:n}(x;\phi) = \frac{1}{B(r, n-r+1)}\left[F(x;\phi\right]^{r-1}\left[1-F(x;\phi\right]^{n-r} f(x;\phi),$$

where $B(.,.)$ stands for beta function. Using the binomial series expansion of $\left[1-F(x;\phi\right]^{n-r}$, then, $f_{r:n}(x;\phi)$, can be written as:

$$f_{r:n}(x;\phi) = \frac{1}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left[F(x;\phi)\right]^{r+i-1} f(x;\phi).$$

Substituting cdf (7), then the probability density function of *rth* order statistics $X_{r:n}$ from *ELG* distribution is derived as follows

$$f_{r:n}(x;\phi) = \frac{1}{B(r, n-r+1)}\sum_{i=0}^{n-r}(-1)^i\binom{n-r}{i}\left[\frac{(1-(1+\lambda x)^{-\theta})^{\alpha}}{1-p[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]}\right]^{r+i-1} f(x;\phi). \qquad (15)$$

Using the power series (8) and binomial expansion, then (15) takes the following form

$$f_{r:n}(x;\phi) = \frac{1}{B(r, n-r+1)}\sum_{i=0}^{n-r}\sum_{j,k=0}^{\infty}(-1)^{i+k}\binom{n-r}{i}\binom{j}{k}\binom{r+i+j-2}{r+i-2}p^j[1-(1+\lambda x)^{-\theta}]^{\alpha(k+r+i-1)} f(x;\phi).$$

Then, by substituting pdf (6), using the series expansion (8) and binomial expansion in the previous equation,we have

$$f_{r:n}(x;\phi) = \frac{1}{B(r, n-r+1)}\sum_{i=0}^{n-r}\sum_{j,k=0}^{\infty}(-1)^{i+k}\binom{n-r}{i}\binom{j}{k}\binom{r+i+j-2}{r+i-2}p^j[1-(1+\lambda x)^{-\theta}]^{\alpha(k+r+i-1)}$$

$$\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\binom{h}{m}(-1)^m(h+1)p^h(1-p)\alpha\lambda\theta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\alpha(m+1)}$$

Therefore, the pdf of rth order statistics of *ELG* can be expressed as follows

$$f_{r:n}(x;\phi)=\sum_{i=0}^{n-r}\sum_{j,k,m,h=0}^{\infty}\xi_{i,h,j,k,m}\, h_{\alpha(m+k+r+i)}(x;\lambda,\theta), \qquad 0<x<\infty, \qquad (16)$$

where,

$$\xi_{i,h,j,k,m}=\frac{(-1)^{i+k+m}(h+1)}{\mathrm{B}(r,n-r+1)(m+k+r+i)}\binom{n-r}{i}\binom{j}{k}\binom{r+i+j-2}{r+i-2}\binom{h}{m}(1-p)n\,p^{h+j},$$

and $h_{\alpha(m+k+r+i)}(x;\lambda,\theta)$ denotes the pdf of *EL* with parameters $\lambda,\alpha(m+k+r+i)$, and θ.

In particular, the pdf of the smallest order statistic $X_{1:n}$ is obtained by substituting r=1 in (16) as follows

$$f_{1:n}(x;\phi)=\sum_{i=0}^{n-1}\sum_{j,k,m,h=0}^{\infty}\varsigma_{i,h,j,k,m}\, h_{\alpha(m+k+i+1)}, \qquad 0<x<\infty,$$

$$\varsigma_{i,j,k,m,h}=\frac{(-1)^{i+k+m}(h+1)}{(m+k+i+1)}\binom{n-1}{i}\binom{j}{k}\binom{i+j-1}{i-1}\binom{h}{m}(1-p)n\,p^{h+j},$$

and $h_{\alpha(m+k+1+i)}(x;\lambda,\theta)$ denotes the pdf of *EL* with parameters $\lambda,\alpha(m+k+1+i)$, and θ.

Also, the pdf of the largest order statistic $X_{n:n}$ is obtained by substituting $r=n$ in (16), as follows

$$f_{n:n}(x;\phi)=\sum_{j,k,m,h=0}^{\infty}\eta_{j,k,m,h}h_{\alpha(k+n+i+m)}(x;\lambda,\theta), \qquad 0<x<\infty,$$

$$\eta_{j,k,m,h}=\frac{(-1)^{k+m}\left(h+1\right)}{(k+n+i+m)}\binom{j}{k}\binom{n+j-2}{n-2}\binom{h}{m}(1-p)p^{h+j}.$$

and again $h_{\alpha(m+k+1+i)}(x;\lambda,\theta)$ denotes the pdf of *EL* with parameters $\lambda,\alpha(m+k+1+i)$, and θ.

3.7 Bonferroni and Lorenz curves

Bonferroni and Lorenz curves are income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance. The Bonferroni curve is calculated by the following form:

$$B_F[F(x)]=\frac{1}{\mu F(x)}\int_0^x uf(u)du$$

Then by using pdf (10), the Bonferroni curve can be expressed as follows:

$$B_F[F(x)]=\frac{1}{\mu F(x)}\sum_{j,k=0}^{\infty}W_{j,k}\int_0^x u\alpha(k+1)\lambda\theta(1+\lambda u)^{-(\theta+1)}[1-(1+\lambda u)^{-\theta}]^{\alpha(k+1)-1}du. \qquad (17)$$

Let, $z = (1+\lambda u)^{-\theta}$, then the integrated part in (17) can be written as follows

$$I = \sum_{j,k=0}^{\infty} W_{j,k} \int_{1}^{(1+\lambda x)^{-\theta}} \frac{\alpha(k+1)}{\lambda} (z^{\frac{-1}{\theta}} - 1)[1-z]^{\alpha(k+1)-1} dz.$$

Using the series expansion, then the previous integeral is as follows

$$I = \sum_{j,k,h=0}^{\infty} (-1)^h W_{j,k} \frac{\alpha(k+1)}{\lambda} \binom{\alpha(k+1)-1}{h} \int_{1}^{(1+\lambda x)^{-\theta}} (z^{h-\frac{1}{\theta}} - z^h),$$

$$= \sum_{j,k,h=0}^{\infty} (-1)^h W_{j,k} \frac{\alpha(k+1)}{\lambda} \binom{\alpha(k+1)-1}{h} \left\{ \left[\frac{(1+\lambda x)^{-\theta\left(h-\frac{1}{\theta}+1\right)} - 1}{h - \frac{1}{\theta} + 1} \right] - \left[\frac{(1+\lambda x)^{-\theta(h+1)} - 1}{h+1} \right] \right\}.$$

For simplicity, put $Q_{j,k,h} = (-1)^h W_{j,k} \frac{\alpha(k+1)}{\lambda} \binom{\alpha(k+1)-1}{h}$, then the Bonferroni curve can be written as follows

$$B_F[F(x)] = \frac{\sum_{j,k,h=0}^{\infty} Q_{j,k,h} \left\{ \left[\frac{(1+\lambda x)^{-\theta\left(h-\frac{1}{\theta}+1\right)} - 1}{h - \frac{1}{\theta} + 1} \right] - \left[\frac{(1+\lambda x)^{-\theta(h+1)} - 1}{h+1} \right] \right\} [1-(1+\lambda x)^{-\theta}]^{\alpha}}{[1-p[1-[1-(1+\lambda x)^{-\theta}]^{\alpha}]] \left(\sum_{k,j=0}^{\infty} W_{k,j} \alpha(k+1\left(\mathrm{B}(1-\frac{1}{\theta}, \alpha(k+1)) - \mathrm{B}(1, \alpha(k+1)) \right) \right)}.$$

Also, the Lorenz curve is calculated by the following form

$$L[Z] = \frac{1}{\mu} \int_{0}^{z} x f(x) dx.$$

The Lorenz curve of *ELG* distribution takes the following form

$$L[Z] = \frac{\sum_{j,k,h=0}^{\infty} Q_{j,k,h} \left\{ \left[\frac{(1+\lambda x)^{-\theta\left(h-\frac{1}{\theta}+1\right)} - 1}{h - \frac{1}{\theta} + 1} \right] - \left[\frac{(1+\lambda x)^{-\theta(h+1)} - 1}{h+1} \right] \right\}}{\left(\sum_{k,j=0}^{\infty} W_{k,j} \alpha(k+1\left(\mathrm{B}(1-\frac{1}{\theta}, \alpha(k+1)) - \mathrm{B}(1, \alpha(k+1)) \right) \right)}.$$

4. Parameter Estimation

In this section, estimation of the *ELG* model parameters; is obtained by using maximum likelihood method of estimation.

Let $X_1, X_2, \ldots X_n$ be a simple random sample from the *ELG* distribution with set of parameters $\phi \equiv (\alpha, \lambda, \theta, p)$. The log-likelihood function, denoted by lnl, based on the observed random sample of size n from density (6) is given by:

$$lnl\,(x;\phi) = n\ln(1-p) + n\ln\alpha + nln\lambda + nln\theta - (\theta+1)\sum_{i=1}^{n}\ln(1+\lambda x_i) +$$

$$(\alpha-1)\sum_{i=1}^{n}\ln S_i - 2\sum_{i=1}^{n}\ln\left\{1-p+pS_i^{\alpha}\right\},$$

where, $S_i = 1-(1+\lambda x_i)^{-\theta}$. The partial derivatives of the log-likelihood function with respect to the unknown parameters are given by:

$$\frac{\partial lnl}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln S_i - 2\sum_{i=1}^{n}\frac{pS_i^{\alpha}\ln(S_i)}{1-p+pS_i^{\alpha}},$$

$$\frac{\partial lnl}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\ln(1+\lambda x_i) + (\alpha-1)\sum_{i=1}^{n}\frac{(1-S_i)\ln(1+\lambda x_i)}{S_i} - 2\sum_{i=1}^{n}\frac{p\alpha S_i^{\alpha-1}(1-S_i)\ln(1+\lambda x_i)}{1-p+pS_i^{\alpha}},$$

$$\frac{\partial lnl}{\partial\lambda} = \frac{n}{\lambda} - (\theta+1)\sum_{i=1}^{n}\frac{x_i}{1+\lambda x_i} + (\alpha-1)\sum_{i=1}^{n}\frac{\theta x_i(1+\lambda x_i)^{-(\theta+1)}}{S_i} - 2\sum_{i=1}^{n}\frac{p\theta\alpha S_i^{\alpha-1}x_i(1+\lambda x_i)^{-(\theta+1)}}{1-p+pS_i^{\alpha}},$$

$$\frac{\partial lnl}{\partial p} = \frac{-n}{(1-p)} + 2\sum_{i=1}^{n}\frac{1-S_i^{\alpha}}{1-p+pS_i^{\alpha}}.$$

The maximum likelihood estimators of the model parameters are determined by solving numerically the non-linear equations $\frac{\partial lnl}{\partial\alpha}=0$, $\frac{\partial lnl}{\partial\theta}=0$, $\frac{\partial lnl}{\partial\lambda}=0$ and $\frac{\partial lnl}{\partial p}=0$ simultaneously.

For interval estimation and hypothesis tests on the model parameters, the observed Fisher's information matrix must be obtained. The 4×4 unit observed information matrix $I(\phi)$ is given as follows

$$I(\phi) = -\begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\lambda} & I_{\alpha\theta} & I_{\alpha p} \\ I_{\lambda\alpha} & I_{\lambda\lambda} & I_{\lambda\theta} & I_{\lambda p} \\ I_{\theta\alpha} & I_{\theta\lambda} & I_{\theta\theta} & I_{\theta p} \\ I_{p\alpha} & I_{p\lambda} & I_{p\theta} & I_{pp} \end{bmatrix},$$

where, $I_{\phi_i\phi_j} = \partial^2 lnl \big/ \partial\phi_i\partial\phi_j$. The entries of Fisher's information matrix for *ELG* are given in the Appendix.

The approximate $100(1-\gamma)\%$ two sided confidence intervals for $\alpha, \lambda, \theta, p$ are respectively, given by:

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\alpha})},\ \hat{\lambda} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\lambda})},\ \hat{\theta} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{\theta})}\ ,\ \hat{p} \pm Z_{\gamma/2}\sqrt{\mathrm{var}(\hat{p})}$$

Here, $Z_{\gamma/2}$ is the upper $\gamma/2\ ^{th}$ percentile of the standard normal distribution and var (.)'s denote the diagonal elements of $I^{-1}(\phi)$ corresponding to the model parameters.

5. Applications

In this section, the flexibility of *ELG* model is examined by comparing it with some other distributions. Two real data sets are used to show that *ELG* distribution can be applied in practice and can be a better model than some others.

For the two sets of data; the *ELG* is compared to Lomax (*L*), exponentiated Lomax, kumaraswamy Lomax (*KL*), Weibull Lomax *(WL)* and exponentiated Pareto (*EP*) distributions. The density functions for kumaraswamy Lomax, Weibull Lomax and exponentiated Pareto distributions are as follows;

$$f_{KL}(x;a,b,\lambda,\theta) = ab\lambda\theta(1+\lambda x)^{b\theta-1}[1-(1+\lambda x)^{-\theta}]^{b-1}\exp[-a[(1+\lambda x)^{\theta}-1]^{b}],$$
$$f_{WL}(x;a,b,\lambda,\theta) = ab\lambda\theta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{a-1}[1-[1-(1+\lambda x)^{-\theta}]^{b-1},$$
$$f_{EP}(x;\theta,\alpha) = \theta(1+x)^{-(\theta+1)}[1-(1+x)^{-\theta}]^{\alpha}.$$

The first data set represents 84 observations of failure times (in hours) for a particular wind shield model reported by (Murthy et al. (2004)):

0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82, 3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652,2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

The maximum likelihood method is employed to obtain the point estimates of the model parameters. To compare the fitted models, some selected measures are applied. The selected measures include; *-2log*-likelihoodfunction evaluated at the parameter estimates, Akaike information criterion (*AIC*), Bayesian information criterion (*BIC*), consistent

Akaike information criterion (*CAIC*), Hannan-Quinn information criterion (*HQIC*) and the Kolmogorov-Smirnov (*k-s*) statistic. The mathematical form of these measures is as follows

$$AIC = 2p - 2\ln L, \qquad CAIC = AIC + \frac{2p(p+1)}{n-p-1},$$

$$BIC = k\ln(n) - 2\ln L, HQIC = 2p\ln[\ln(n)] - 2\ln L,$$

$$k - s = \sup_y [F_n(y) - F(y)],$$

where k is the number of models parameter, n is the sample size and $\ln L$ is the maximized value of the log- likelihood function under the fitted models. The better model is corresponding to the lower values of, *AIC*, *CAIC*, *BIC*, and *k-s* statistics. The results for the previous measures to the mentioned models are listed in Table 1.

Table1: Measurements for all models based on the first data set

Models	Statistics					
	-2logl	***AIC***	***CAIC***	***BIC***	***HQIC***	***k-s***
ELG	**256.718**	**264.718**	**265.212**	**274.441**	**268.627**	**0.073**
EL	**326.846**	**330.846**	**330.991**	**335.708**	**332.801**	**0.965**
EP	**389.005**	**395.005**	**395.297**	**402.297**	**397.936**	**0.309**
KL	**599.38**	**607.380**	**607.873**	**617.103**	**611.288**	**0.528**
WL	**260.277**	**268.277**	**268.771**	**278.00**	**272.186**	**0.812**
L	**322.473**	**326.473**	**326.618**	**331.335**	**328.427**	**0.185**

The values in Table1 indicate that the most fitted distribution to the data is *ELG* distribution compared to other distributions considered here (*EL, EP, KL, WL, L).*

Plots of the estimated cumulative and estimated densities of the fitted models for the first set of data are described below,

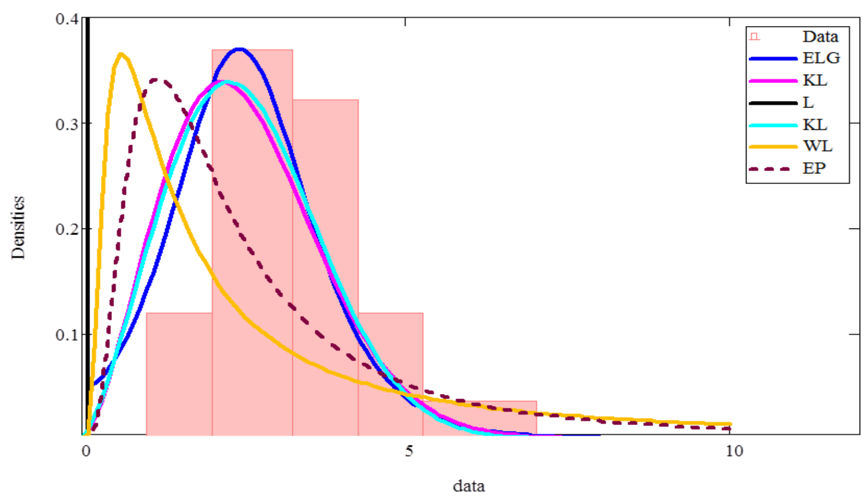


Figure 5. Estimated densitiesof models for the first data set

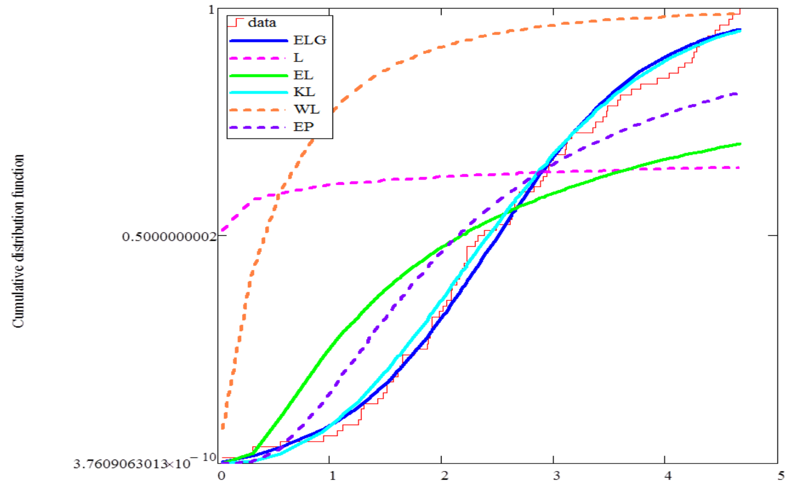


Figure 6. Estimated cumulative densities for the first data set

Again from Figures 5 and 6 we can notice that the most fitted distribution compared with the other models, to the first set of data is *ELG*.

The second data set contains 100 observations on breaking stress of carbon fibers (in Gba) studied by Nichols and Padgett (2006). The second set of data are as follows:

3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11,4.42,2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43,2.95, 2.97, 3.39, 2.96,2.53,2.67, 2.93, 3.22, 3.39, 2.81, 4.2,3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55,2.59,2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36,0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19,1.57, 0.81, 5.56, 1.73,1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17,1.69,1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7,2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12,1.89, 2.88, 2.82, 2.05,3.65.

The same models (*ELG*, *EL*, *EP*, *KL*, *WL*, *L*) are fitted for the second set of data, and the values of the measurements are listed in Table 2. It is clearfrom Table 2, that the *ELG* is the most fitted distribution compared with the other distributions for fitting the second set of data.

Table2: Measurements for all models based on the second data set

Models	Statistics					
	-2logl	*AIC*	*CAIC*	*BIC*	*HQIC*	*k-s*
ELG	**283.042**	**291.042**	**291.454**	**301.463**	**295.259**	**0.067**
EL	**394.113**	**389.113**	**398.234**	**403.323**	**400.222**	**0.965**
EP	**339.575**	**405.198**	**405.443**	**408.361**	**408.361**	**0.334**
KL	**669.575**	**677.575**	**677.987**	**681.792**	**681.792**	**0.519**
WL	**287.229**	**295.229**	**295.642**	**299.477**	**299.477**	**0.811**
L	**318.588**	**322.588**	**322.709**	**327.798**	**324.696**	**0.055**

Furthermore, the graphical comparison corresponding to these fittedmodelsto conform our claim is illustrated in Figures 7 and 8.

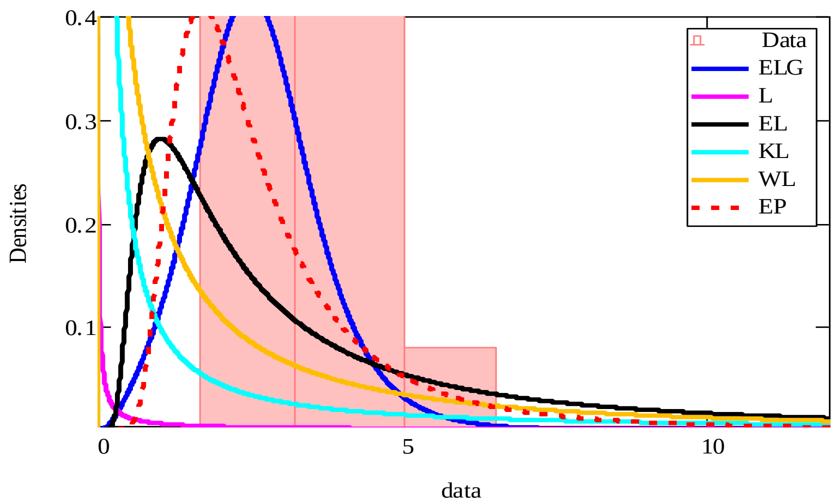


Figure 7. Estimated densities of models for the second data

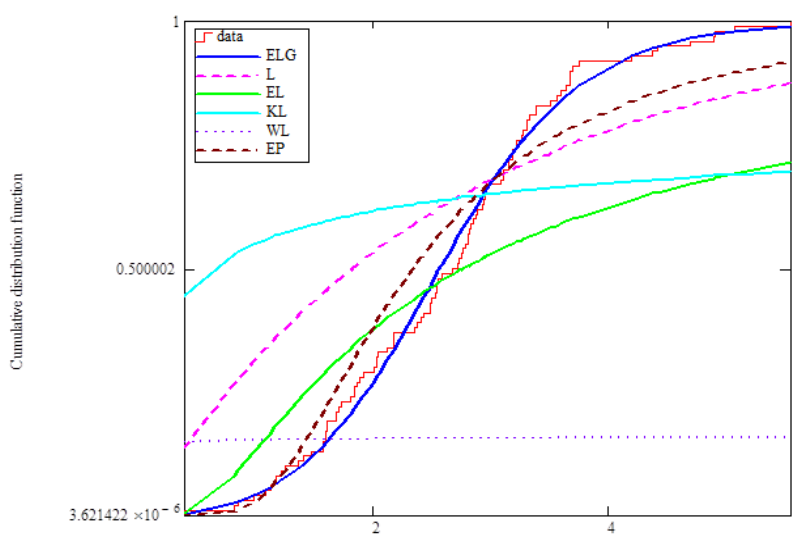


Figure 8. Estimated cumulative densities for the second data set

Again, Figures 7 and 8show that *ELG* model is the best fitted model for the second data set.

6. Conclusion

In the present study, we propose a new distribution called exponentiated Lomax geometric distribution. The subject distribution is derived by compounding exponentiated Lomax and geometric distributions. The density function of *ELG* can be expressed as a mixture of *EL* density functions. The *ELG* distribution includes the Lomax geometric and exponentiated Pareto geometric as new distributions. Explicit expressions for moments, probability weighted moments, Bonferroni and Lorenz curves, order statistics and R´enyi's entropy are derived. The estimation of parameters along with the information matrix is derived. Applications of the exponentiated Lomax geometric distribution to realdata show that the new distribution can be used quite effectively to provide better fits as compared to Lomax, exponentiated Lomax, Kumaraswamy Lomax, Weibull Lomax and exponentiated Pareto distributions.

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Appendix: Entries of Observed Information Matrix for *ELG.*

Distribution

$$\frac{\partial^2 lnl}{\partial \alpha^2} = -\frac{n}{\alpha^2} - 2p(1-p)\sum_{i=1}^{n} \frac{S_i^{\alpha}(lnS_i)^2}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial^2 lnl}{\partial \theta^2} = -\frac{n}{\theta^2} - (\alpha-1)\sum_{i=1}^{n} \frac{(\ln(1+\lambda x_i))^2(1-S_i)}{S_i^2} -$$

$$2p\alpha\sum_{i=1}^{n} \frac{(\ln(1+\lambda x_i))^2(1-S_i)\left[(1-p)\left((\alpha-1)S_i^{\alpha-2} - S_i^{\alpha-1}\right) - pS_i^{2\alpha-2}\right]}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial^2 lnl}{\partial \lambda^2} = -\frac{n}{\lambda^2} + (\theta+1)\sum_{i=1}^{n} \frac{x_i^2}{(1+\lambda x_i)^2} + (\alpha-1)\sum_{i=1}^{n} \frac{1}{S_i}\frac{\partial^2 S_i}{\partial \lambda^2} - \frac{1}{S_i^2}\left(\frac{\partial S_i}{\partial \lambda}\right)^2 +$$

$$\alpha p\sum_{i=1}^{n} \frac{\left[1-p+pS_i^{\alpha}\right]\left[S_i^{\alpha-1}\frac{\partial^2 S_i}{\partial \lambda^2} + (\alpha-1)S_i^{\alpha-2}\left(\frac{\partial S_i}{\partial \lambda}\right)^2\right] - S_i^{2\alpha-2}\left(\frac{\partial S_i}{\partial \lambda}\right)^2}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial^2 lnl}{\partial p^2} = \frac{-n}{(1-p)^2} + \frac{2\left[1-S_i^{\alpha}\right]^2}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial^2 lnl}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n} \frac{1}{S_i}\frac{\partial S_i}{\partial \lambda} - 2p\sum_{i=1}^{n} \frac{\frac{\partial S_i}{\partial \lambda}\left\{\left(1-p\left[1-S_i^{\alpha}\right]\right)S_i^{\alpha-1}(1+\alpha\ln S_i) - \alpha pS_i^{2\alpha-1}\ln S_i\right\}}{\left\{1-p\left[1-S_i^{\alpha}\right]\right\}^2},$$

$$\frac{\partial^2 lnl}{\partial \alpha \partial \theta} = \sum_{i=1}^{n} \frac{(1-S_i)\ln(1+\lambda x_i)}{S_i} - 2\sum_{i=1}^{n} \frac{\left\{pS_i^{\alpha-1}(1-S_i)\ln(1+\lambda x_i)\right\}\left\{(1-p)(1+\alpha\ln S_i) + pS_i^{\alpha}\right\}}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial^2 lnl}{\partial \alpha \partial p} = -2\sum_{i=1}^{n} \frac{(S_i)^{\alpha}\ln(S_i)}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial^2 lnl}{\partial\theta\partial\lambda}=-\sum_{i=1}^{n}\frac{x_i}{\left(1+\lambda x_i\right)}-\theta\left(\alpha-1\right)\sum_{i=1}^{n}\frac{x_i\left(1+\lambda x_i\right)^{-1}\frac{\partial S_i}{\partial\theta}}{\left[S_i\right]^2}$$

$$-2p\alpha\sum_{i=1}^{n}x_i\frac{\left[1-p+pS_i^{\alpha}\right].\left[S_i^{\alpha-1}.\left(1+\lambda x_i\right)^{-(\theta+1)}.\left(1-\theta\ln\left(1+\lambda x_i\right)+\frac{\theta\left(\alpha-1\right)}{S_i}\frac{\partial S_i}{\partial\theta}\right)\right]-\alpha\theta p.S_i^{2(\alpha-1)}\frac{\partial S_i}{\partial\theta}.\left(1+\lambda x_i\right)^{-(\theta+1)}}{\left[1-p+pS_i^{\alpha}\right]^2}$$

$$\frac{\partial^2 lnl}{\partial p\partial\lambda}=-2\sum_{i=1}^{n}\frac{\alpha\theta x_i\left(S_i\right)^{\alpha-1}\left(1+\lambda x_i\right)^{-\theta-1}}{\left[1-p+pS_i^{\alpha}\right]^2},\quad \frac{\partial^2 lnl}{\partial p\partial\theta}=-2\sum_{i=1}^{n}\frac{\alpha\left(S_i\right)^{\alpha-1}(1-S_i)\ln(1+\lambda x_i)}{\left[1-p+pS_i^{\alpha}\right]^2},$$

$$\frac{\partial S_i}{\partial\theta}=\left(1+\lambda x_i\right)^{-\theta}\ln(\left(1+\lambda x_i\right)=(1-S_i)\ln(\left(1+\lambda x_i\right),$$

$$\frac{\partial S_i}{\partial\lambda}=x_i\theta\left(1+\lambda x_i\right)^{-\theta-1}.$$