Bayesian Estimation of $R = P[Y < X]$ for Burr Type XII Distribution Using Median Ranked Set Sampling

By

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Abstract

This article deals with the estimation of $R = P[Y < X]$ when X and Y are two independent Burr type XII distributions having the same scale parameter. Bayesian technique is used to estimate R based on median ranked set sampling, ranked set sampling and simple random sampling. Bayesian estimators of R are obtained under squared error and linear exponential loss functions. The Bayesian estimator of R can not be obtained in explicit form, and therefore it has been implemented using a numerical technique. Analysis of a simulated data have been presented to compare the performances of the Bayes estimates based on median ranked set sampling with the corresponding Bayes estimates based on simple random sampling and ranked set sampling.

Keywords: *Burr XII distribution, ranked set sampling, median ranked set sampling, Bayesian estimation.*

Introduction

The concept of ranked set sampling (RSS) was first proposed by McIntyre (1952) as a useful sampling technique for improving the accuracy of estimating mean pasture yield. He found that the estimator based on RSS is more efficient than the corresponding estimator based on simple random sampling (SRS). Takahasi and Wakimoto (1968) provided the necessary mathematical theory of RSS. Dell and Clutter (1972) showed that, regardless of ranking errors, the RSS estimator of a population mean is unbiased and at least as precise as the SRS estimator with the same number of quantifications. RSS procedure involves randomly drawing n sets of n units each from the population. It is assumed that the units in each set can be ranked visually. From the first set of n units, the smallest ranked unit is measured; the second smallest ranked unit is measured from the second set of n units. The procedure is continued until the largest unit from the n^{th} sample is selected for measurement. Thus, a total of n-measured units, which represent one cycle, are obtained. This process may be repeated r times until a sample number of nr units are obtained. These nr units produce the RSS data.

Muttlak (1997) suggested the median ranked set sampling (MRSS) as a modification of ranked set sampling, which reduces the errors in ranking and increase the efficiency over RSS with perfect ranking for some probability distribution functions. The MRSS procedure can be summarized as follows. Select n random samples of size n units from the target population and select the median element of each ordered set if n is odd. However if n is even, it selects the $(n/2)^{th}$ smallest observation from each of the first $(n/2)$ ordered sets and the second smallest $(n + 2/2)^{th}$ observation from the second $(n/2)$ sets. This selection procedure yields a MRSS of size n, and may be repeated r times to give a MRSS of size nr.

The two-parameter Burr type XII distribution was first introduced in the literature by Burr (1942), and has been gained special attention in the last two decades due to its importance in practical situations. This distribution is widely used in some areas such as business, engineering, quality control, and reliability. The cumulative distribution function (cdf) of the Burr XII distribution denoted by Burr (c, b) is given as

$$F(x; c, b) = 1 - (1 + x^c)^{-b}, x > 0, c > 0, b > 0,$$

where c and b are shape parameters. The corresponding probability density function (pdf) is given by

$$f(x; c, b) = \mathrm{b}cx^{c-1}(1 + x^c)^{-(b+1)}, x > 0, c > 0, b > 0.$$

In the context of reliability, the stress strength model describes the life of a component which has a random strength X and is subjected to random stress Y. The component fails at the instant that the stress applied to it exceeds strength and the component will function satisfactorily whenever $(Y < X)$. Thus, $P(Y < X)$ stress-strength model is a measure of the reliability of the component. This model first considered by Birnbaum (1956) and the formal term stress strength model appeared in the title of Church and Harris (1970), Kotz et al. (2003) collected the theoretical and practical results of the theory and applications of the stress strength relationships during the last decades. Estimation problem of $R = P(Y < X)$ based on SRS technique have been studied by several authors (see, for example, Awad and Charraf (1986), McCool (1991), Surles and Padgett (1998), Raqab and Kundu (2005), Kundu and Gupta (2006), Raqab et al. (2008), Kundu and Raqab (2009), Rezaei et al. (2010) and Ali et al. (2012), Al-Mutairi et al. (2015)).

In the literature, there were few works for the estimating problems of $P(Y < X)$ based on RSS. Sengupta and Mukhuti (2008) considered the estimation of R using RSS data, they proved that the RSS provides an unbiased estimator with smaller variance as compared with SRS of same sizes, even when the rankings are imperfect. Muttlak et al. (2010) considered the estimation of R when X and Y are independently distributed exponential random variables with different scale parameters using RSS data. Hussian (2014) studied the estimation of R when X and Y are two generalized inverted exponential random variables with different parameters using SRS and RSS techniques. Hassan et al. (2014) proposed the estimation problem of $P(Y < X)$ for Burr type XII distribution based on RSS. They showed that the estimators based on RSS are more efficient than the estimators based on SRS data. Hassan et al. (2015) introduced the estimation of $P(Y < X)$ for Burr XII distribution under several modifications of RSS using maximum likelihood method; they proposed MRSS, extreme ranked set sampling, and percentile ranked set sampling. They concluded that the maximum likelihood estimates (MLEs) based on RSS and its modifications are more efficient than the estimators based on SRS data.

To our knowledge, in the literature, there were no studies that have been performed about stress-strength model based on RSS and MRSS using Bayesian approach. Therefore, this article aims to estimate R based on SRS, RSS and MRSS techniques. Bayesian estimators of R using non-informative priors under symmetric and asymmetric loss functions will be considered. Numerical study will be performed to compare the different estimates.

2. Bayesian Estimator for *R* Based on SRS

In this section, Bayesian estimator of R using non-informative priors will be obtained under the assumptions that the shape parameter c is known and the shape parameters b and a have independent non-informative priors with the following probability density functions:

$$\pi(b) \propto \frac{1}{\mathrm{b}}; \qquad b > 0, \tag{1}$$

and

$$\pi(a) \propto \frac{1}{\mathrm{a}}; \qquad a > 0. \tag{2}$$

Let $\underline{X} = \{X_1, \dots, X_p\}$ be a simple random sample from Burr (c, b) and $\underline{Y} = \{Y_1, \dots, Y_q\}$ be a simple random sample from Burr (c, a), therefore the likelihood function for the observed sample is given by

$$L\left(\underline{x}, \underline{y} \middle| b, a\right) \propto \prod_{i=1}^{p} b\,(1 + x_i^c)^{-b} \prod_{j=1}^{q} a\left(1 + y_j^c\right)^{-a}. \tag{3}$$

The posterior density functions of b and a can be obtained by combining the priors density functions in (1) and (2) with the likelihood function in (3) as follows

$$\pi\left(b, a|\underline{x}, \underline{y}\right) \propto b^{p-1} a^{q-1} e^{-b\lambda_1} e^{-a\lambda_2}, \tag{4}$$

where $\lambda_1 = \sum_{i=1}^{p} \ln(1 + x_i^c)$ and $\lambda_2 = \sum_{j=1}^{q} \ln(1 + y_j^c)$.

Bayesian estimator of R can be obtained using the following traditional transformation technique as follows: Let

$$r_1 = \frac{a}{a+b} \ and \ A_1 = a + b, \text{ then } a = r_1 A_1 \ and \ b = A_1(1 - r_1), \quad 0 < r_1 < 1, \quad A_1 > 0, \tag{5}$$

then, the posterior density function (4) will be written as follows

$$\pi\left(r_1, A_1|\underline{x}, \underline{y}\right) \propto {A_1}^{p+q-1}(1 - r_1)^{p-1}\, {r_1}^{q-1} e^{-A_1[(1-r_1)\lambda_1 + r_1\lambda_2]},$$

integrate out of A_1

$$\pi\left(r_1|\underline{x}, \underline{y}\right) = \Psi \frac{(1-r_1)^{p-1}\, {r_1}^{q-1}}{[(1-r_1)\lambda_1 + r_1\lambda_2]^{p+q}}, \tag{6}$$

here Ψ is a constant and it can be obtained by integrate out of r_1 as follows

$$\Psi^{-1} = \int_0^1 \frac{(1 - r_1)^{p-1}\, {r_1}^{-p-1}}{[\lambda_2]^{p+q}[\frac{(1 - r_1)\lambda_1}{r_1\lambda_2} + 1]^{p+q}} dr_1.$$

$$\text{Let } \eta_1 = \frac{(1-r_1)}{r_1}, r_1 = \frac{1}{1+\eta_1} \text{ and } 1 - r_1 = \frac{\eta_1}{1+\eta_1}, \tag{7}$$

therefore,

$$\Psi^{-1} = \int_0^\infty \frac{(\eta_1)^{p-1}}{[\lambda_2]^{p+q}\left[\eta_1\left(\frac{\lambda_1}{\lambda_2}\right) + 1\right]^{p+q}} d\eta_1, \tag{8}$$

Let $\eta_2 = \eta_1\left(\frac{\lambda_1}{\lambda_2}\right)$, then (8) will be as follows

$$\Psi^{-1} = \frac{1}{(\lambda_1)^p(\lambda_2)^q} \int_0^\infty \frac{(\eta_2)^{p-1}}{[\eta_2 + 1]^{p+q}} d\eta_2 = \frac{\mathrm{B}(p, q)}{(\lambda_1)^p(\lambda_2)^q},$$

where $\mathrm{B}(p,q)$ is a beta function. So, the posterior probability density function of R denoted by $\pi\left(r_1|\underline{x},\underline{y}\right)$, for $0<r<1$, takes the following form

$$\pi\left(r_1|\underline{x},\underline{y}\right)=\frac{(\lambda_1)^p(\lambda_2)^q}{\mathrm{B}(p,q)}\frac{(1-r_1)^{p-1}\,r_1{}^{q-1}}{[(1-r_1)\lambda_1+r_1\lambda_2]^{p+q}}. \tag{9}$$

Using (9), Bayesian estimator of R under squared error (SE) loss function, denoted by $\hat{R}_{SE}$ is given by

$$\hat{R}_{SE}=\frac{\lambda_1{}^p\,\lambda_2{}^q}{\mathrm{B}(p,q)}\int_0^1\frac{(1-r_1)^{p-1}\,r_1{}^q}{[(1-r_1)\lambda_1+r_1\lambda_2]^{p+q}}\,dr_1. \tag{10}$$

Under linear exponential (LINEX) loss function, Bayesian estimator of R, denoted by $\hat{R}_{LIN}$, can be obtained as follows

$$\hat{R}_{LIN}=\frac{-1}{\vartheta}ln\left[E\left(e^{-\vartheta r_1}\right)\right],$$

$$\hat{R}_{LIN}=\frac{-1}{\vartheta}ln\left\{\frac{\lambda_1{}^p\,\lambda_2{}^q}{\mathrm{B}(p,q)}\int_0^1\frac{e^{-\vartheta r_1}(1-r_1)^{p-1}\,r_1{}^{q-1}}{[(1-r_1)\lambda_1+r_1\lambda_2]^{p+q}}\,dr_1\right\}. \tag{11}$$

Clearly, the computations of $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ in (10) and (11) are complicated. So numerical technique will be used to evaluate $\hat{R}_{SE}$ and $\hat{R}_{LIN}$.

3. Bayesian Estimator for R Based on RSS

In this section, Bayesian estimator of R under SE and LINEX loss functions based on RSS data using non-informative Jeffrey priors will be obtained. Let $\{\dot{X}_{i(i)s},\ i=1,2,\dots n;s=1,2,\dots,r\}$ be a ranked set sample with sample size $p=nr$, where n is the set size and r is the number of cycles from Burr (c,b). Then the pdf of $\dot{X}_{i(i)s}$ is given by:

$$f_i\left(\dot{x}_{i(i)s}\right)=\frac{n!}{(i-1)!(n-i)!}bc\dot{x}_{i(i)s}^{c-1}\left(1+\dot{x}_{i(i)s}^{c}\right)^{-[b(n-i+1)+1]}\left(1-\left(1+\dot{x}_{i(i)s}^{c}\right)^{-b}\right)^{i-1},\quad \dot{x}_{i(i)s}>0.$$

Let $\underline{\dot{X}}=\{\dot{X}_{1(1)s},\dot{X}_{2(2)s},\dots,\ \dot{X}_{n(n)s},s=1,2,..r\}$ be a ranked set sample from Burr (c,b) and the prior density of b as in (1). Then the likelihood function is as follow

$$L\left(\underline{\dot{x}}|b\right)\propto b^p e^{-b\sum_{s=1}^r\sum_{i=1}^n(n-i+1)ln\left(1+\dot{x}_{i(i)s}^c\right)}\prod_{s=1}^r\prod_{i=1}^n\left(1-\left(1+\dot{x}_{i(i)s}^c\right)^{-b}\right)^{i-1},$$

By using binomial expansion, then $L\left(\underline{\dot{x}}|b\right)$ can be written as

$$L(\underline{\dot{x}}|b)\propto b^p e^{-b\sum_{s=1}^r\sum_{i=1}^n(n-i+1)ln\left(1+\dot{x}_{i(i)s}^c\right)}\prod_{s=1}^r\prod_{i=1}^n\left[\sum_{L=0}^{i-1}(-1)^L\binom{i-1}{L}\left(1+\dot{x}_{i(i)s}^c\right)^{-bL}\right],$$

Then,

$$L\left(\underline{\dot{x}}|b\right)\propto$$

$$\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\left[\prod_{s=1}^r\prod_{i=1}^n\left(\Phi_{L_i^s}(i)\right)\right]b^p e^{-b\sum_{s=1}^r\sum_{i=1}^n\left(T_{L_i^s}(i)\right)}, \tag{12}$$

where $T_{L_i^s}(i)=(n+L_i^s-i+1)ln\left(1+\dot{x}_{i(i)s}^c\right)$ and $\Phi_{L_i^s}(i)=(-1)^{L_i^s}\binom{i-1}{L_i^s}$.

Combine the prior density (1) and the likelihood function (12), to obtain the posterior density function of b, denoted by $\pi_{RSS}\left(b|\underline{\dot{x}}\right)$, as follows

$$\pi_{RSS}\left(b|\underline{\dot{x}}\right)\propto$$

$$\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\left[\prod_{s=1}^r\prod_{i=1}^n\left(\Phi_{L_i^s}(i)\right)\right]b^{p-1}e^{-b\sum_{s=1}^r\sum_{i=1}^n\left(T_{L_i^s}(i)\right)}. \tag{13}$$

By similar way, let $\{\dot{Y}_{j(j)s}, j = 1, \dots, m; s = 1, \dots, r\}$ be the ranked set sample of size $q = mr$ from Burr (c, a). Then the pdf of $\dot{Y}_{j(j)s}$ is given by:

$$f_j(\dot{y}_{j(j)s}) = \frac{m!}{(j-1)!\,(m-j)!} ac\dot{y}_{j(j)s}^{c-1}\left(1+\dot{y}_{j(j)s}^{c}\right)^{-[a(m-j+1)+1]}\left(1-\left(1+\dot{y}_{j(j)s}^{c}\right)^{-a}\right)^{j-1}, \dot{y}_{j(j)s} > 0\,,$$

Let $\underline{\dot{Y}} = \{\dot{Y}_{1(1)s}, \dot{Y}_{2(2)s}, \dots, \dot{Y}_{m(m)s}; s = 1,2,..r\}$ be a ranked set sample from Burr (c, a), where m is the set size and r is the number of cycles. The likelihood function will be obtained as follows

$$L\left(\underline{\dot{y}}|a\right) \propto \sum_{\tau_1^1=0}^{0} \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^{0} \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \cdots \sum_{\tau_1^r=0}^{0} \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right] a^{q} e^{-a\left(\sum_{s=1}^{r}\sum_{j=1}^{m} T_{\tau_j^s}(j)\right)}, \quad (14)$$

where $T_{\tau_j^s}(j) = \left(m+\tau_j^s-j+1\right)ln\left(1+\dot{y}_{j(j)s}^{c}\right)$ and $\Phi_{\tau_j^s}(j) = (-1)^{\tau_j^s}\binom{j-1}{\tau_j^s}$.

Combine the prior density (2) and the likelihood function (14) to obtain the posterior density function of a, denoted by $\pi_{RSS}\left(a|\underline{\dot{y}}\right)$, as follows

$$\pi_{RSS}\left(a\,|\underline{\dot{y}}\right) \propto \sum_{\tau_1^1=0}^{0} \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^{0} \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \cdots \sum_{\tau_1^r=0}^{0} \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right] a^{q-1} e^{-a\left(\sum_{s=1}^{r}\sum_{j=1}^{m} T_{\tau_j^s}(j)\right)}. \quad (15)$$

Assuming that b and a are independent, from (13) and (15), the joint bivariate posterior density function of b and a will be

$$\pi_{RSS}\left(b, a|\underline{\dot{x}}, \underline{\dot{y}}\right) \propto \sum_{L_1^1=0}^{0} \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^{0} \cdots \sum_{L_n^2=0}^{n-1} \cdots \cdots \sum_{L_1^r=0}^{0} \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^{0} \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^{0} \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \cdots \sum_{\tau_1^r=0}^{0} \cdots$$

$$\sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right] b^{p-1} a^{q-1} e^{-b\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^s}(i)\right)} e^{-a\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^s}(j)\right)}.$$

Bayesian estimator of R can be obtained using the standard transformation technique defined in (5) as follows

$$\pi_{RSS}\left(A_1, r_1|\underline{\dot{x}}, \underline{\dot{y}}\right) \propto \sum_{L_1^1=0}^{0} \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^{0} \cdots \sum_{L_n^2=0}^{n-1} \cdots \cdots \sum_{L_1^r=0}^{0} \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^{0} \cdots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^{0} \cdots$$

$$\times \sum_{\tau_m^2=0}^{m-1} \cdots \cdots \sum_{\tau_1^r=0}^{0} \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]$$

$$\times\ (1-r_1)^{p-1} {r_1}^{q-1} A_1^{p+q-1} e^{-A_1\left((1-r_1)\left[\sum_{s=1}^{r}\sum_{i=1}^{n} T_{L_i^s}(i)\right]+r_1\left[\sum_{s=1}^{r}\sum_{j=1}^{m} T_{\tau_j^s}(j)\right]\right)}.$$

Integrate out of A_1, the posterior density function of R using non-informative prior for $0 < r_1 < 1$ is given by

$$\pi_{RSS}\left(r_1|\underline{\dot{x}}, \underline{\dot{y}}\right) = \dot{\Psi} \sum_{L_1^1=0}^{0} \cdots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^{0} \cdots \sum_{L_n^2=0}^{n-1} \cdots \cdots \sum_{L_1^r=0}^{0} \cdots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^{0} \cdots \sum_{\tau_m^1=0}^{m-1}$$

$$\times \sum_{\tau_1^2=0}^{0} \cdots \sum_{\tau_m^2=0}^{m-1} \cdots \cdots \sum_{\tau_1^r=0}^{0} \cdots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]$$

$$\times (1-r_1)^{p-1} {r_1}^{q-1} \left[(1-r_1)\left[\sum_{s=1}^{r}\sum_{i=1}^{n} T_{L_i^s}(i)\right] + r_1\left[\sum_{s=1}^{r}\sum_{j=1}^{m} T_{\tau_j^s}(j)\right]\right]^{-(p+q)},$$

where $\dot{\Psi}$ is a constant and it can be obtained as follows

$$\dot{\Psi}^{-1}=\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}\cdots\sum_{\tau_1^r=0}^{0}\cdots$$

$$\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]\int_0^1\frac{(1-r_1)^{p-1}(r_1)^{-p-1}}{\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]^{p+q}\left[\frac{(1-r_1)\left[\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right]}{r_1\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]}+1\right]^{p+q}}dr_1. \tag{16}$$

Using transformation technique defined in (7), then (16) takes the following form

$$\dot{\Psi}^{-1}=\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}\cdots\cdots\sum_{\tau_1^r=0}^{0}\cdots$$

$$\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]\int_0^\infty\frac{(\eta_1)^{p-1}}{\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]^{p+q}\left[\eta_1\frac{\left[\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right]}{\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]}+1\right]^{p+q}}d\eta_1. \tag{17}$$

Let, $\eta_3=\eta_1\left(\frac{\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)}{\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)}\right)$, then (17) takes the following form

$$\dot{\Psi}^{-1}=\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}\cdots$$

$$\times\cdots\sum_{\tau_1^r=0}^{0}\cdots\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]\frac{\mathrm{B}(p,q)}{\left(\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right)^{q}\left(\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right)^{p}},$$

Therefore, the posterior density function of R based on RSS, denoted by $\pi_{RSS}\left(r_1|\underline{\dot{x}},\underline{\dot{y}}\right)$, for $0<r<1$ is given by

$$\pi_{RSS}\left(r_1|\underline{\dot{x}},\underline{\dot{y}}\right)=\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}\cdots$$

$$\cdots\sum_{\tau_1^r=0}^{0}\cdots\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right](1-r_1)^{p-1}{r_1}^{q-1}$$

$$\times\left[(1-r_1)\left[\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right]+r_1\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]\right]^{-(p+q)}\left[\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\quad\sum_{L_1^2=0}^{0}\cdots\right.$$

$$\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}\cdots\cdots\sum_{\tau_1^r=0}^{0}\cdots\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]$$

$$\left.\times\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]\mathrm{B}(p,q)\left[\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right]^{-p}\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]^{-q}\right]^{-1}. \tag{18}$$

Using (18), Bayesian estimator of R, under SE loss function , denoted by $\dot{R}_{SE}$, is obtained as follows

$$\dot{R}_{SE}=\int_0^1 r_1\pi_{RSS}\left(r_1|\underline{\dot{x}},\underline{\dot{y}}\right)dr_1,$$

$$\dot{R}_{SE}=\int_0^1\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}$$

$$\cdots\cdots\sum_{\tau_1^r=0}^{0}\cdots\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right](1-r_1)^{p-1}{r_1}^{q}$$

$$\times\left[(1-r_1)\left[\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right]+r_1\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]\right]^{-(p+q)}\left[\sum_{L_1^1=0}^{0}\cdots\sum_{L_n^1=0}^{n-1}\sum_{L_1^2=0}^{0}\cdots\sum_{L_n^2=0}^{n-1}\cdots\right.$$

$$\cdots\sum_{L_1^r=0}^{0}\cdots\sum_{L_n^r=0}^{n-1}\sum_{\tau_1^1=0}^{0}\cdots\sum_{\tau_m^1=0}^{m-1}\sum_{\tau_1^2=0}^{0}\cdots\sum_{\tau_m^2=0}^{m-1}\cdots\cdots\sum_{\tau_1^r=0}^{0}\cdots\sum_{\tau_m^r=0}^{m-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]$$

$$\left.\times\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]\mathrm{B}(p,q)\left[\sum_{s=1}^{r}\sum_{i=1}^{n}T_{L_i^s}(i)\right]^{-p}\left[\sum_{s=1}^{r}\sum_{j=1}^{m}T_{\tau_j^s}(j)\right]^{-q}\right]^{-1}dr_1. \tag{19}$$

Bayesian estimator of R using non-informative prior under LINEX loss function denoted by $\dot{R}_{LIN}$, is given as follows

$$\dot{R}_{LIN} = \frac{-1}{\vartheta} ln\left\{\int_0^1 e^{-\vartheta r_1}\pi_{RSS}\left(r_1|\underline{\dot{x}},\underline{\dot{y}}\right)dr_1\right\},$$

$$\dot{R}_{LIN} = \frac{-1}{\vartheta} ln\Bigg\{\int_0^1 \sum_{L_1^1=0}^{0} \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^{0} \dots \sum_{L_n^2=0}^{n-1} \dots \dots \sum_{L_1^r=0}^{0} \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^{0} \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^{0} \dots \sum_{\tau_m^2=0}^{m-1} \dots$$

$$\dots \sum_{\tau_1^r=0}^{0} \dots \sum_{\tau_m^r=0}^{m-1} \left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right] e^{-\vartheta r_1 + ln\left((1-r_1)^{p-1}{r_1}^{q-1}\right)}$$

$$\times\left[(1-r_1)\left[\sum_{s=1}^{r}\sum_{i=1}^{n} T_{L_i^s}(i)\right] + r_1\left[\sum_{s=1}^{r}\sum_{j=1}^{m} T_{\tau_j^s}(j)\right]\right]^{-(p+q)} \Bigg[\sum_{L_1^1=0}^{0} \dots \sum_{L_n^1=0}^{n-1} \sum_{L_1^2=0}^{0} \dots \sum_{L_n^2=0}^{n-1} \dots$$

$$\dots \sum_{L_1^r=0}^{0} \dots \sum_{L_n^r=0}^{n-1} \sum_{\tau_1^1=0}^{0} \dots \sum_{\tau_m^1=0}^{m-1} \sum_{\tau_1^2=0}^{0} \dots \sum_{\tau_m^2=0}^{m-1} \dots \dots \sum_{\tau_1^r=0}^{0} \dots \sum_{\tau_m^r=0}^{m-1} \mathrm{B}(p,q)\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^s}(i)\right)\right]$$

$$\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^s}(j)\right)\right]\left[\sum_{s=1}^{r}\sum_{i=1}^{n} T_{L_i^s}(i)\right]^{-p}\left[\sum_{s=1}^{r}\sum_{j=1}^{m} T_{\tau_j^s}(j)\right]^{-q}\Bigg]^{-1} dr_1\Bigg\}. \qquad (20)$$

Bayesian estimators $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ of R under SE and LINEX loss functions using Jeffrey priors based on ranked set sample data for the simple case of a single cycle can be obtained by substituting $(r = 1)$ in (19) and (20) respectively as follows

$$\dot{R}_{SE} = \Big[\mathrm{B}(n,m)\sum_{L_1=0}^{0} \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^{0} \dots \sum_{\tau_m=0}^{m-1}\left(\prod_{i=1}^{n}\Phi_{L_i}(i)\right)\left(\prod_{j=1}^{m}\Phi_{\tau_j}(j)\right)\left(\sum_{i=1}^{n} T_{L_i}(i)\right)^{-n}$$

$$\times\left(\sum_{j=1}^{m} T_{\tau_j}(j)\right)^{-m}\Big]^{-1}\int_0^1 \sum_{L_1=0}^{0} \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^{0} \dots \sum_{\tau_m=0}^{m-1}\left(\prod_{i=1}^{n}\Phi_{L_i}(i)\right)\ \left(\prod_{j=1}^{m}\Phi_{\tau_j}(j)\right)$$

$$\times (1-r_1)^{n-1}{r_1}^{m}\ \left[(1-r_1)\left[\sum_{i=1}^{n} T_{L_i}(i)\right] + r_1\left[\sum_{j=1}^{m} T_{\tau_j}(j)\right]\right]^{-(n+m)} dr_1. \qquad (21)$$

and,

$$\dot{R}_{LIN} = \frac{-1}{\vartheta} ln\Bigg\{\int_0^1 \Big[\sum_{L_1=0}^{0} \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^{0} \dots \sum_{\tau_m=0}^{m-1}\left(\prod_{i=1}^{n}\Phi_{L_i}(i)\right)\left(\prod_{j=1}^{m}\Phi_{\tau_j}(j)\right)\mathrm{B}(n,m)\left(\sum_{i=1}^{n} T_{L_i}(i)\right)^{-n}$$

$$\left(\sum_{j=1}^{m} T_{\tau_j}(j)\right)^{-m}\Big]^{-1}\sum_{L_1=0}^{0} \dots \sum_{L_n=0}^{n-1} \sum_{\tau_1=0}^{0} \dots \sum_{\tau_m=0}^{m-1}\left(\prod_{i=1}^{n}\Phi_{L_i}(i)\right)\left(\prod_{j=1}^{m}\Phi_{\tau_j}(j)\right)e^{-\vartheta r + ln\left((1-r_1)^{n-1}{r_1}^{m-1}\right)}$$

$$\times\left[(1-r_1)\left[\sum_{i=1}^{n} T_{L_i}(i)\right] + r_1\left[\sum_{j=1}^{m} T_{\tau_j}(j)\right]\right]^{-(n+m)} dr_1\Bigg\}. \qquad (22)$$

Clearly, it is not easy to obtain a closed form solution of (21) and (22). Therefore, an iterative technique must be applied to solve these equations numerically to obtain an estimate of R.

4. Bayesian Estimator for R Based on MRSS

In this section, Bayesian estimator of $R = P(Y < X)$ based on MRSS technique will be obtained in a case of even and odd set sizes under SE as a symmetric loss function and LINEX as asymmetric loss function using non-informative priors.

4.1 Bayesian estimator for R with odd set size

In this subsection, Bayesian estimator for R based on MRSS with odd set size will be derived. Let $\{X^*_{i(g)s}; i = 1,2,..n, s = 1,2,...r\}$ is a MRSS from Burr (c, b) with sample size $p = nr$, where n is the set size, r is the number of cycles and $g = \frac{n+1}{2}$. Then the pdf of $X^*_{i(g)s}$ will be as follows:

$$f_g\left(x^*_{i(g)s}\right) = \frac{n!}{[(g-1)!]^2}\ bcx^{*c-1}_{i(g)s}\left[1 + x^{*c}_{i(g)s}\right]^{-(bg+1)}\left[1 - \left(1 + x^{*c}_{i(g)s}\right)^{-b}\right]^{g-1}, x^*_{i(g)s} > 0.$$

Let $\underline{X}^* = \{ X^*_{1(g)s},\ X^*_{2(g)s}, \dots ,\ X^*_{n(g)s}; s = 1,2,..r\}$ where $g = (n+1)/2$ be a median ranked set sample data from Burr (c, b) and the prior density of b as in (1). The likelihood function denoted by $L(\underline{x}^*|b)$ will be as follows

$$L(\underline{x}^*|b) \propto b^p e^{-bg\sum_{s=1}^{r}\sum_{i=1}^{n} ln\left(1+x^{*c}_{i(g)s}\right)} \prod_{s=1}^{r}\prod_{i=1}^{n}\left[1-\left(1+x^{*c}_{i(g)s}\right)^{-b}\right]^{g-1},$$

by using binomial expansion,

$$L(\underline{x}^*|b) \propto b^p e^{-bg\sum_{s=1}^{r}\sum_{i=1}^{n} ln\left(1+x^{*c}_{i(g)s}\right)} \prod_{s=1}^{r}\prod_{i=1}^{n}\left[\sum_{L^*=0}^{g-1}(-1)^{L^*}\binom{g-1}{L^*}\left(1+x^{*c}_{i(g)s}\right)^{-bL^*}\right],$$

$$L(\underline{x}^*|b) \propto b^p \sum_{L_1^{1*}=0}^{g-1}\sum_{L_2^{1*}=0}^{g-1}\dots\sum_{L_n^{1*}=0}^{g-1}\sum_{L_1^{2*}=0}^{g-1}\sum_{L_2^{2*}=0}^{g-1}\dots\sum_{L_n^{2*}=0}^{g-1}\dots\dots\sum_{L_1^{r*}=0}^{g-1}\sum_{L_2^{r*}=0}^{g-1}\dots\sum_{L_n^{r*}=0}^{g-1}\dots$$

$$\times\left[\prod_{s=1}^{r}\prod_{i=1}^{n}(-1)^{L_i^{s*}}\binom{g-1}{L_i^{s*}}\right] e^{-b\sum_{s=1}^{r}\sum_{i=1}^{n}\left(g+L_i^{s*}\right)ln\left(1+x^{*c}_{i(g)s}\right)}. \tag{23}$$

Combine the prior density (1) and the likelihood function in (23) to obtain the posterior density of b denoted by $\pi_{MRSS}(b|\underline{x}^*)$ as follows

$$\pi_{MRSS}(b|\underline{x}^*) \propto b^{p-1}\sum_{L_1^{1*}=0}^{g-1}\sum_{L_2^{1*}=0}^{g-1}\dots\sum_{L_n^{1*}=0}^{g-1}\sum_{L_1^{2*}=0}^{g-1}\sum_{L_2^{2*}=0}^{g-1}\dots\sum_{L_n^{2*}=0}^{g-1}\dots\sum_{L_1^{r*}=0}^{g-1}\sum_{L_2^{r*}=0}^{g-1}\dots$$

$$\sum_{L_n^{r*}=0}^{g-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\Phi_{L_i^{s*}}(i)\right] e^{-b\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)}. \tag{24}$$

where $\Phi_{L_i^{s*}}(i) = (-1)^{L_i^{s*}}\binom{g-1}{L_i^{s*}}$ and $T_{L_i^{s*}}(i) = (g+L_i^{s*})ln\left(1+x^{*c}_{i(g)s}\right)$.

Similarly, Let $\{Y^*_{j(h)s}; j = 1,2,\dots,m, s = 1,2,\dots r\}$ is a MRSS from Burr (c, a) with sample size $q = mr$, where m is the set size, r is the number of cycles. Then the pdf of $Y^*_{j(h)s}$will be as follows:

$$f_h\left(y^*_{j(h)s}\right) = \frac{m!}{[(h-1)!]^2} acy^{*c-1}_{j(h)s}\left[1+y^{*c}_{j(h)s}\right]^{-(ah+1)}\left[1-\left(1+y^{*c}_{j(h)s}\right)^{-a}\right]^{h-1}, y^*_{j(h)s} > 0.$$

Let $\underline{Y}^* = \{Y^*_{1(h)s},\ Y^*_{2(h)s}, \dots ,\ Y^*_{m(h)s}; s = 1,2,\dots r\}$ be a median ranked set sample from Burr (c, a) and the prior density of a as in (2). The likelihood function is as follows

$$L\left(\underline{y}^*|a\right) \propto a^q e^{-ah\sum_{s=1}^{r}\sum_{j=1}^{m} ln\left(1+y^{*c}_{j(h)s}\right)}\prod_{s=1}^{r}\prod_{j=1}^{m}\left[1-\left(1+y^{*c}_{j(h)s}\right)^{-a}\right]^{h-1},$$

by using binomial expansion,

$$L\left(\underline{y}^*|a\right) \propto a^q e^{-ah\sum_{s=1}^{r}\sum_{j=1}^{m} ln\left(1+y^{*c}_{j(h)s}\right)}\prod_{s=1}^{r}\prod_{j=1}^{m}\left[\sum_{\tau^*=0}^{h-1}(-1)^{\tau^*}\binom{h-1}{\tau^*}\left(1+y^{*c}_{j(h)s}\right)^{-a\tau^*}\right],$$

$$L\left(\underline{y}^*|a\right) \propto a^q \sum_{\tau_1^{1*}=0}^{h-1}\sum_{\tau_2^{1*}=0}^{h-1}\dots\sum_{\tau_m^{1*}=0}^{h-1}\sum_{\tau_1^{2*}=0}^{h-1}\sum_{\tau_2^{2*}=0}^{h-1}\dots\sum_{\tau_m^{2*}=0}^{h-1}\dots\dots\sum_{\tau_1^{r*}=0}^{h-1}\sum_{\tau_2^{r*}=0}^{h-1}\dots$$

$$\sum_{\tau_m^{r*}=0}^{h-1}\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\Phi_{\tau_j^{s*}}(j)\right] e^{-a\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)}, \tag{25}$$

where $T_{\tau_j^{s*}}(j) = \left(h+\tau_j^{s*}\right)ln\left(1+y^{*c}_{j(h)s}\right)$ and $\Phi_{\tau_j^{s*}}(j) = (-1)^{\tau_j^{s*}}\binom{h-1}{\tau_j^{s*}}$.

Combine the prior density (2) and the likelihood function (25) to obtain the posterior density of a denoted by $\pi_{MRSS}\left(a\,\middle|\underline{y}^*\right)$ as follows

$$\pi_{MRSS}\left(a\,\middle|\underline{y}^*\right) \propto a^{q-1}\sum_{\tau_1^{1*}=0}^{h-1}\sum_{\tau_2^{1*}=0}^{h-1}\dots\sum_{\tau_m^{1*}=0}^{h-1}\sum_{\tau_1^{2*}=0}^{h-1}\sum_{\tau_2^{2*}=0}^{h-1}\dots\sum_{\tau_m^{2*}=0}^{h-1}\dots\dots\sum_{\tau_1^{r*}=0}^{h-1}\sum_{\tau_2^{r*}=0}^{h-1}\dots$$

$$\sum_{\tau_m^{r*}=0}^{h-1}\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\Phi_{\tau_j^{s*}}(j)\right] e^{-a\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)}, \tag{26}$$

Assume that b and a are independent, from posterior densities in (24) and (26), the joint bivariate posterior density of b and a based on MRSS data will be

$$\pi_{MRSS}\left(b, a|\underline{x}^*, \underline{y}^*\right) \propto \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots$$

$$\sum_{L_n^{r*}=0}^{g-1} \cdots \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^{r} \prod_{i=1}^{n} \Phi_{L_i^{s*}}(i)\right]$$

$$\left[\prod_{s=1}^{r} \prod_{j=1}^{m} \left(\Phi_{\tau_j^{s*}}(j)\right)\right] b^{p-1} e^{-b \sum_{s=1}^{r} \sum_{i=1}^{n} \left(T_{L_i^{s*}}(i)\right)} a^{q-1} e^{-a \sum_{s=1}^{r} \sum_{j=1}^{m} \left(T_{\tau_j^{s*}}(j)\right)}. \qquad (27)$$

By applying the transformation technique (5) in (27) the joint posterior density function of b and a based on MRSS in case of odd set size will be as follows

$$\pi_{MRSS}\left(A_1, r_1|\underline{x}^*, \underline{y}^*\right) \propto \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1}$$

$$\sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \quad \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^{r} \prod_{i=1}^{n} \left(\Phi_{L_i^{s*}}(i)\right)\right]$$

$$\left[\prod_{s=1}^{r} \prod_{j=1}^{m} \left(\Phi_{\tau_j^{s*}}(j)\right)\right] r_1^{q-1} (1-r_1)^{p-1} A_1^{p+q-1} e^{-A_1\left[(1-r_1) \sum_{s=1}^{r} \sum_{i=1}^{n} \left(T_{L_i^{s*}}(i)\right) + r_1 \sum_{s=1}^{r} \sum_{j=1}^{m} \left(T_{\tau_j^{s*}}(j)\right)\right.}.$$

Integrate out of A_1, the posterior density function of r_1 based on MRSS in case of odd set size for $0 < r_1 < 1$ will be as follows

$$\pi_{MRSS}\left(r_1|\underline{x}^*, \underline{y}^*\right) = \Psi^* \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1}$$

$$\sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots \sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^{r} \prod_{i=1}^{n} \left(\Phi_{L_i^{s*}}(i)\right)\right]$$

$$\left[\prod_{s=1}^{r} \prod_{j=1}^{m} \left(\Phi_{\tau_j^{s*}}(j)\right)\right] r_1^{q-1} (1-r_1)^{p-1} \left[(1-r_1) \sum_{s=1}^{r} \sum_{i=1}^{n} \left(T_{L_i^{s*}}(i)\right) + r_1 \sum_{s=1}^{r} \sum_{j=1}^{m} \left(T_{\tau_j^{s*}}(j)\right)\right]^{-(p+q)}, \qquad (28)$$

where Ψ^* is a constant and is obtained as follows

$$\Psi^{*-1} = \int_0^1 \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots$$

$$\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^{r} \prod_{i=1}^{n} \left(\Phi_{L_i^{s*}}(i)\right)\right] \left[\prod_{s=1}^{r} \prod_{j=1}^{m} \left(\Phi_{\tau_j^{s*}}(j)\right)\right]$$

$$\times \frac{r_1^{-p-1} (1-r_1)^{p-1}}{\left(\sum_{s=1}^{r} \sum_{j=1}^{m} \left(T_{\tau_j^{s*}}(j)\right)\right)^{p+q} \left[\frac{(1-r_1) \sum_{s=1}^{r} \sum_{i=1}^{n} \left(T_{L_i^{s*}}(i)\right)}{r_1 \left(\sum_{s=1}^{r} \sum_{j=1}^{m} \left(T_{\tau_j^{s*}}(j)\right)\right)} + 1\right]^{(p+q)}} dr_1, \qquad (29)$$

Using transformation (7), defined in previous section, then (29) takes the following form

$$\Psi^{*-1} = \sum_{L_1^{1*}=0}^{g-1} \sum_{L_2^{1*}=0}^{g-1} \cdots \sum_{L_n^{1*}=0}^{g-1} \cdots \sum_{L_1^{2*}=0}^{g-1} \sum_{L_2^{2*}=0}^{g-1} \cdots \sum_{L_n^{2*}=0}^{g-1} \cdots \sum_{L_1^{r*}=0}^{g-1} \sum_{L_2^{r*}=0}^{g-1} \cdots \sum_{L_n^{r*}=0}^{g-1} \sum_{\tau_1^{1*}=0}^{h-1} \sum_{\tau_2^{1*}=0}^{h-1} \cdots$$

$$\sum_{\tau_m^{1*}=0}^{h-1} \sum_{\tau_1^{2*}=0}^{h-1} \sum_{\tau_2^{2*}=0}^{h-1} \cdots \sum_{\tau_m^{2*}=0}^{h-1} \cdots \sum_{\tau_1^{r*}=0}^{h-1} \sum_{\tau_2^{r*}=0}^{h-1} \cdots \sum_{\tau_m^{r*}=0}^{h-1} \left[\prod_{s=1}^{r} \prod_{i=1}^{n} \left(\Phi_{L_i^{s*}}(i)\right)\right] \left[\prod_{s=1}^{r} \prod_{j=1}^{m} \left(\Phi_{\tau_j^{s*}}(j)\right)\right]$$

$$\times \int_0^\infty \frac{(\eta_1)^{p-1}}{\left(\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)\right)^{p+q}\left[\eta_1\frac{\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)}{\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)}+1\right]^{(p+q)}} d\eta_1, \tag{30}$$

Let, $\eta_4 = \eta_1\left(\frac{\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)}{\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)}\right)$, then (30) will be written as follows

$$\Psi^{*-1} = \sum_{L_1^{1*}=0}^{g-1}\sum_{L_2^{1*}=0}^{g-1}\dots\sum_{L_n^{1*}=0}^{g-1}\dots\sum_{L_1^{2*}=0}^{g-1}\sum_{L_2^{2*}=0}^{g-1}\dots\sum_{L_n^{2*}=0}^{g-1}\dots\sum_{L_1^{r*}=0}^{g-1}\sum_{L_2^{r*}=0}^{g-1}\dots\sum_{L_n^{r*}=0}^{g-1}\quad\sum_{\tau_1^{1*}=0}^{h-1}\sum_{\tau_2^{1*}=0}^{h-1}\dots$$

$$\sum_{\tau_m^{1*}=0}^{h-1}\sum_{\tau_1^{2*}=0}^{h-1}\sum_{\tau_2^{2*}=0}^{h-1}\dots\sum_{\tau_m^{2*}=0}^{h-1}\dots\sum_{\tau_1^{r*}=0}^{h-1}\sum_{\tau_2^{r*}=0}^{h-1}\dots\sum_{\tau_m^{r*}=0}^{h-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^{s*}}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^{s*}}(j)\right)\right]$$

$$\times \frac{1}{\left(\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)\right)^{p}\left(\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)\right)^{q}}\int_0^\infty \frac{(\eta_4)^{p-1}}{[\eta_4+1]^{(p+q)}}d\eta_4,$$

$$\Psi^{*-1} = \sum_{L_1^{1*}=0}^{g-1}\sum_{L_2^{1*}=0}^{g-1}\dots\sum_{L_n^{1*}=0}^{g-1}\dots\sum_{L_1^{2*}=0}^{g-1}\sum_{L_2^{2*}=0}^{g-1}\dots\sum_{L_n^{2*}=0}^{g-1}\dots\sum_{L_1^{r*}=0}^{g-1}\sum_{L_2^{r*}=0}^{g-1}\dots\sum_{L_n^{r*}=0}^{g-1}\sum_{\tau_1^{1*}=0}^{h-1}\sum_{\tau_2^{1*}=0}^{h-1}\dots$$

$$\sum_{\tau_m^{1*}=0}^{h-1}\sum_{\tau_1^{2*}=0}^{h-1}\sum_{\tau_2^{2*}=0}^{h-1}\dots\sum_{\tau_m^{2*}=0}^{h-1}\dots\sum_{\tau_1^{r*}=0}^{h-1}\sum_{\tau_2^{r*}=0}^{h-1}\dots\sum_{\tau_m^{r*}=0}^{h-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^{s*}}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^{s*}}(j)\right)\right]$$

$$\times \frac{\mathrm{B}(p,q)}{\left(\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)\right)^{p}\left(\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)\right)^{q}}.$$

Therefore, the posterior density function of R based on MRSS in case of odd set size, for $0 < r < 1$ is given by

$$\pi_{MRSS}\left(r_1|\underline{x}^*,\underline{y}^*\right) = \sum_{L_1^{1*}=0}^{g-1}\sum_{L_2^{1*}=0}^{g-1}\dots\sum_{L_n^{1*}=0}^{g-1}\dots\sum_{L_1^{2*}=0}^{g-1}\sum_{L_2^{2*}=0}^{g-1}\dots\sum_{L_n^{2*}=0}^{g-1}\dots\sum_{L_1^{r*}=0}^{g-1}\sum_{L_2^{r*}=0}^{g-1}\dots\sum_{L_n^{r*}=0}^{g-1}$$

$$\sum_{\tau_1^{1*}=0}^{h-1}\sum_{\tau_2^{1*}=0}^{h-1}\dots\sum_{\tau_m^{1*}=0}^{h-1}\sum_{\tau_1^{2*}=0}^{h-1}\sum_{\tau_2^{2*}=0}^{h-1}\dots\sum_{\tau_m^{2*}=0}^{h-1}\dots\sum_{\tau_1^{r*}=0}^{h-1}\sum_{\tau_2^{r*}=0}^{h-1}\dots\sum_{\tau_m^{r*}=0}^{h-1}\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^{s*}}(i)\right)\right]$$

$$\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^{s*}}(j)\right)\right] r_1^{q-1}(1-r_1)^{p-1}\left[(1-r_1)\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)\right.$$

$$\left.+r_1\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)\right]^{-(p+q)}\left[\sum_{L_1^{1*}=0}^{g-1}\sum_{L_2^{1*}=0}^{g-1}\dots\sum_{L_n^{1*}=0}^{g-1}\dots\sum_{L_1^{2*}=0}^{g-1}\sum_{L_2^{2*}=0}^{g-1}\dots\sum_{L_n^{2*}=0}^{g-1}\dots\sum_{L_1^{r*}=0}^{g-1}\right.$$

$$\sum_{L_2^{r*}=0}^{g-1}\dots\sum_{L_n^{r*}=0}^{g-1}\sum_{\tau_1^{1*}=0}^{h-1}\sum_{\tau_2^{1*}=0}^{h-1}\dots\sum_{\tau_m^{1*}=0}^{h-1}\sum_{\tau_1^{2*}=0}^{h-1}\sum_{\tau_2^{2*}=0}^{h-1}\dots\sum_{\tau_m^{2*}=0}^{h-1}\dots\sum_{\tau_1^{r*}=0}^{h-1}\sum_{\tau_2^{r*}=0}^{h-1}\dots\sum_{\tau_m^{r*}=0}^{h-1}\mathrm{B}(p,q)$$

$$\left.\left[\prod_{s=1}^{r}\prod_{i=1}^{n}\left(\Phi_{L_i^{s*}}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{m}\left(\Phi_{\tau_j^{s*}}(j)\right)\right]\left(\sum_{s=1}^{r}\sum_{i=1}^{n}\left(T_{L_i^{s*}}(i)\right)\right)^{-p}\left(\sum_{s=1}^{r}\sum_{j=1}^{m}\left(T_{\tau_j^{s*}}(j)\right)\right)^{-q}\right]^{-1}. \tag{31}$$

Using (31) Bayesian estimator of R based on MRSS for odd set size using non-informative prior under SE and LINEX loss functions respectively, can be obtained as follows

$$R_{SE}^* = \int_0^1 r_1\,\pi_{MRSS}\left(r_1|\underline{x}^*,\underline{y}^*\right)dr_1, \tag{32}$$

and

$$R_{LIN}^* = \frac{-1}{\vartheta}\ln\int_0^1 e^{-\vartheta r_1}\pi_{MRSS}\left(r_1|\underline{x}^*,\underline{y}^*\right)dr_1. \tag{33}$$

respectively.

Bayesian estimators R_{SE}^* and R_{LIN}^* of R in case of simple case when $(r = 1)$ based on MRSS data for odd set size under SE and LINEX loss functions using non-informative prior can be obtained by substituting $(r = 1)$ in (32) and (33) as follows

$$R_{SE}^{*}=\int_0^1\Big[\mathrm{B}(n,m)\sum_{L_1^*=0}^{g-1}\sum_{L_2^*=0}^{g-1}\dots\sum_{L_n^*=0}^{g-1}\Big[\prod_{i=1}^{n}(-1)^{L_i^*}\binom{g-1}{L_i^*}\Big]\big[\sum_{i=1}^{n}(g+L_i^*)ln\big(1+x_{i(g)}^{*c}\big)\big]^{-n}$$

$$\times\sum_{\tau_1^*=0}^{h-1}\sum_{\tau_2^*=0}^{h-1}\dots\sum_{\tau_m^*=0}^{h-1}\Big[\prod_{j=1}^{m}(-1)^{\tau_j^*}\binom{h-1}{\tau_j^*}\Big]\big[\sum_{j=1}^{m}\big(h+\tau_j^*\big)ln\big(1+y_{j(h)}^{*c}\big)\big]^{-m}\Big]^{-1}\sum_{L_1^*=0}^{g-1}\sum_{L_2^*=0}^{g-1}\dots$$

$$\times\sum_{L_n^*=0}^{g-1}\Big[\prod_{i=1}^{n}(-1)^{L_i^*}\binom{g-1}{L_i^*}\Big]\sum_{\tau_1^*=0}^{h-1}\sum_{\tau_2^*=0}^{h-1}\dots\sum_{\tau_m^*=0}^{h-1}\Big[\prod_{j=1}^{m}(-1)^{\tau_j^*}\binom{h-1}{\tau_j^*}\Big]r_1^m(1-r_1)^{n-1}$$

$$\times\big[(1-r_1)\sum_{i=1}^{n}(g+L_i^*)ln\big(1+x_{i(g)}^{*c}\big)+r_1\sum_{j=1}^{m}\big(h+\tau_j^*\big)ln\big(1+y_{j(h)}^{*c}\big)\big]^{-n+m}dr_1. \quad (34)$$

and

$$R_{LIN}^{*}=\frac{-1}{\vartheta}\ln\Big\{\Big[\mathrm{B}(n,m)\sum_{L_1^*=0}^{g-1}\sum_{L_2^*=0}^{g-1}\dots\sum_{L_n^*=0}^{g-1}\Big[\prod_{i=1}^{n}(-1)^{L_i^*}\binom{g-1}{L_i^*}\Big]\big[\sum_{i=1}^{n}(g+L_i^*)\,ln\big(1+x_{i(g)}^{*c}\big)\big]^{-n}$$

$$\sum_{\tau_1^*=0}^{h-1}\sum_{\tau_2^*=0}^{h-1}\dots\sum_{\tau_m^*=0}^{h-1}\Big[\prod_{j=1}^{m}(-1)^{\tau_j^*}\binom{h-1}{\tau_j^*}\Big]\big[\sum_{j=1}^{m}\big(h+\tau_j^*\big)ln\big(1+y_{j(h)}^{*c}\big)\big]^{-m}\Big]^{-1}\int_0^1\sum_{L_1^*=0}^{g-1}\sum_{L_2^*=0}^{g-1}\dots$$

$$\sum_{L_n^*=0}^{g-1}\Big[\prod_{i=1}^{n}(-1)^{L_i^*}\binom{g-1}{L_i^*}\Big]\sum_{\tau_1^*=0}^{h-1}\sum_{\tau_2^*=0}^{h-1}\dots\sum_{\tau_m^*=0}^{h-1}\Big[\prod_{j=1}^{m}(-1)^{\tau_j^*}\binom{h-1}{\tau_j^*}\Big]e^{-\vartheta r_1\ln[(1-r_1)^{n-1}r_1^{m-1}]}$$

$$\times\big[(1-r_1)[\sum_{i=1}^{n}(g+L_i^*)ln\big(1+x_{i(g)}^{*c}\big)]+r_1[\sum_{j=1}^{m}\big(h+\tau_j^*\big)ln\big(1+y_{j(h)}^{*c}\big)]\big]^{-(n+m)}dr_1\Big\}. \quad (35)$$

As it seems, it is not easy to obtain a closed form solution to (34) and (35). Therefore, an iterative technique must be applied to solve these equations numerically.

4.2 Bayesian estimator for R with even set size

The main aim of this subsection is to obtain Bayes estimators of R based on MRSS in case of even set size using non-informative Jeffery priors under SE and LINEX loss functions.

Let $\{X_{1(u)s}^{**},X_{2(u)s}^{**},..,X_{u(u)s}^{**}\}\cup\{X_{u+1(u+1)s}^{**},\dots,X_{n(u+1)s}^{**}\}$ be a MRSS from Burr (c,b), where $u=n/2$, then the pdfs of $X_{i(u)s}^{**}$ and $X_{i(u+1)s}^{**}$ are given as follows:

$$f_u\big(x_{i(u)s}^{**}\big)=D_5bcx_{i(u)s}^{**c-1}\big[1+x_{i(u)s}^{**c}\big]^{-(b(u+1)+1)}\Big[1-\big(1+x_{i(u)s}^{**c}\big)^{-b}\Big]^{u-1},x_{i(u)s}^{**}>0,$$

and

$$f_{u+1}\big(x_{i(u+1)s}^{**}\big)=D_5bcx_{i(u+1)s}^{**c-1}\big[1+x_{i(u+1)s}^{**c}\big]^{-(bu+1)}\Big[1-\big(1+x_{i(u+1)s}^{**c}\big)^{-b}\Big]^{u},x_{i(u+1)s}^{**}>0.$$

Let the set $\underline{X}^{**}=\{X_{1(u)s}^{**},\dots,X_{u(u)s}^{**},\ X_{u+1(u+1)s}^{**},\dots,X_{n(u+1)s}^{**};s=1,2,..r\}$ be a MRSS from Burr (c,b), where $u=n/2$. The likelihood function denoted by $L\big(\underline{x}^{**}|b\big)$ with even set size will be as follows

$$L(\underline{x}^{**}|b)\propto\prod_{s=1}^{r}\Big[\prod_{i=1}^{u}b\big[1+x_{i(u)s}^{**c}\big]^{-b(u+1)}\Big[1-\big(1+x_{i(u)s}^{**c}\big)^{-b}\Big]^{u-1}\prod_{i=u+1}^{n}b\big[1+x_{i(u+1)s}^{**c}\big]^{-bu}$$

$$\times\Big[1-\big(1+x_{i(u+1)s}^{**c}\big)^{-b}\Big]^{u}\Big],$$

by using binomial expansion,

$$L(\underline{x}^{**}|b)\propto b^pe^{-b[(u+1)\sum_{s=1}^{r}\sum_{i=1}^{u}\ln\left(1+x_{i(u)s}^{**c}\right)+u\sum_{s=1}^{r}\sum_{i=u+1}^{n}\ln\left(1+x_{i(u+1)s}^{**c}\right)]}\prod_{s=1}^{r}\prod_{i=1}^{u}\big[\sum_{L^{**}=0}^{u-1}(-1)^{L^{**}}$$

$$\times\binom{u-1}{L^{**}}\big(1+x_{i(u)s}^{**c}\big)^{-bL^{**}}\big]\prod_{s=1}^{r}\prod_{i=u+1}^{n}\big[\sum_{l^{**}=0}^{u}(-1)^{l^{**}}\binom{u}{l^{**}}\big(1+x_{i(u+1)s}^{**c}\big)^{-bl^{**}}\big],$$

$$L(\underline{x}^{**}|b)\propto b^p\sum_{L_1^{1**}=0}^{u-1}\sum_{L_2^{1**}=0}^{u-1}\dots\sum_{L_u^{1**}=0}^{u-1}\sum_{L_1^{2**}=0}^{u-1}\sum_{L_2^{2**}=0}^{u-1}\dots\sum_{L_u^{2**}=0}^{u-1}\dots\sum_{L_1^{r**}=0}^{u-1}\sum_{L_2^{r**}=0}^{u-1}\dots\sum_{L_u^{r**}=0}^{u-1}\dots$$

$$\sum_{l_{u+1}^{1**}=0}^{u}\sum_{l_{u+2}^{1**}=0}^{u}\dots\sum_{l_n^{1**}=0}^{u}\sum_{l_{u+1}^{2**}=0}^{u}\sum_{l_{u+2}^{2**}=0}^{u}\dots\sum_{l_n^{2**}=0}^{u}\dots\sum_{l_{u+1}^{r**}=0}^{u}\sum_{l_{u+2}^{r**}=0}^{u}\dots\sum_{l_n^{r**}=0}^{u}\dots$$

$$\left[\prod_{s=1}^{r}\prod_{i=1}^{u}\left(\Phi_{L_i^{s**}}(i)\right)\right]\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\left(\Phi_{l_i^{s**}}(i)\right)\right]e^{-b\sum_{s=1}^{r}\left[\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right]}, \quad (36)$$

where $\Phi_{L_i^{s**}}(i)=(-1)^{L_i^{s**}}\binom{u-1}{L_i^{s**}}$, $\Phi_{l_i^{s**}}(i)=(-1)^{l_i^{s**}}\binom{u}{l_i^{s**}}$,

$T_{L_i^{s**}}(i)=(u+L_i^{s**}+1)\ln\left(1+x_{i(u)s}^{**c}\right)$ and $T_{l_i^{s**}}(i)=(u+l_i^{s**})\ln\left(1+x_{i(u+1)s}^{**c}\right)$,

Combine the prior density in (1) and the likelihood function in (36), the posterior density of b based on MRSS in case of even set size will be obtained as follows

$$\pi_{MRSS}(b|\underline{x}^{**}) \propto b^{p-1}\sum_{L_1^{1**}=0}^{u-1}\sum_{L_2^{1**}=0}^{u-1}\cdots\sum_{L_u^{1**}=0}^{u-1}\sum_{L_1^{2**}=0}^{u-1}\sum_{L_2^{2**}=0}^{u-1}\cdots\sum_{L_u^{2**}=0}^{u-1}\cdots\sum_{L_1^{r**}=0}^{u-1}\sum_{L_2^{r**}=0}^{u-1}\cdots$$

$$\sum_{L_u^{r**}=0}^{u-1}\cdots\sum_{l_{u+1}^{1**}=0}^{u}\sum_{l_{u+2}^{1**}=0}^{u}\cdots\sum_{l_n^{1**}=0}^{u}\sum_{l_{u+1}^{2**}=0}^{u}\sum_{l_{u+2}^{2**}=0}^{u}\cdots\sum_{l_n^{2**}=0}^{u}\cdots\sum_{l_{u+1}^{r**}=0}^{u}\sum_{l_{u+2}^{r**}=0}^{u}\cdots\sum_{l_n^{r**}=0}^{u}\cdots$$

$$\left[\prod_{s=1}^{r}\prod_{i=1}^{u}\Phi_{L_i^{s**}}(i)\right]\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{l_i^{s**}}(i)\right]e^{-b\sum_{s=1}^{r}\left[\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right]}, \quad (37)$$

Let $\{Y_{1(v)s}^{**},Y_{2(v)s}^{**},\ldots,Y_{v(v)s}^{**}\}\cup\{Y_{v+1(v+1)s}^{**},\ldots,Y_{m(v+1)s}^{**}\}$ be a MRSS from Burr (c,a), where $v=m/2$. Then the pdfs of $Y_{j(v)s}^{**}$ and $Y_{j(v+1)s}^{**}$ are given as follows:

$$f_v\left(y_{j(v)s}^{**}\right)=\frac{n!}{(u-1)!(u)!}acy_{j(v)s}^{**c-1}\left[1+y_{j(v)s}^{**c}\right]^{-(a(v+1)+1)}\left[1-\left(1+y_{j(v)s}^{**c}\right)^{-a}\right]^{v-1}, y_{j(v)s}^{**}>0$$

and

$$f_{v+1}\left(y_{j(v+1)s}^{**}\right)=\frac{m!}{(v-1)!(v)!}acy_{j(v+1)s}^{**c-1}\left[1+y_{j(v+1)s}^{**c}\right]^{-(av+1)}\left[1-\left(1+y_{j(v+1)s}^{**c}\right)^{-a}\right]^{v}, y_{j(v+1)s}^{**}>0,$$

Let $\underline{Y}^{**}=\{Y_{1(v)s}^{**},\ Y_{2(v)s}^{**},\ldots,\ Y_{v(v)s}^{**},\ Y_{v+1(v+1)s}^{**},\ldots,\ Y_{m(v+1)s}^{**};s=1,..,r\}$ be a median ranked set sample from Burr (c,a), where $v=m/2$.

$$L_{MRSS}(\underline{y}^{**}|a)\propto a^{q}\sum_{\tau_1^{1**}=0}^{v-1}\sum_{\tau_2^{1**}=0}^{v-1}\cdots\sum_{\tau_v^{1**}=0}^{v-1}\sum_{\tau_1^{2**}=0}^{v-1}\sum_{\tau_2^{2**}=0}^{v-1}\cdots\sum_{\tau_v^{2**}=0}^{v-1}\cdots\sum_{\tau_1^{r**}=0}^{v-1}\sum_{\tau_2^{r**}=0}^{v-1}\cdots\sum_{\tau_v^{r**}=0}^{v-1}\cdots$$

$$\sum_{t_{v+1}^{1**}=0}^{v}\sum_{t_{v+2}^{1**}=0}^{v}\cdots\sum_{t_m^{1**}=0}^{v}\sum_{t_{v+1}^{2**}=0}^{v}\sum_{t_{v+2}^{2**}=0}^{v}\cdots\sum_{t_m^{2**}=0}^{v}\cdots\sum_{t_{v+1}^{r**}=0}^{v}\sum_{t_{v+2}^{r**}=0}^{v}\cdots\sum_{t_m^{r**}=0}^{v}\cdots\left[\prod_{s=1}^{r}\prod_{j=1}^{v}\Phi_{\tau_j^{s**}}(j)\right]$$

$$\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{t_j^{s**}}(j)\right]e^{-a\sum_{s=1}^{r}\left[\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{j=v+1}^{m}\left(\mathrm{T}_{t_j^{s**}}(j)\right)\right]},$$

where $\Phi_{\tau_j^{s**}}(j)=(-1)^{\tau_j^{s**}}\binom{v-1}{\tau_j^{s**}}$, $\Phi_{t_j^{s**}}(j)=(-1)^{t_j^{s**}}\binom{v}{t_j^{s**}}$,

$\mathrm{T}_{\tau_j^{s**}}(j)=\left(v+\tau_j^{s**}+1\right)\ln(1+y_{j(v)s}^{**c})$ and $T_{t_j^{s**}}(j)=\left(v+t_j^{s**}\right)\ln\left(1+y_{j(v+1)s}^{**c}\right)$, (38)

By combining the prior density (2) and the likelihood function (38), the posterior density of a denoted by $\pi_{MRSS}(a|\underline{y}^{**})$ based on MRSS in case of even set size will be obtained as follows

$$\pi_{MRSS}(a|\underline{y}^{**})\propto a^{q-1}\sum_{\tau_1^{1**}=0}^{v-1}\sum_{\tau_2^{1**}=0}^{v-1}\cdots\sum_{\tau_v^{1**}=0}^{v-1}\sum_{\tau_1^{2**}=0}^{v-1}\sum_{\tau_2^{2**}=0}^{v-1}\cdots\sum_{\tau_v^{2**}=0}^{v-1}\cdots\sum_{\tau_1^{r**}=0}^{v-1}$$

$$\sum_{\tau_2^{r**}=0}^{v-1}\cdots\sum_{\tau_v^{r**}=0}^{v-1}\cdots\sum_{t_{v+1}^{1**}=0}^{v}\sum_{t_{v+2}^{1**}=0}^{v}\cdots\sum_{t_m^{1**}=0}^{v}\sum_{t_{v+1}^{2**}=0}^{v}\sum_{t_{v+2}^{2**}=0}^{v}\cdots\sum_{t_m^{2**}=0}^{v}\cdots\sum_{t_{v+1}^{r**}=0}^{v}\sum_{t_{v+2}^{r**}=0}^{v}\cdots$$

$$\sum_{t_m^{r**}=0}^{v}\cdots\left[\prod_{s=1}^{r}\prod_{j=1}^{v}\Phi_{\tau_j^{s**}}(j)\right]\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{t_j^{s**}}(j)\right]e^{-a\sum_{s=1}^{r}\left[\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right]}. \quad (39)$$

From posterior densities (37) and (39), the joint posterior density function of b and a will be as follows

$$\pi_{MRSS}\left(b,a|\underline{x}^{**},\underline{y}^{**}\right)\propto\sum_{L_1^{1**}=0}^{u-1}\sum_{L_2^{1**}=0}^{u-1}\cdots\sum_{L_u^{1**}=0}^{u-1}\sum_{L_1^{2**}=0}^{u-1}\sum_{L_2^{2**}=0}^{u-1}\cdots\sum_{L_u^{2**}=0}^{u-1}\cdots\sum_{L_1^{r**}=0}^{u-1}\sum_{L_2^{r**}=0}^{u-1}\cdots$$

$\sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{l_{u+1}^{1**}=0}^{u} \sum_{l_{u+2}^{1**}=0}^{u} \cdots \sum_{l_n^{1**}=0}^{u} \sum_{l_{u+1}^{2**}=0}^{u} \sum_{l_{u+2}^{2**}=0}^{u} \cdots \sum_{l_n^{2**}=0}^{u} \cdots \sum_{l_{u+1}^{r**}=0}^{u} \sum_{l_{u+2}^{r**}=0}^{u} \cdots \sum_{l_n^{r**}=0}^{u} \cdots$

$\sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^{v} \sum_{t_{v+2}^{1**}=0}^{v}$

$\cdots \sum_{t_m^{1**}=0}^{v} \sum_{t_{v+1}^{2**}=0}^{v} \quad \sum_{t_{v+2}^{2**}=0}^{v} \cdots \sum_{t_m^{2**}=0}^{v} \cdots \sum_{t_{v+1}^{r**}=0}^{v} \sum_{t_{v+2}^{r**}=0}^{v} \cdots \sum_{t_m^{r**}=0}^{v} \cdots \left[\prod_{s=1}^{r} \prod_{i=1}^{u} \Phi_{L_i^{s**}}(i)\right]$

$\times \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{l_i^{s**}}(i)\right] \left[\prod_{s=1}^{r} \prod_{j=1}^{v} \Phi_{\tau_j^{s**}}(j)\right] \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{t_j^{s**}}(j)\right] b^{p-1} a^{q-1}$

$$\times e^{-b \sum_{s=1}^{r}\left[\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right]} e^{-a \sum_{s=1}^{r}\left[\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right]}. \tag{40}$$

By substituting the transformation technique (5) in the posterior density function (40), the Bayes estimator of R based on MRSS with even set size under non-informative prior can be obtained as follows

$\pi_{MRSS}\left(A_1, r_1 | \underline{x}^{**}, \underline{y}^{**}\right) \propto \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots$

$\sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{l_{u+1}^{1**}=0}^{u} \sum_{l_{u+2}^{1**}=0}^{u} \cdots \sum_{l_n^{1**}=0}^{u} \sum_{l_{u+1}^{2**}=0}^{u} \sum_{l_{u+2}^{2**}=0}^{u} \cdots \sum_{l_n^{2**}=0}^{u} \cdots \sum_{l_{u+1}^{r**}=0}^{u} \sum_{l_{u+2}^{r**}=0}^{u} \cdots \sum_{l_n^{r**}=0}^{u} \cdots$

$\sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^{v}$

$\sum_{t_{v+2}^{1**}=0}^{v} \cdots \sum_{t_m^{1**}=0}^{v} \sum_{t_{v+1}^{2**}=0}^{v} \sum_{t_{v+2}^{2**}=0}^{v} \cdots \sum_{t_m^{2**}=0}^{v} \cdots \sum_{t_{v+1}^{r**}=0}^{v} \sum_{t_{v+2}^{r**}=0}^{v} \cdots \sum_{t_m^{r**}=0}^{v} \cdots (1-r_1)^{p-1} {r_1}^{q-1}$

$\times {A_1}^{p+q-1} \left[\prod_{s=1}^{r} \prod_{i=1}^{u} \Phi_{L_i^{s**}}(i)\right] \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{l_i^{s**}}(i)\right] \left[\prod_{s=1}^{r} \prod_{j=1}^{v} \Phi_{\tau_j^{s**}}(j)\right] \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{t_j^{s**}}(j)\right]$

$e^{-A_1\left[(1-r_1) \sum_{s=1}^{r}\left[\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right]+r_1 \sum_{s=1}^{r}\left[\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right]\right]}.$

Integrate out of A_1, the marginal posterior density function of r_1 is as follows

$\pi_{MRSS}\left(r_1 | \underline{x}^{**}, \underline{y}^{**}\right) = \Psi^{**} \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots$

$\sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{l_{u+1}^{1**}=0}^{u} \sum_{l_{u+2}^{1**}=0}^{u} \cdots \sum_{l_n^{1**}=0}^{u} \sum_{l_{u+1}^{2**}=0}^{u} \sum_{l_{u+2}^{2**}=0}^{u} \cdots \sum_{l_n^{2**}=0}^{u} \cdots \sum_{l_{u+1}^{r**}=0}^{u} \sum_{l_{u+2}^{r**}=0}^{u} \cdots \sum_{l_n^{r**}=0}^{u} \cdots \sum_{\tau_1^{1**}=0}^{v-1}$

$\sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1} \sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_{v+1}^{r**}=0}^{v-1} \sum_{\tau_{v+2}^{r**}=0}^{v-1} \cdots \sum_{\tau_m^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^{v} \sum_{t_{v+2}^{1**}=0}^{v} \cdots \sum_{t_m^{1**}=0}^{v}$

$\sum_{t_{v+1}^{2**}=0}^{v} \sum_{t_{v+2}^{2**}=0}^{v} \cdots \sum_{t_m^{2**}=0}^{v} \cdots \sum_{t_{v+1}^{r**}=0}^{v} \sum_{t_{v+2}^{r**}=0}^{v} \cdots \sum_{t_m^{r**}=0}^{v} \cdots \left[\prod_{s=1}^{r} \prod_{i=1}^{u} \Phi_{L_i^{s**}}(i)\right]\left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{l_i^{s**}}(i)\right]$

$\left[\prod_{s=1}^{r} \prod_{j=1}^{v} \Phi_{\tau_j^{s**}}(j)\right] \left[\prod_{s=1}^{r} \prod_{j=v+1}^{m} \Phi_{t_j^{s**}}(j)\right] (1-r_1)^{p-1} {r_1}^{q-1} [(1-r_1) \left(\sum_{s=1}^{r} \sum_{i=1}^{u} \left(T_{L_i^{s**}}(i)\right)\right.$

$$\left. + \sum_{s=1}^{r} \sum_{i=u+1}^{n} \left(T_{l_i^{s**}}(i)\right)\right) + r_1 \left(\sum_{s=1}^{r} \sum_{j=1}^{v} \left(\mathrm{T}_{\tau_j^{s**}}(j)\right) + \sum_{s=1}^{r} \sum_{j=v+1}^{m} \left(\mathrm{T}_{t_j^{s**}}(j)\right)\right)]^{-(p+q)}, \tag{41}$$

where Ψ^{**} is a constant and it can be obtained as follows

$\Psi^{**-1} = \int_0^1 \sum_{L_1^{1**}=0}^{u-1} \sum_{L_2^{1**}=0}^{u-1} \cdots \sum_{L_u^{1**}=0}^{u-1} \sum_{L_1^{2**}=0}^{u-1} \sum_{L_2^{2**}=0}^{u-1} \cdots \sum_{L_u^{2**}=0}^{u-1} \cdots \sum_{L_1^{r**}=0}^{u-1} \sum_{L_2^{r**}=0}^{u-1} \cdots \sum_{L_u^{r**}=0}^{u-1} \cdots \sum_{l_{u+1}^{1**}=0}^{u}$

$\sum_{l_{u+2}^{1**}=0}^{u} \cdots \sum_{l_n^{1**}=0}^{u} \sum_{l_{u+1}^{2**}=0}^{u} \sum_{l_{u+2}^{2**}=0}^{u} \cdots \sum_{l_n^{2**}=0}^{u} \cdots \sum_{l_{u+1}^{r**}=0}^{u} \sum_{l_{u+2}^{r**}=0}^{u} \cdots \sum_{l_n^{r**}=0}^{u} \cdots \sum_{\tau_1^{1**}=0}^{v-1} \sum_{\tau_2^{1**}=0}^{v-1} \cdots \sum_{\tau_v^{1**}=0}^{v-1}$

$\sum_{\tau_1^{2**}=0}^{v-1} \sum_{\tau_2^{2**}=0}^{v-1} \cdots \sum_{\tau_v^{2**}=0}^{v-1} \cdots \sum_{\tau_1^{r**}=0}^{v-1} \sum_{\tau_2^{r**}=0}^{v-1} \cdots \sum_{\tau_v^{r**}=0}^{v-1} \cdots \sum_{t_{v+1}^{1**}=0}^{v} \sum_{t_{v+2}^{1**}=0}^{v} \cdots \sum_{t_m^{1**}=0}^{v} \sum_{t_{v+1}^{2**}=0}^{v} \sum_{t_{v+2}^{2**}=0}^{v}$

$\cdots \sum_{t_m^{2**}=0}^{v} \cdots \sum_{t_{v+1}^{r**}=0}^{v} \sum_{t_{v+2}^{r**}=0}^{v} \cdots \sum_{t_m^{r**}=0}^{v} \cdots \left[\prod_{s=1}^{r} \prod_{i=1}^{u} \Phi_{L_i^{s**}}(i)\right] \left[\prod_{s=1}^{r} \prod_{i=u+1}^{n} \Phi_{l_i^{s**}}(i)\right]$

$$\left[\prod_{s=1}^{r}\prod_{j=1}^{v}\Phi_{\tau_j^{s**}}(j)\right]\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{t_j^{s**}}(j)\right](1-r_1)^{p-1}{r_1}^{-p-1}\left(\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)\right.$$

$$\left.+\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right)^{-(p+q)}\left[\frac{(1-r_1)\left(\sum_{s=1}^{r}\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{s=1}^{r}\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right)}{r_1\left(\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right)}+1\right]^{-(p+q)} dr_1, \quad (42)$$

using (7), then (42) will be written as follows

$$\Psi^{**-1}=\int_0^\infty\sum_{L_1^{1**}=0}^{u-1}\sum_{L_2^{1**}=0}^{u-1}\cdots\sum_{L_u^{1**}=0}^{u-1}\sum_{L_1^{2**}=0}^{u-1}\sum_{L_2^{2**}=0}^{u-1}\cdots\sum_{L_u^{2**}=0}^{u-1}\cdots\sum_{L_1^{r**}=0}^{u-1}\sum_{L_2^{r**}=0}^{u-1}\cdots\sum_{L_u^{r**}=0}^{u-1}\cdots\sum_{l_{u+1}^{1**}=0}^{u}$$

$$\sum_{l_{u+2}^{1**}=0}^{u}\cdots\sum_{l_n^{1**}=0}^{u}\sum_{l_{u+1}^{2**}=0}^{u}\sum_{l_{u+2}^{2**}=0}^{u}\cdots\sum_{l_n^{2**}=0}^{u}\cdots\sum_{l_{u+1}^{r**}=0}^{u}\sum_{l_{u+2}^{r**}=0}^{u}\cdots\sum_{l_n^{r**}=0}^{u}\cdots\sum_{\tau_1^{1**}=0}^{v-1}\sum_{\tau_2^{1**}=0}^{v-1}\cdots\sum_{\tau_v^{1**}=0}^{v-1}$$

$$\sum_{\tau_1^{2**}=0}^{v-1}\sum_{\tau_2^{2**}=0}^{v-1}\cdots\sum_{\tau_v^{2**}=0}^{v-1}\cdots\sum_{\tau_1^{r**}=0}^{v-1}\sum_{\tau_2^{r**}=0}^{v-1}\cdots\sum_{\tau_v^{r**}=0}^{v-1}\cdots\sum_{t_{v+1}^{1**}=0}^{v}\sum_{t_{v+2}^{1**}=0}^{v}\cdots\sum_{t_m^{1**}=0}^{v}\sum_{t_{v+1}^{2**}=0}^{v}\sum_{t_{v+2}^{2**}=0}^{v}$$

$$\cdots\sum_{t_m^{2**}=0}^{v}\cdots\sum_{t_{v+1}^{r**}=0}^{v}\sum_{t_{v+2}^{r**}=0}^{v}\cdots\sum_{t_m^{r**}=0}^{v}\cdots\left[\prod_{s=1}^{r}\prod_{i=1}^{u}\Phi_{L_i^{s**}}(i)\right]\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{l_i^{s**}}(i)\right]$$

$$\left[\prod_{s=1}^{r}\prod_{j=1}^{v}\Phi_{\tau_j^{s**}}(j)\right]\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{t_j^{s**}}(j)\right]\left(\frac{\eta_1}{1+\eta_1}\right)^{p-1}\left(\frac{1}{1+\eta_1}\right)^{-p-1}\left(\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)\right.$$

$$\left.+\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right)^{-(p+q)}\left[\eta_1\left(\frac{\sum_{s=1}^{r}\left(\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right)}{\sum_{s=1}^{r}\left(\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right)}\right)+1\right]^{-(p+q)}\frac{d\eta_1}{(1+\eta_1)^2},$$

then,

$$\Psi^{**-1}=\int_0^\infty\sum_{L_1^{1**}=0}^{u-1}\sum_{L_2^{1**}=0}^{u-1}\cdots\sum_{L_u^{1**}=0}^{u-1}\sum_{L_1^{2**}=0}^{u-1}\sum_{L_2^{2**}=0}^{u-1}\cdots\sum_{L_u^{2**}=0}^{u-1}\cdots\sum_{L_1^{r**}=0}^{u-1}\sum_{L_2^{r**}=0}^{u-1}\cdots\sum_{L_u^{r**}=0}^{u-1}\cdots$$

$$\sum_{l_{u+1}^{1**}=0}^{u}\sum_{l_{u+2}^{1**}=0}^{u}\cdots\sum_{l_n^{1**}=0}^{u}\sum_{l_{u+1}^{2**}=0}^{u}\sum_{l_{u+2}^{2**}=0}^{u}\cdots\sum_{l_n^{2**}=0}^{u}\cdots\sum_{l_{u+1}^{r**}=0}^{u}\sum_{l_{u+2}^{r**}=0}^{u}\cdots\sum_{l_n^{r**}=0}^{u}\cdots\sum_{\tau_1^{1**}=0}^{v-1}\sum_{\tau_2^{1**}=0}^{v-1}$$

$$\cdots\sum_{\tau_v^{1**}=0}^{v-1}\quad\sum_{\tau_1^{2**}=0}^{v-1}\sum_{\tau_2^{2**}=0}^{v-1}\cdots\sum_{\tau_v^{2**}=0}^{v-1}\cdots\sum_{\tau_1^{r**}=0}^{v-1}\sum_{\tau_2^{r**}=0}^{v-1}\cdots\sum_{\tau_v^{r**}=0}^{v-1}\cdots\sum_{t_{v+1}^{1**}=0}^{v}\sum_{t_{v+2}^{1**}=0}^{v}\cdots\sum_{t_m^{1**}=0}^{v}\sum_{t_{v+1}^{2**}=0}^{v}$$

$$\sum_{t_{v+2}^{2**}=0}^{v}\cdots\sum_{t_m^{2**}=0}^{v}\cdots\sum_{t_{v+1}^{r**}=0}^{v}\sum_{t_{v+2}^{r**}=0}^{v}\cdots\sum_{t_m^{r**}=0}^{v}\cdots\left[\prod_{s=1}^{r}\prod_{i=1}^{u}\Phi_{L_i^{s**}}(i)\right]\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{l_i^{s**}}(i)\right]$$

$$\times\left[\prod_{s=1}^{r}\prod_{j=1}^{v}\Phi_{\tau_j^{s**}}(j)\right]\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{t_j^{s**}}(j)\right](\eta_1)^{p-1}\left(\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\right.$$

$$\left.\times\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right)^{-(p+q)}\left[\eta_1\left(\frac{\sum_{s=1}^{r}\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{s=1}^{r}\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)}{\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)}\right)+1\right]^{-(p+q)} d\eta_1, \quad (43)$$

Let, $\eta_5=\eta_1\left(\frac{\sum_{s=1}^{r}\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{s=1}^{r}\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)}{\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)}\right)$, then (43) takes the following form

$$\Psi^{**-1}=\sum_{L_1^{1**}=0}^{u-1}\sum_{L_2^{1**}=0}^{u-1}\cdots\sum_{L_u^{1**}=0}^{u-1}\sum_{L_1^{2**}=0}^{u-1}\sum_{L_2^{2**}=0}^{u-1}\cdots\sum_{L_u^{2**}=0}^{u-1}\cdots\sum_{L_1^{r**}=0}^{u-1}\sum_{L_2^{r**}=0}^{u-1}\cdots\sum_{L_u^{r**}=0}^{u-1}\cdots\sum_{l_{u+1}^{1**}=0}^{u}$$

$$\sum_{l_{u+2}^{1**}=0}^{u}\cdots\sum_{l_n^{1**}=0}^{u}\sum_{l_{u+1}^{2**}=0}^{u}\sum_{l_{u+2}^{2**}=0}^{u}\cdots\sum_{l_n^{2**}=0}^{u}\cdots\sum_{l_{u+1}^{r**}=0}^{u}\sum_{l_{u+2}^{r**}=0}^{u}\cdots\sum_{l_n^{r**}=0}^{u}\cdots\sum_{\tau_1^{1**}=0}^{v-1}\sum_{\tau_2^{1**}=0}^{v-1}\cdots\sum_{\tau_v^{1**}=0}^{v-1}$$

$$\sum_{\tau_1^{2**}=0}^{v-1}\sum_{\tau_2^{2**}=0}^{v-1}\cdots\sum_{\tau_v^{2**}=0}^{v-1}\cdots\sum_{\tau_1^{r**}=0}^{v-1}\sum_{\tau_2^{r**}=0}^{v-1}\cdots\sum_{\tau_v^{r**}=0}^{v-1}\cdots\sum_{t_{v+1}^{1**}=0}^{v}\sum_{t_{v+2}^{1**}=0}^{v}\cdots\sum_{t_m^{1**}=0}^{v}\sum_{t_{v+1}^{2**}=0}^{v}\sum_{t_{v+2}^{2**}=0}^{v}\cdots$$

$$\sum_{t_m^{2**}=0}^{v}\cdots\sum_{t_{v+1}^{r**}=0}^{v}\sum_{t_{v+2}^{r**}=0}^{v}\cdots\sum_{t_m^{r**}=0}^{v}\cdots\left[\prod_{s=1}^{r}\prod_{i=1}^{u}\Phi_{L_i^{s**}}(i)\right]\left[\prod_{s=1}^{r}\prod_{i=u+1}^{n}\Phi_{l_i^{s**}}(i)\right]\left[\prod_{s=1}^{r}\prod_{j=1}^{v}\Phi_{\tau_j^{s**}}(j)\right]$$

$$\times\left[\prod_{s=1}^{r}\prod_{j=v+1}^{m}\Phi_{t_j^{s**}}(j)\right]\left(\sum_{s=1}^{r}\sum_{i=1}^{u}\left(T_{L_i^{s**}}(i)\right)+\sum_{s=1}^{r}\sum_{i=u+1}^{n}\left(T_{l_i^{s**}}(i)\right)\right)^{-p}\mathrm{B}(p,q)$$

$$\times\left(\sum_{s=1}^{r}\sum_{j=1}^{v}\left(\mathrm{T}_{\tau_j^{s**}}(j)\right)+\sum_{s=1}^{r}\sum_{j=v+1}^{m}\left(T_{t_j^{s**}}(j)\right)\right)^{-q}. \quad (44)$$

So, the posterior density function of R denoted by $\pi_{MRSS}\left(r_1|\underline{x}^{**},\underline{y}^{**}\right)$, can be obtained by using (44) and (41), for $0<r<1$. Then Bayes estimator of R denoted by R_{SE}^{**} under SE loss function is as follows

$$R_{SE}^{**}=\int_0^1 r_1\,\pi_{MRSS}\left(r_1|\underline{x}^{**},\underline{y}^{**}\right)dr_1, \quad (45)$$

Additionally, the Bayes estimator of R under LINEX loss function, denoted by R_{LIN}^{**}, Based on MRSS in case of even set size is given as follows

$$R_{LIN}^{**}=\frac{-1}{\vartheta}\ln E\left(e^{-\vartheta r_1}\right),$$

$$E\left(e^{-\vartheta r_1}\right)=\int_0^1 e^{-\vartheta r_1}\,\pi_{MRSS}\left(r_1|\underline{x}^{**},\underline{y}^{**}\right)dr_1. \quad (46)$$

By using (45), Bayes estimator of R under SE in the simple case of a single cycle when $(r=1)$ is as follows

$$R_{SE}^{**}=\int_0^1\left[\sum_{L_1^{**}=0}^{u-1}\sum_{L_2^{**}=0}^{u-1}\cdots\sum_{L_u^{**}=0}^{u-1}\left[\prod_{i=1}^{u}(-1)^{L_i^{**}}\binom{u-1}{L_i^{**}}\right]\sum_{l_{u+1}^{**}=0}^{u}\sum_{l_{u+2}^{**}=0}^{u}\cdots\sum_{l_n^{**}=0}^{u}\left[\prod_{i=u+1}^{n}(-1)^{l_i^{**}}\binom{u}{l_i^{**}}\right]\right.$$

$$\sum_{\tau_1^{**}=0}^{v-1}\sum_{\tau_2^{**}=0}^{v-1}\cdots\sum_{\tau_v^{**}=0}^{v-1}\left[\prod_{j=1}^{v}\left((-1)^{\tau_j^{**}}\binom{v-1}{\tau_j^{**}}\right)\right]\sum_{t_{v+1}^{**}=0}^{v}\sum_{t_{v+2}^{**}=0}^{v}\cdots\sum_{t_m^{**}=0}^{v}\left[\prod_{j=v+1}^{m}\left((-1)^{t_j^{**}}\binom{v}{t_j^{**}}\right)\right]$$

$$\times(1-r_1)^{n-1}{r_1}^{m}\left[(1-r_1)\left[\sum_{i=1}^{u}(u+L_i^{**}+1)\ln\left(1+x_{i(u)}^{**c}\right)+\sum_{i=u+1}^{n}(u+l_i^{**})\ln\left(1+x_{i(u+1)}^{**c}\right)\right]\right.$$

$$\left.+r_1\left[\sum_{j=1}^{v}\left(v+\tau_j^{**}+1\right)\ln\left(1+y_{j(v)}^{**c}\right)+\sum_{j=v+1}^{m}(v+t_j^{**})\ln\left(1+y_{j(v+1)}^{**c}\right)\right]\right]^{-(n+m)}\left[\sum_{L_1^{**}=0}^{u-1}\sum_{L_2^{**}=0}^{u-1}\cdots\right.$$

$$\sum_{L_u^{**}=0}^{u-1}\sum_{l_{u+1}^{**}=0}^{u}\sum_{l_{u+2}^{**}=0}^{u}\cdots\sum_{l_n^{**}=0}^{u}\sum_{\tau_1^{**}=0}^{v-1}\sum_{\tau_2^{**}=0}^{v-1}\cdots\sum_{\tau_v^{**}=0}^{v-1}\quad\sum_{t_{v+1}^{**}=0}^{v}\sum_{t_{v+2}^{**}=0}^{v}\cdots\sum_{t_m^{**}=0}^{v}$$

$$\left[\prod_{i=1}^{u}(-1)^{L_i^{**}}\binom{u-1}{L_i^{**}}\right]\left[\prod_{i=u+1}^{n}(-1)^{l_i^{**}}\binom{u}{l_i^{**}}\right]\left[\prod_{j=1}^{v}\left((-1)^{\tau_j^{**}}\binom{v-1}{\tau_j^{**}}\right)\right]\left[\prod_{j=v+1}^{m}\left((-1)^{t_j^{**}}\binom{v}{t_j^{**}}\right)\right]$$

$$\times\mathrm{B}(n,m)\left[\sum_{i=1}^{u}(u+L_i^{**}+1)\ln\left(1+x_{i(u)}^{**c}\right)+\sum_{i=u+1}^{n}(u+l_i^{**})\ln\left(1+x_{i(u+1)}^{**c}\right)\right]^{-n}$$

$$\left.\times\left[\sum_{j=1}^{v}\left(v+\tau_j^{**}+1\right)\ln\left(1+y_{j(v)}^{**c}\right)+\sum_{j=v+1}^{m}(v+t_j^{**})\ln\left(1+y_{j(v+1)}^{**c}\right)\right]^{-m}\right]^{-1}dr_1. \quad (47)$$

Bayes estimator of R under LINEX loss function, based on MRSS in case of even set size when $(r=1)$ is given by

$$R_{LIN}^{**}=\frac{-1}{\vartheta}ln\left\{\int_0^1\left[\sum_{L_1^{**}=0}^{u-1}\sum_{L_2^{**}=0}^{u-1}\cdots\sum_{L_u^{**}=0}^{u-1}\left[\prod_{i=1}^{u}(-1)^{L_i^{**}}\binom{u-1}{L_i^{**}}\right]\sum_{l_{u+1}^{**}=0}^{u}\sum_{l_{u+2}^{**}=0}^{u}\cdots\right.\right.$$

$$\sum_{l_n^{**}=0}^{u}\left[\prod_{i=u+1}^{n}(-1)^{l_i^{**}}\binom{u}{l_i^{**}}\right]\sum_{\tau_1^{**}=0}^{v-1}\sum_{\tau_2^{**}=0}^{v-1}\cdots\sum_{\tau_v^{**}=0}^{v-1}\left[\prod_{j=1}^{v}\left((-1)^{\tau_j^{**}}\binom{v-1}{\tau_j^{**}}\right)\right]$$

$$\times\sum_{t_{v+1}^{**}=0}^{v}\sum_{t_{v+2}^{**}=0}^{v}\cdots\sum_{t_m^{**}=0}^{v}\left[\prod_{j=v+1}^{m}\left((-1)^{t_j^{**}}\binom{v}{t_j^{**}}\right)\right]e^{-\vartheta r_1\ln((1-r_1)^{n-1}{r_1}^{m-1})}$$

$$\times\left[(1-r_1)\left[\sum_{i=1}^{u}(u+L_i^{**}+1)\ln\left(1+x_{i(u)}^{**c}\right)+\sum_{i=u+1}^{n}(u+l_i^{**})\ln\left(1+x_{i(u+1)}^{**c}\right)\right]\right.$$

$$\left.+r_1\left[\sum_{j=1}^{v}\left(v+\tau_j^{**}+1\right)\ln\left(1+y_{j(v)}^{**c}\right)+\sum_{j=v+1}^{m}(v+t_j^{**})\ln\left(1+y_{j(v+1)}^{**c}\right)\right]\right]^{-(n+m)}$$

$$\times \left[\left[\sum_{L_1^{**}=0}^{u-1} \sum_{L_2^{**}=0}^{u-1} \cdots \sum_{L_u^{**}=0}^{u-1} \left[\prod_{i=1}^{u} (-1)^{L_i^{**}} \binom{u-1}{L_i^{**}}\right] \sum_{l_{u+1}^{**}=0}^{u} \sum_{l_{u+2}^{**}=0}^{u} \cdots \sum_{l_n^{**}=0}^{u} \left[\prod_{i=u+1}^{n} (-1)^{l_i^{**}} \binom{u}{l_i^{**}}\right]\right.\right.$$

$$\sum_{\tau_1^{**}=0}^{v-1} \sum_{\tau_2^{**}=0}^{v-1} \cdots \sum_{\tau_v^{**}=0}^{v-1} \left[\prod_{j=1}^{v} \left((-1)^{\tau_j^{**}} \binom{v-1}{\tau_j^{**}}\right)\right] \sum_{t_{v+1}^{**}=0}^{v} \sum_{t_{v+..2}^{**}=0}^{v} \cdots \sum_{t_m^{**}=0}^{v} \left[\prod_{j=v+1}^{m} \left((-1)^{t_j^{**}} \binom{v}{t_j^{**}}\right)\right]$$

$$\times \mathrm{B}(n,m)\left[\sum_{i=1}^{u} (u + L_i^{**} + 1) \ln\left(1 + x_{i(u)}^{**c}\right) + \sum_{i=u+1}^{n} (u + l_i^{**}) \ln\left(1 + x_{i(u+1)}^{**c}\right)\right]^{-n}$$

$$\times \left[\sum_{j=1}^{v} \left(v + \tau_j^{**} + 1\right) \ln\left(1 + y_{j(v)}^{**c}\right) + \sum_{j=v+1}^{m} \left(v + t_j^{**}\right) \ln\left(1 + y_{j(v+1)}^{**c}\right)\right]^{-m}\Big]^{-1} dr_1\Big\}. \qquad (48)$$

Clearly, it is not easy to obtain a closed form solution for Bayesian estimators under non-informative Jeffery prior in (47) and (48) using SE and LINEX loss functions respectively, so an iterative procedure can be used to evaluate these equations.

5. Numerical Study

In this section, an extensive numerical investigation will be carried out using MathCAD (14) to compare the performance of the Bayes estimates $\hat{R}_{SE}, \hat{R}_{LIN}, R_{SE}^{*}, R_{LIN}^{*}, R_{SE}^{**}$ and R_{LIN}^{**} in case of odd and even set sizes respectively, based on MRSS data under SE and LINEX loss functions. The investigated properties are biases and mean square errors (MSEs). Without loss of generality, the number of cycles r will be one in all cases.

Several studies have shown that if one chooses the parameters b and c within the ranges (1-10) and (1-20) respectively, along with a sample size (2-10), the cumulative distribution function of Burr (c, b) covers the three main Pearson system types I, IV and VI, as well as many traditional types such that gamma distribution (see Burr (1968) and Burr and Cislak (1968)), keeping the above points in mind, 1000 random samples of Burr (c, b) and Burr (c, a) are generated with sample sizes $n = m = 2,3,4$. Without loss of generality, the shape parameter c will be assumed to be one in all experiments, the ratio ρ is selected as $0.25, 0.33, 0.5, 1$ $2,3,4,5,6$ depend on the different values of the shape parameters $b = 3,4,5,6,9,12$ and $a = 2,3,6,9,10,12$ where $\rho = \frac{b}{a}$.

Numerical results are summarized in Tables $(1) - (9)$ and represented through Figures$(1) - (4)$ from these tables and figures the following conclusions can be observed as follows

1. The biases of all estimators $\hat{R}_{SE}, \hat{R}_{LIN}, \dot{R}_{SE}, \dot{R}_{LIN}, R_{SE}^{*}, R_{LIN}^{*}, R_{SE}^{**}$ and R_{LIN}^{**} based on SRS, RSS, MRSS in the case of odd and even set sizes respectively are small. (See Tables (1) – (9)).
2. MSEs of all estimators $\hat{R}_{SE}, \hat{R}_{LIN}, \dot{R}_{SE}, \dot{R}_{LIN}, R_{SE}^{*}, R_{LIN}^{*}, R_{SE}^{**}$ and R_{LIN}^{**} are increasing as ρ increases up to $\rho = 1$, then MSEs are decreasing as ρ increases, in almost all cases.
3. In almost all cases, MSEs of all estimators $\hat{R}_{SE}, \hat{R}_{LIN}, \dot{R}_{SE}, \dot{R}_{LIN}, R_{SE}^{*}$, R_{LIN}^{*}, R_{SE}^{**} and R_{LIN}^{**} are decreasing as the set size increases at the same value of ρ. (See for example Figure (1)).

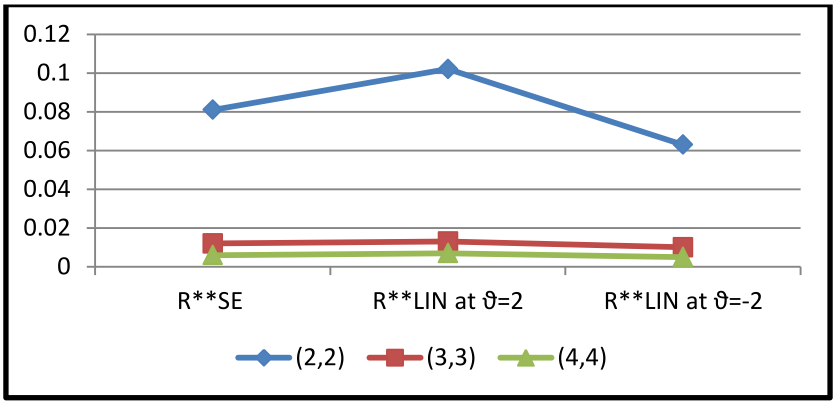

Figure (1): MSEs of R_{SE}^{**} and R_{LIN}^{**} when $\rho = 0.25$ based on MRSS data with even set size.

4. In almost all cases, in case of $\rho < 1$, MSEs of $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^{*}$ and R_{LIN}^{**} under LINEX loss function when $\vartheta = -2$, have the smallest MSEs comparing with the other estimators $\hat{R}_{SE}, \dot{R}_{SE}, R_{SE}^{*}$ and R_{SE}^{**} based on SE loss function and $\hat{R}_{LIN}, \dot{R}_{LIN}$, R_{LIN}^{*} and R_{LIN}^{**} based on LINEX loss function when $\vartheta = 2$. (See for example Figures (2) and (3)).

5. In almost all cases, in case of $\rho > 1$, MSEs of $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^{*}$ and R_{LIN}^{**} under LINEX loss function when $\vartheta = 2$, have the smallest MSEs comparing with the other estimators $\hat{R}_{SE}, \dot{R}_{SE}, R_{SE}^{*}$ and R_{SE}^{**} based on SE and $\hat{R}_{LIN}, \dot{R}_{LIN}, R_{LIN}^{*}$ and R_{LIN}^{**}based on LINEX loss function when $\vartheta = -2$. (See for example Figures (2) and(3)).

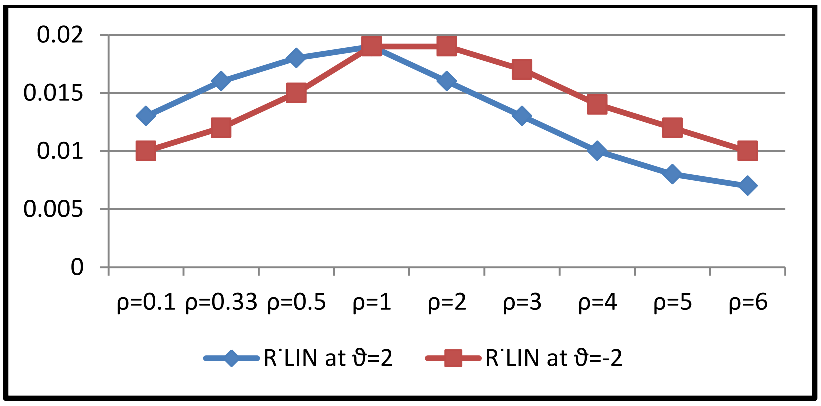

Figure (2): MSEs of the estimates $\dot{R}_{LIN}$ based on RSS data when $\vartheta = 2$ and $\vartheta = -2$.

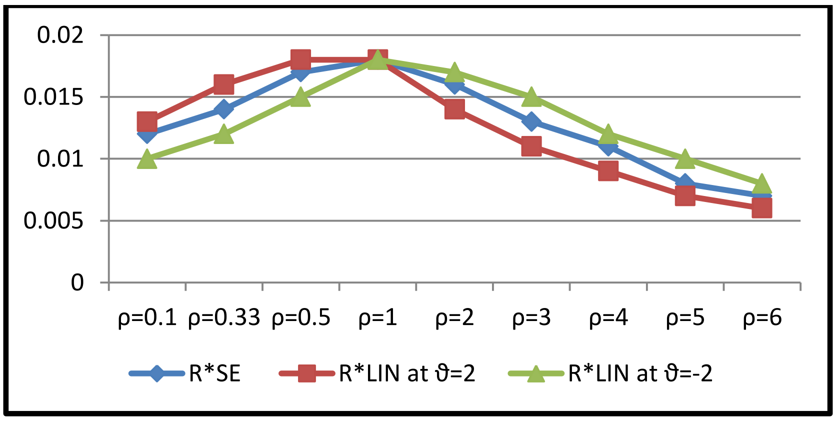

Figure (3):MSEs of the estimates $R_{SE}^{*}, R_{SE}^{**}, R_{LIN}^{*}$ and R_{LIN}^{**} based on MRSS data when $(n, m) = (3,3)$.

6. MSEs of the estimators R^*_{SE}, R^*_{LIN} , R^{**}_{SE} and R^{**}_{LIN} , $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on MRSS and RSS data are smaller than MSEs of the estimators $\hat{R}_{SE}, \hat{R}_{LIN}$ based on SRS data, in almost all cases. (See for example Figure (4)).

7. MSEs of the estimators R^*_{SE}, R^*_{LIN} , R^{**}_{SE} and R^{**}_{LIN} based on MRSS in case of odd and even set sizes respectively, are smaller than the estimators $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS, in almost all cases. (See for example Figure (4)).

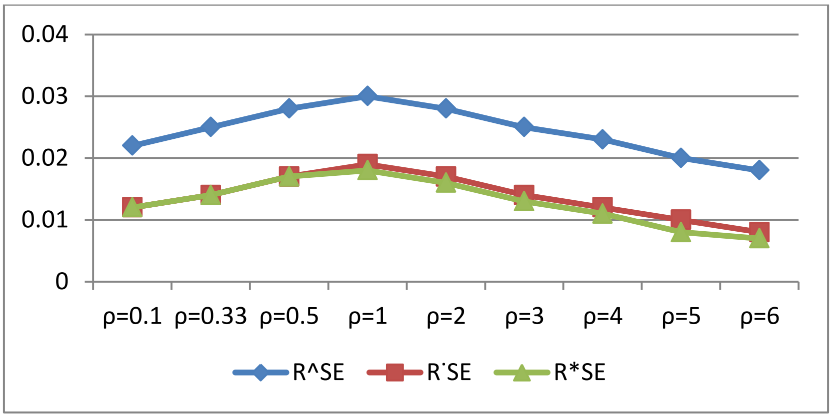

Figure (4): MSEs of the estimates $\hat{R}_{SE}$, $\dot{R}_{SE}$ and R^*_{SE} based on SRS, RSS and MRSS data at $(n, m) = (3,3)$.

6. Conclusions

In this article, the estimation problem of $R = P(Y < X)$ when X and Y follow Burr XII distribution using Bayesian approach based on SRS, RSS and MRSS techniques using non-informative Jeffery prior under symmetric and asymmetric loss functions is discussed.

From the numerical results, it is observed that, the MSEs of all estimators based on SRS, RSS and MRSS are decreasing as the set size increases. In almost all cases, MSEs of RSS and MRSS based on RSS and MRSS techniques are smaller than MSEs based on SRS. In almost all cases, the estimators based on MRSS have the smallest MSEs under SE and LINEX loss functions. The estimators based on MRSS approach dominate the corresponding based on RSS and SRS. Finally, this study revealed that the estimators based on MRSS are more efficient than the other two sampling techniques.

Table (1): Biases and MSEs for the estimates $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ based on SRS $at\ (n,m) = (2,2)$.

ρ	Biases			MSEs		
	$\hat{R}_{SE}$	$\hat{R}_{LIN}$		$\hat{R}_{SE}$	$\hat{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.087	-0.121	-0.059	0.036	0.047	0.028
0.33	-0.081	-0.117	-0.048	0.038	0.048	0.031
0.50	-0.062	-0.102	-0.026	0.039	0.047	0.033
1	-0.014	-0.054	0.026	0.038	0.031	0.038
2	0.036	0.001	0.075	0.035	0.030	0.041
3	0.056	0.025	0.091	0.032	0.026	0.041
4	0.064	0.037	0.097	0.030	0.023	0.039
5	0.068	0.043	0.097	0.027	0.021	0.036
6	0.068	0.046	0.069	0.025	0.019	0.033

Table (2): Biases and MSEs for the estimates $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ based on SRS $at\ (n,m) = (3,3)$.

ρ	Biases			MSEs		
	$\hat{R}_{SE}$	$\hat{R}_{LIN}$		$\hat{R}_{SE}$	$\hat{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.062	-0.085	-0.042	0.023	0.029	0.018
0.33	-0.0258	-0.084	-0.036	0.025	0.031	0.021
0.50	-0.004	-0.075	-0.019	0.028	0.033	0.024
1	-0.010	-0.040	0.020	0.028	0.029	0.028
2	0.027	0.001	0.055	0.026	0.023	0.030
3	0.041	0.019	0.066	0.023	0.019	0.028
4	0.046	0.028	0.068	0.020	0.017	0.025
5	0.048	0.032	0.067	0.018	0.014	0.022
6	0.048	0.034	0.065	0.016	0.013	0.020

Table (3): Biases and MSEs for the estimates $\hat{R}_{SE}$ and $\hat{R}_{LIN}$ based on SRS data $at\ (n,m) = (4,4)$.

ρ	Biases			MSEs		
	$\hat{R}_{SE}$	$\hat{R}_{LIN}$		$\hat{R}_{SE}$	$\hat{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.042	-0.058	-0.028	0.017	0.020	0.014
0.33	-0.038	-0.057	-0.021	0.019	0.023	0.017
0.50	-0.026	-0.048	-0.005	0.021	0.024	0.019
1	0.004	-0.020	0.028	0.024	0.024	0.024
2	0.033	0.013	0.056	0.023	0.020	0.026
3	0.043	0.026	0.063	0.020	0.017	0.023
4	0.046	0.032	0.063	0.017	0.014	0.021
5	0.047	0.034	0.061	0.015	0.012	0.018
6	0.046	0.035	0.058	0.013	0.011	0.016

Table (4): Biases and MSEs for the estimates $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS data $at\ (n,m) = (2,2)$.

ρ	Biases			MSEs		
	$\dot{R}_{SE}$	$\dot{R}_{LIN}$		$\dot{R}_{SE}$	$\dot{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.059	-0.080	-0.039	0.025	0.031	0.021
0.33	-0.054	-0.079	-0.031	0.028	0.034	0.023
0.50	-0.039	-0.068	-0.012	0.031	0.035	0.027
1	0.001	-0.030	0.033	0.030	0.031	0.031
2	0.039	0.012	0.068	0.030	0.026	0.035
3	0.053	0.030	0.079	0.027	0.022	0.033
4	0.058	0.038	0.081	0.024	0.019	0.030
5	0.060	0.042	0.080	0.021	0.017	0.027
6	0.059	0.043	0.078	0.019	0.015	0.024

Table (5): Biases and MSEs for the estimates $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS data $at\ (n,m) = (3,3)$.

ρ	Biases			MSEs		
	$\dot{R}_{SE}$	$\dot{R}_{LIN}$		$\dot{R}_{SE}$	$\dot{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.032	-0.043	-0.022	0.011	0.013	0.010
0.33	-0.030	-0.044	-0.017	0.013	0.016	0.012
0.50	-0.022	-0.039	-0.006	0.016	0.018	0.015
1	-0.0001	-0.018	0.018	0.018	0.018	0.018
2	0.023	0.006	0.039	0.016	0.015	0.018
3	0.030	0.017	0.044	0.014	0.012	0.016
4	0.032	0.022	0.044	0.011	0.009	0.013
5	0.032	0.023	0.042	0.009	0.008	0.011
6	0.031	0.024	0.039	0.008	0.006	0.009

Table (6): Biases and MSEs for the estimates $\dot{R}_{SE}$ and $\dot{R}_{LIN}$ based on RSS data $at\ (n,m) = (4,4)$.

ρ	Biases			MSEs		
	$\dot{R}_{SE}$	$\dot{R}_{LIN}$		$\dot{R}_{SE}$	$\dot{R}_{LIN}$	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.105	-0.114	-0.096	0.015	0.017	0.013
0.33	-0.093	-0.103	-0.083	0.011	0.015	0.007
0.50	-0.065	-0.076	-0.053	0.010	0.011	0.009
1	-0.001	-0.014	0.010	0.012	0.005	0.012
2	0.061	0.050	0.072	0.008	0.007	0.010
3	0.090	0.080	0.100	0.007	0.010	0.013
4	0.102	0.930	0.122	0.006	0.012	0.008
5	0.106	0.062	0.155	0.009	0.008	0.011
6	0.109	0.103	0.015	0.015	0.013	0.017

Table (7): Biases and MSEs for the estimates R_{SE}^{**} and R_{LIN}^{**} based on MRSS data $at\ (n,m) = (2,2)$.

ρ	Biases			MSEs		
	R_{SE}^{**}	R_{LIN}^{**}		R_{SE}^{**}	R_{LIN}^{**}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.055	-0.078	-0.035	0.023	0.029	0.019
0.33	-0.050	-0.075	-0.027	0.026	0.032	0.021
0.50	-0.035	-0.064	-0.008	0.029	0.033	0.025
1	0.003	-0.027	0.033	0.030	0.031	0.031
2	0.041	0.014	0.070	0.029	0.025	0.034
3	0.055	0.031	0.081	0.026	0.021	0.032
4	0.059	0.039	0.083	0.023	0.018	0.029
5	0.060	0.043	0.081	0.020	0.016	0.026
6	0.060	0.044	0.079	0.018	0.014	0.023

Table (7): Biases and MSEs for the estimates R_{SE}^{**} and R_{LIN}^{**} based on MRSS data $at\ (n,m) = (3,3)$.

ρ	Biases			MSEs		
	R_{SE}^{**}	R_{LIN}^{*}		R_{SE}^{**}	R_{LIN}^{*}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.031	-0.042	-0.021	0.010	0.012	0.009
0.33	-0.029	-0.042	-0.017	0.012	0.015	0.011
0.50	-0.022	-0.038	-0.007	0.015	0.017	0.014
1	-0.001	-0.018	0.017	0.017	0.018	0.017
2	0.021	0.006	0.037	0.015	0.014	0.017
3	0.028	0.016	0.041	0.013	0.011	0.015
4	0.030	0.020	0.041	0.010	0.008	0.012
5	0.030	0.022	0.039	0.008	0.007	0.010
6	0.029	0.023	0.007	0.006	0.005	0.008

Table (9): Biases and MSEs for the estimates R_{SE}^{**} and R_{LIN}^{**} based on MRSS data $at\ (n,m) = (4,4)$.

ρ	Biases			MSEs		
	R_{SE}^{**}	R_{LIN}^{**}		R_{SE}^{**}	R_{LIN}^{**}	
		$\vartheta = 2$	$\vartheta = -2$		$\vartheta = 2$	$\vartheta = -2$
0.25	-0.020	-0.026	-0.014	0.006	0.007	0.005
0.33	-0.019	-0.007	-0.012	0.007	0.004	0.006
0.50	-0.015	-0.025	-0.005	0.010	0.010	0.008
1	-0.001	-0.012	-0.011	0.012	0.012	0.011
2	0.013	0.004	0.023	0.010	0.009	0.011
3	0.018	0.010	0.026	0.008	0.007	0.008
4	0.019	0.013	0.025	0.005	0.005	0.007
5	0.019	0.014	0.024	0.005	0.004	0.005
6	0.018	0.014	0.022	0.004	0.003	0.004

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