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BAYESIAN ANALYSIS FOR MIXTURE OF BURR XII AND BURR X DISTRIBUTIONS

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Abstract

A finite mixture of probability distributions provides an important gadget in modeling wide range of random phenomena. A mixed model occurs when two or more different causes of failure are presented each with parametric form of life distributions with unknown proportions. Burr type XII and Burr type X distributions are of great importance and extensively used in many practical applications. This study is concerned with Bayesian analysis of a newly developed two-component mixture model of Burr XII and Burr X (BXIIBX) distributions. Based on type I censored samples, the parameters of a mixture model are estimated in view of Bayesian approach. The Bayesian estimators are computed using the idea of Markov Chain Monte Carlo (MCMC) method. A simulation study is performed for investigating the accuracy of estimators for different sample sizes.

1. Introduction

Mixtures of lifetime distributions appear in various practical applications. In many situations, available data can be considered such that

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the data come from a mixture population of two or more distributions. This idea enables to mix statistical distributions to get a new distribution carrying the properties of its components. Therefore, finite mixture models receive increasing attention over the recent years in many practical applications, for good review, see for example, Everitt and Hand [4], Titterington et al. [11], Lindsay [7], Al-Hussaini [1], McLachlan and Peel [8], Sultan et al. [10], Kazmi et al. [6], Ateya and Alharthi [3], Saieed [9], Ali et al. [2] and Karakoca et al. [5].

In 1942, Burr developed a family of twelve cumulative distribution functions, based on generating the Pearson differential equation, for modelling lifetime data. The Burr XII and Burr X distributions are widely used models and most desirable for statistical modelling from this family. The probability density function (pdf) of Burr XII distribution with shape parameters c and k is given by

$$g_1(x;\, c,\, k) = ckx^{c-1}(1 + x^c)^{-(k+1)}, \quad x > 0, \quad c,\, k > 0. \tag{1}$$

The pdf of Burr X distribution with shape parameter α and scale parameter λ is given by

$$g_2(x;\, \alpha,\, \lambda) = 2\lambda^2 \alpha x[1 - e^{-(\lambda x)^2}]^{\alpha-1} e^{-(\lambda x)^2}, \quad x > 0, \quad \alpha,\, \lambda > 0. \tag{2}$$

A finite mixture density function with the m-component densities of specified parametric form with unknown mixing weights p_i is defined by

$$f(x) = \sum_{i=1}^{m} p_i g_i(x),\; i = 1,\, 2,\, \ldots,\, m, \quad p_i \geq 0, \quad \text{and} \quad \sum_{i=1}^{m} p_i = 1.$$

A finite mixture of two sub-populations with mixed proportions $p,\, (1-p)$ is of practical importance and so, our interest here is with a finite mixture with two components. Hence, the pdf of 2-component mixture of Burr type XII and type X distributions, denoted by BXIIBX, is defined mathematically as follows:

$$f(x; \underline{\psi}) = pckx^{(c-1)}(1 + x^{c})^{-(k+1)}$$

$$+ 2(1 - p)\alpha\lambda^{2}xe^{-(\lambda x)^{2}}[1 - e^{-(\lambda x)^{2}}]^{(\alpha-1)}, \tag{3}$$

where $x, c, k, \alpha, \lambda > 0, 0 < p < 1,$ and $\psi \equiv (c, k, \alpha, \lambda, p)$ is the parameter vector of the BXIIBX model. The reliability function of mixture (3) denoted by $R(x; \underline{\psi})$ is given by

$$R(x; \underline{\psi}) = p(1 + x^{c})^{-k} + (1 - p)[1 - (1 - e^{-(\lambda x)^{2}})^{\alpha}]. \tag{4}$$

In most life-testing experiments, we cannot continue the experiment until the last failure is observed due to time limitations, lack of funds and other constraints on the data collection. The data obtained in such situations are called censored data. Type I and type II censoring schemes are the most popular censoring schemes. In this study, we concern with the estimation and evaluation of BXIIBX mixture distribution in a Bayesian approach, which has not been considered in the literature yet, based on type I censored samples. Due to the complexity of the model, Bayesian estimates are obtained via MCMC algorithm. This article can be organized as follows. In Section 2, maximum likelihood method of estimation is performed based on type I censored samples. In Section 3, the posterior distribution has been derived under informative priors. Section 4 presents a numerical study to examine Bayes estimates of BXIIBX mixture distribution using MCMC algorithm. A conclusion is summarized in last section.

2. Likelihood Function Based on Type I Censored Sample

In this section, the likelihood function of the parameters for BXIIBX distribution is obtained based on type I censored samples. In the type I censoring scheme, the experiment of n items is placed on a test with fixed test termination time T. The experiment is performed and its observed r out of n units failed up to time T instead of continuing experiments until all n items have failed. In many real life situations, failed objects can be recognized as belonging to subpopulation 1 or subpopulation 2, so depending on the cause of failure it may be observed r_1 and r_2. The number of uncensored observations is $r = r_1 + r_2$, where r_1 failure units from subpopulation (1), r_2 failure units from subpopulation (2), and the remaining $n - r$ observations are censored giving no information about to which subpopulation they belong. Furthermore, let x_{ij} as the failure time of the *j*th unit to the *i*th subpopulation, where $j = 1, 2, ..., r_i$, $i = 1, 2;$ $0 \leq x_{1j}, x_{2j} \leq T$. The likelihood function for two-component BXIIBX mixture model with *n* items under study in which r_1 will fail due to cause 1, r_2 will fail due to cause 2 and the remaining $(n - r_1 - r_2)$ will survive at time *T*, is given as follows:

$$L(\psi \mid x) \propto \prod_{j=1}^{r_1} p g_1(x_{1j}) \prod_{j=1}^{r_2} (1-p) g_2(x_{2j}) [R(T)]^{n-r}. \tag{5}$$

The likelihood function (5) can be written as follows:

$$\begin{aligned} L(\psi \mid x) \propto\ & 2^{r_2} p^{n-r-\delta_1+r_1} (1-p)^{r_2+\delta_1-\delta_2} c^{r_1} k^{r_1} \lambda^{2r_2} \alpha^{r_2} \\ & \cdot \exp\left[\sum_{j=1}^{r_1} (c-1) \ln x_{1j} - (k+1) \ln(1 + x_{1j}^c)\right] \\ & \cdot \exp\left[\sum_{j=1}^{r_2} \ln x_{2j} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 + (\alpha - 1) \ln[1 - e^{-(\lambda x_{2j})^2}]\right] \\ & \cdot \sum_{\delta_1=0}^{n-r} \sum_{\delta_2=0}^{\delta_1} \binom{n-r}{\delta_1} \binom{\delta_1}{\delta_2} (-1)^{\delta_1} [(1+T^c)^{-k}]^{\delta_2} [[1 - [1 - e^{-(\lambda T)^2}]^{\alpha}]], \end{aligned} \tag{6}$$

where $x = (x_{11}, x_{12}, ..., x_{1r_1}, x_{21}, x_{22}, ..., x_{2r_2})$ is the observed failure time for the non-censored observations.

3. Prior and Posterior Distribution

In this subsection, we assume that the prior distribution for p has uniform distribution over the interval (0, 1). Further, assuming that the prior of parameter c has a gamma distribution with parameters (a, b), while k, α and λ have exponential distribution with parameters β_1, β_2 and β_3, respectively. Hence, assuming independence of parameters, the joint prior distribution of parameters is as follows:

$$\pi(\psi) \propto \frac{b^a}{\beta_1\beta_2\beta_3\Gamma(a)} c^{a-1} e^{-\left(\frac{k}{\beta_1}+\frac{\alpha}{\beta_2}+\frac{\lambda}{\beta_3}\right)-b}, \quad c, k, \lambda, \alpha, a, b, \beta_1, \beta_2, \beta_3 > 0.$$

The posterior distribution of c, k, λ, α and p for BXIIBX under type I censored takes the following form:

$$\prod(\psi \mid x)$$

$$= \frac{1}{H} \frac{b^a e^{-b} 2^{r_2}}{\beta_1\beta_2\beta_3\Gamma(a)} p^{n-r-\delta_1+r_1}(1-p)^{r_2+\delta_1-\delta_2} c^{a+r_1-1} k^{r_1} \lambda^{2r_2} \alpha^{r_2}$$

$$\cdot \exp\left\{\sum_{j=1}^{r_1}(c-1)\ln x_{1j} - \frac{k}{\beta_1} - \sum_{j=1}^{r_1}(k+1)\ln(1+x_{1j}^c)\right\}$$

$$\cdot \exp\left\{\sum_{j=1}^{r_2}\ln x_{2j} - \sum_{j=1}^{r_2}(\lambda x_{2j})^2 - \frac{\alpha}{\beta_2} - \frac{\lambda}{\beta_2} + (\alpha-1)\ln[1-e^{-(\lambda x_{2j})^2}]\right\}$$

$$\cdot \sum_{\delta_1=0}^{n-r}\sum_{\delta_2=0}^{\delta_1}\binom{n-r}{\delta_1}\binom{\delta_1}{\delta_2}(-1)^{\delta_1}[(1+T^c)^{-k}]\{[1-[1-e^{-(\lambda T)^2}]^{\alpha}]\}^{\delta_1-\delta_2},$$

$$H = \frac{b^a e^{-b} 2^{r_2}}{\beta_1\beta_2\beta_3\Gamma(a)} B(n-r-\delta_1+r_1+1, r_2+\delta_1-\delta_2+1)$$

$$\cdot \exp\left(\sum_{j=1}^{r_2}\ln x_{2j} - \sum_{j=1}^{r_1}\ln x_{1j}\right)\sum_{\delta_1=0}^{n-r}\sum_{\delta_2=0}^{\delta_1}\binom{n-r}{\delta_1}\binom{\delta_1}{\delta_2}(-1)^{\delta_1} I,$$

$$I = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^1 c^{a+r_1-1} k^{r_1} \lambda^{2r_2} \alpha^{r_2}$$

$$\cdot \exp\left[\sum_{j=1}^{r_1} c \ln x_{1j} - \frac{k}{\beta_1} - \frac{\alpha}{\beta_2} - \frac{\lambda}{\beta_3} - \sum_{j=1}^{r_2} (\lambda x_{2j})^2 \right.$$

$$\left. + (\alpha - 1) \ln[1 - e^{-(\lambda x_{2j})^2}] \right]$$

$$- \sum_{j=1}^{r_1} k \ln(1 + x_{1j}^c) [(1 - (1 - e^{-(\lambda T)^2})^\alpha)]^{\delta_1 - \delta_2} \, dc dk d\alpha d\lambda dp.$$

The Bayesian estimators of the parameters c, k, α, λ, p, denoted by $\hat{c}$, $\hat{k}$, $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{p}$, are obtained, respectively, as follows:

$$\hat{c} \propto \int_0^\infty c \prod (\psi \mid x) dc, \quad \hat{k} \propto \int_0^\infty k \prod (\psi \mid x) dk, \quad \hat{\alpha} \propto \int_0^\infty \alpha \prod (\psi \mid x) d\alpha,$$

$$\hat{\lambda} \propto \int_0^\infty \lambda \prod (\psi \mid x) d\lambda \quad \text{and} \quad \hat{p} \propto \int_0^1 p \prod (\psi \mid x) dp.$$

Generally, it is difficult for $\hat{c}$, $\hat{k}$, $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{p}$ to have explicit solution. Therefore; the MCMC approach is applied to approximate evaluation for the previous integrals depending on Open BUGS software.

4. Numerical Study

In this section, numerical study is performed to investigate the performance of Bayesian estimator for c, k, λ, α and p under type I censored samples. The Gibbs sampling may be one of the best known MCMC sampling algorithms in Bayesian literature. This technique aims to find a Markov chain which has a limiting distribution of the target posterior, and then the simulated sample or chain can be used to compute any desired characteristic. The Gibbs sampling can be carried out using the Open

BUGS software, a specialized software package for implementing MCMC simulation and Gibbs sampling. The numerical study is carried out as follows:

Step (1) Generate random samples of sizes 5, 15, 25, 30, 50 and 100, from the BXIIBX mixture distribution.

Step (2) The posterior density functions for the unknown parameters *c*, *k*, λ, α and *p* are determined.

Step (3) Selected two chains with different initials as follows: $(c = 0.5,\ k = 0.5,\ \lambda = 1,\ \alpha = 0.5,\ p = 0.2)(c = 1.3,\ k = 1.5,\ \lambda = 1.3,\ \alpha = 0.75,\ p = 0.5)$. Fixed censoring times are chosen as $T = 0.5$.

Step (4) For each of the 10000 samples, the Bayes estimates are computed.

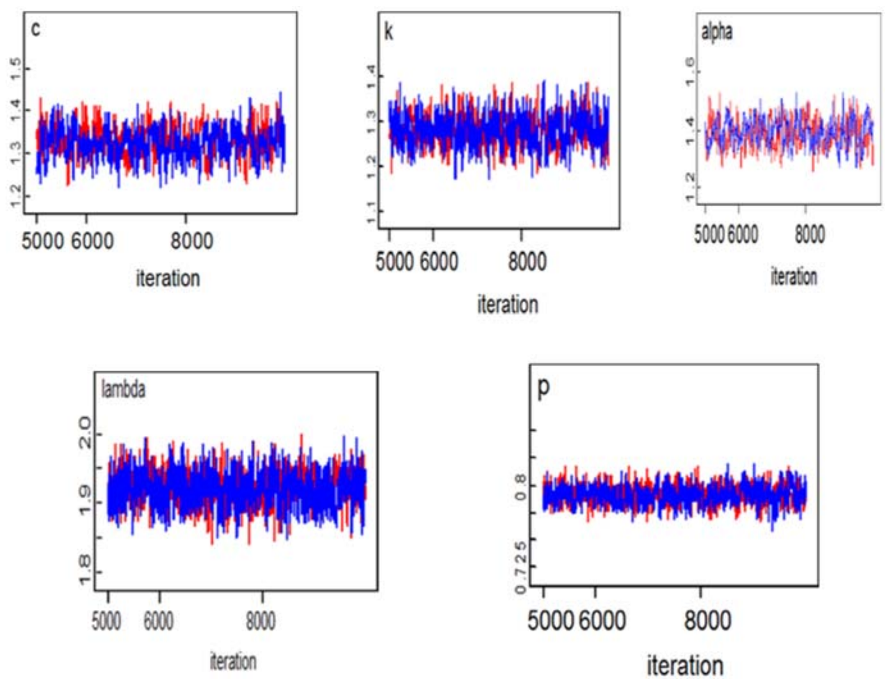

Figure 1. History plot for *c*, *k*, α, λ and *p* at $n = 50$.

Table 1. Simulation results

Sample size	T	Parameter	Posterior mean	S. D.	MC error	Median	2.50%	97.50%
5	0.5	$\hat{c}$	0.06174	0.00989	0.00075	0.06123	0.04355	0.08133
		$\hat{k}$	0.00683	0.00273	0.00009	0.00646	0.00237	0.01310
		$\hat{\alpha}$	0.04471	0.00700	0.00052	0.04433	0.03228	0.04433
		$\hat{\lambda}$	0.14000	0.03844	0.00303	0.13800	0.07747	0.22080
		$\hat{p}$	0.32480	0.11100	0.00874	0.30560	0.15720	0.55300
15		$\hat{c}$	0.25160	0.01468	0.00091	0.25080	0.22380	0.28170
		$\hat{k}$	0.19920	0.01420	0.00087	0.19860	0.17240	0.22860
		$\hat{\alpha}$	0.07831	0.01006	0.00046	0.07756	0.05963	0.10030
		$\hat{\lambda}$	0.21930	0.03190	0.00192	0.21880	0.15840	0.28530
		$\hat{p}$	0.85950	0.02288	0.00102	0.86040	0.81190	0.90280
25		$\hat{c}$	0.44320	0.02177	0.00109	0.44330	0.40110	0.48600
		$\hat{k}$	0.40890	0.02029	0.00083	0.40900	0.36970	0.44850
		$\hat{\alpha}$	0.46750	0.02546	0.00158	0.46630	0.42150	0.52050
		$\hat{\lambda}$	1.70200	0.08529	0.00447	1.70100	1.53700	1.86800
		$\hat{p}$	0.70620	0.01895	0.00088	0.70700	0.66700	0.74240
30		$\hat{c}$	0.40870	0.02054	0.00111	0.40810	0.36890	0.45060
		$\hat{k}$	0.37000	0.02012	0.00112	0.36920	0.33340	0.41040
		$\hat{\alpha}$	0.34770	0.02308	0.00139	0.34780	0.30300	0.39270
		$\hat{\lambda}$	1.03100	0.07871	0.00461	1.02800	0.88810	1.19900
		$\hat{p}$	0.82700	0.01524	0.00054	0.82740	0.79600	0.85550
50		$\hat{c}$	1.32800	0.03796	0.00195	1.32900	1.25200	1.40300
		$\hat{k}$	1.28300	0.03297	0.00167	1.28300	1.22100	1.34900
		$\hat{\alpha}$	1.38500	0.04263	0.00238	1.38400	1.30200	1.47100
		$\hat{\lambda}$	1.98850	0.09540	0.00570	1.96300	1.92300	2.12000
		$\hat{p}$	0.79220	0.00846	0.00035	0.79240	0.77540	0.80800
100		$\hat{c}$	1.49800	0.07406	0.00419	1.49800	1.35400	1.64000
		$\hat{k}$	1.79000	0.07763	0.00456	1.79100	1.63500	1.94000
		$\hat{\alpha}$	1.03600	0.08662	0.00510	1.03600	1.86900	1.20100
		$\hat{\lambda}$	1.99400	0.07257	0.00120	1.99400	1.94200	2.04300
		$\hat{p}$	0.69970	0.00446	0.00014	0.69980	0.69100	0.70830

The posterior descriptive summary results are displayed in Table 1 and represented in Figures 1 to 3. Table 1 contains posterior mean, standard deviation (S. D.), MC error (the computational accuracy of the mean), median, 2.5% (an approximation of the lower endpoint of 95% credible interval), and 97.5% (an approximation of the upper endpoint of the 95% credible interval). The following conclusions can be observed on the properties of estimated parameters:

As an example, for $n = 50;$ history plots for unknown parameters in Figure 1 looks like a horizontal band with no long upward or downward trends and two chains are mixed so these are indicators to convergence.

The autocorrelation for parameters c, k, α, λ and p in Figure 2 displays that the correlation is approximately omitted, in the sense that, the samples are independent.

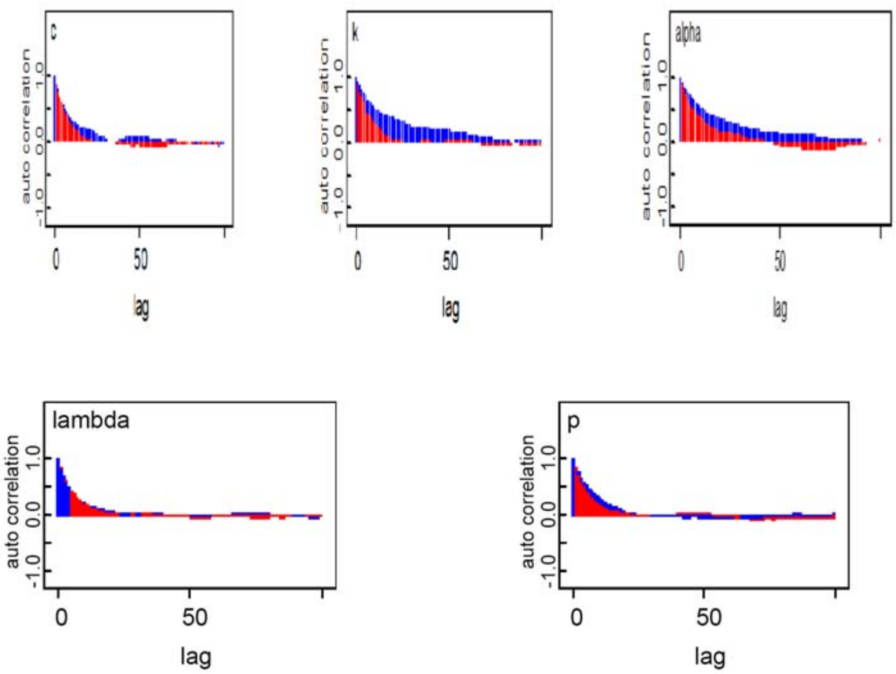

Figure 2. Autocorrelation plot for c, k, α, λ and p at $n = 50$.

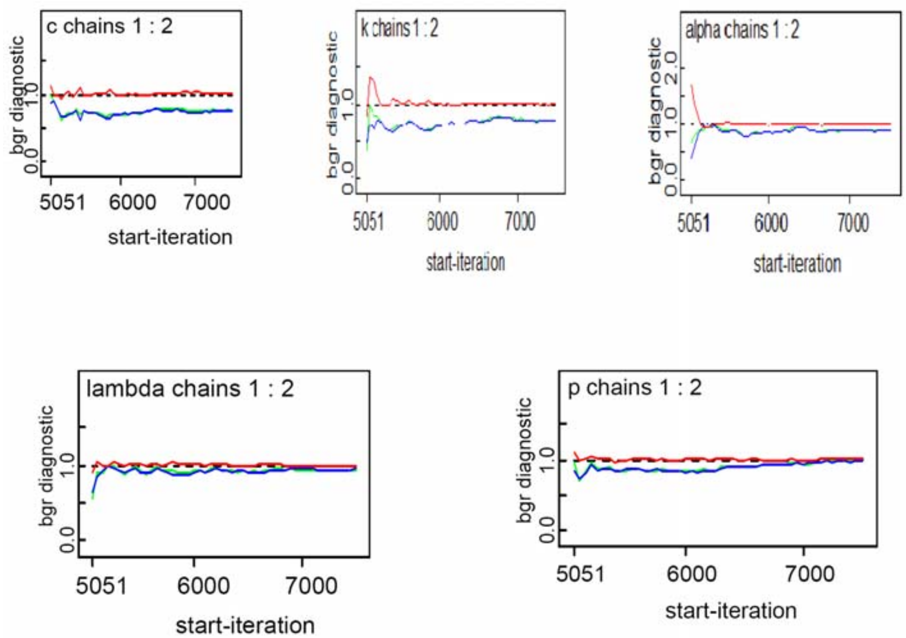

Figure 3. Gelman-Rubin convergence plot of c, k, α, λ and p at $n = 50$.

Figure 3 gives plots of the ratio for Gelman-Rubin convergence for parameters c, k, α, λ and p of the two chains. Plots show that the chains approximate to equal one, i.e., convergence is available.

5. Concluding Remarks

In this article, Bayesian analysis of the mixed Burr type XII and Burr type X distributions was developed under type I censored samples. The simulation study was performed for two chains with different initial values. Bayesian estimates for unknown parameter of BXIIBX model are calculated numerically using Open BUGS software. Bayesian analysis was conducted to estimate posterior descriptive for unknown parameter for different sample sizes. The Bayesian estimators for unknown parameters c, k, α, λ and p have good statistical properties. Further, the MC error is less than 5% of the sample standard deviation.

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