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Estimation and Prediction from Inverse Rayleigh Distribution Based on Lower Record Values

A. Soliman[1], Essam A. Amin[1,a] and Alaa A. Abd-El Aziz[1]

[a] e-mail: ess_amin@yahoo.com

Abstract

This article discusses Bayesian and non-Bayesian estimation problem of the unknown parameter for the inverse Rayleigh distribution based on lower record values. Maximum likelihood estimator of the unknown parameters were obtained. Also, Bayes estimator have been developed under squared error and zero one loss functions. These estimators are derived using the informative prior distribution. Bayesian and non Bayesian interval estimation for the inverse Rayleigh parameters are obtained. Furthermore, Bayesian prediction interval of the future record values are discussed and obtained. Finally practical example using simulated record values are given to illustrate the theoretical results of prediction interval.

Keywords: Bayesian estimation, Lower record values, Maximum likelihood procedure, Inverse Rayleigh distribution, Bayesian Prediction

1. Introduction

The inverse Rayleigh distribution has many applications in the area of reliability studies. Voda (1972) mentioned that the distribution of lifetimes of several types of experimental units can be approximated by the inverse Rayleigh distribution.

The probability density function of the inverse Rayleigh distribution with scale parameter θ is

$$f(x) = (\frac{2\theta}{x^3})\exp(-\frac{\theta}{x^2}) \qquad x,\theta > 0 \qquad (1.1)$$

The corresponding cumulative distribution function is,

[1] I.S.S.R. Cairo univrsity

$$F(x) = \exp(-\frac{\theta}{x^2}) \qquad x, \theta > 0 \qquad (1.2)$$

Voda (1972) presented some properties of the maximum likelihood estimator, for inverse Rayleigh distribution, furthermore confidence intervals and tests of hypotheses are developed. Gharraph (1993) derived five measures of location for the inverse Rayleigh distribution. These measures are the mean, harmonic mean, geometric mean, mode, and the median. He also, estimated the unknown parameter using different methods of estimation. A comparison of these estimators was discussed numerically in term of their bias and root mean square error. Abdel-Monem (2003) developed some estimation and prediction results for the inverse Rayleigh distribution. El-Helbawy and Abdel-Monem (2005) obtained Bayesian estimators of the parameter of the inverse Rayleigh distribution under four loss functions. Bayesian of one and two sample predictions are also developed including point predictions and prediction intervals.

Record values and associated statistics are of great importance in several real live problems involving weather, economic, and support data. The statistical study of record values started with Chandler (1952) and now spread in different directions. Interested readers may refer to Foster and Stuart (1954), Galambos (1978), Dunsmore(1983), Nagaraja (1988), Ahsanullah (1994) and Aronld et al (1992) Ahmed and Fahad (2007) for a review of developments in this area of research. While a lot of work has been done on characterizations, asymptotic theory and generalizations, not much has been done on statistical inference based on record values.

The objective of this paper is to show how record values can be used to develop a methodology to construct and compute Bayesian and non Bayesian estimation and prediction. The lower record values from inverse Rayleigh population based on a set of lower record values will be considered. The Bayes and maximum likelihood estimator for the scale parameter were derived. Bayes point and interval estimators are derived assuming informative priors on the parameter. This can achieved with respect to squared error and zero one error loss functions. Bayesian prediction bounds for the future record values on the basis of first observed records are obtained. Simulated record values are used to illustrate the application of the results through Mathcad (2001).

2. Record Values and Maximum Likelihood Estimation

Let $X_1, X_2, ...$ be an infinite sequence of independent and identically distributed random variables having probability density function (1.1). Consider $R_0, R_1, ..., R_m$ represent the first $(m+1)$ lower record values from the same density function the joint probability density function of $R_0, R_1, ..., R_m$ is;

$$f(R_0, R_1, ...R_m) = \frac{(2\theta)^{m+1}}{\prod_{i=0}^{m} r_i^3} \exp(\frac{-\theta}{r_m^2}), \qquad 0 < r_m < r_{m-1} < \cdots < r_0 < \infty, \qquad (2.1)$$

which is the likelihood function based on the first $(m+1)$ lower record values. Therefore the probability density function of the first $(m+1)$ lower record values from inverse Rayleigh distribution will be

$$f_{R_m}(r_m) = \frac{2}{m!\, r_m}(\frac{\theta}{r_m^2})^{m+1} \exp(-\frac{\theta}{r_m^2}) \qquad r_m > 0, \quad \theta > 0 \qquad (2.2)$$

In addition, the joint probability density R_i and R_j based on the inverse Rayleigh distribution is

$$f_{R_i, R_j}(r_i, r_j) = \frac{4\theta^{j+1}}{i!(j-i-1)!} \cdot \frac{1}{r_i^{2i+3}\, r_j^3}[\frac{1}{r_j^2} - \frac{1}{r_i^2}]^{j-i-1} e^{-\frac{\theta}{r_j^2}} \qquad 0 < r_j < r_i < \infty \qquad (2.3)$$

Also the r-th moment about origin for $(m+1)$ lower record values from inverse Rayleigh distribution is given as follows

$$E(r_m^k) = \frac{\theta^{\frac{k}{2}}}{m!} \Gamma\left(\frac{2m-k+2}{2}\right), \qquad k < 2m+2 \qquad (2.4)$$

The mean and variance of the $(m+1)$ lower record values can easily obtained using equation (2.4). Taking the logarithm of the likelihood function (2.1),

$$\ln(L(\theta; \underline{r})) = (m+1)\ln(2\theta) - \frac{\theta}{r_m^2} - \ln(\prod_{i=0}^{m} r_i^3). \qquad (2.5)$$

Differentiate both side of equation (2.8) with respect to the parameter θ and equating with zero , then the maximum likelihood estimate of θ under lower record value, say $\hat{\theta}$, is given by

$$\hat{\theta} = (m+1) r_m^2 . \qquad (2.6)$$

In addition, the expected value for the estimated parameter $\hat{\theta}$ and its variance are given as follows,

$$E(\hat{\theta}) = (\frac{m+1}{m})\theta, \qquad (2.7)$$

and,

$$\mathrm{var}(\hat{\theta}) = (\frac{1}{m-1})(\frac{m+1}{m}\theta)^2. \qquad (2.8)$$

It is clear from equation (2.7) that the maximum likelihood estimate $\hat{\theta}$ is biased estimate for the parameter θ.

Now consider the following pivotal quantity

$$y = (\frac{m+1}{\hat{\theta}}\theta).$$

It is easy to prove that y has a gamma distribution with parameters $(m+1,1)$. Using the fact that $2y$ has a chi-square distribution with $2(m+1)$ degrees of freedom $\chi^2_{2(m+1)}$, then $100(1-\alpha)\%$ confidence interval for the parameter θ based on the first m+1 lower record values is (L_1, U_1) where

$$L_1 = \frac{\hat{\theta}}{(m+1)}\chi^2_{2(m+1),1-\frac{\alpha}{2}} \quad \text{and} \quad U_1 = \frac{\hat{\theta}}{(m+1)}\chi^2_{2(m+1),\frac{\alpha}{2}} \tag{2.9}$$

where, $\hat{\theta}$ is the maximum likelihood given by equation (2.6)

3. Bayesian Estimation

This section is concerned with the problem of obtaining Bayesian estimators for the scale parameter from the inverse Rayleigh distribution. The prior knowledge which is adequately represented by the natural conjugate prior distribution under two loss functions will be developed.

Let $R_0, R_1, ..., R_m$ be the available $(m+1)$ lower record values from the inverse Rayleigh distribution. Consider the following informative prior distribution for the scale parameter θ

$$\pi_1(\theta) = ae^{-a\theta} \qquad , a > 0 \ , \theta > 0 \tag{3.1}$$

Since, the posterior probability density function, $\pi_1(\theta|\underline{r})$ is obtained by combining the likelihood given in equation (2.4) and the prior probability density $\pi_1(\theta)$, then

$$\pi_1(\theta|\underline{r}) = k\frac{a(2\theta)^{m+1}}{\prod_{i=0}^{m} r_i^3}\exp[-(\frac{1}{r_m^2}+a)\theta] \qquad ,\theta > 0 \tag{3.2}$$

where, the normalizing constant k is given by $k = 1/\int_0^\infty \pi_1(\theta|\underline{r})\,d\theta$.

Put, $A = (\frac{1}{r_m^2}+a)$. Therefore,

$$\pi_1(\theta|\underline{r}) = \frac{A(A\theta)^{m+1}}{\Gamma(m+2)}\exp(-A\theta), \quad \theta > 0. \tag{3.3}$$

where, $\Gamma(.)$ is the gamma function. It follows that the scale parameter θ has gamma distribution with parameters, $(m+2, A)$.

The squared error loss is appropriate when decision becomes gradually more damaging for larger errors. The Bayesian estimator of θ under squared error loss function is the posterior mean and is given by

$$\tilde{\theta}_1 = E(\theta|\underline{r}) = \frac{(m+2)}{A}. \tag{3.4}$$

The Bayes estimator of θ with respect to zero one loss function is the posterior mode which is given by

$$\tilde{\theta}_2 = \frac{d(\ln \Pi_1(\theta|\underline{r}))}{d\theta} = \frac{(m+1)}{A}. \tag{3.5}$$

The highest posterior density interval is such that the posterior density for every point inside the interval is greater than that for every point outside of it so that the interval includes the more probable values of the parameter and excludes the less probable ones. From the posterior density for the parameter θ obtained in equation (3.3) and using the fact that $2A\theta$ has $\chi^2_{2(m+2)}$, then a $100(1-\alpha)\%$ Bayesian interval estimation for the parameter θ based on the first m+1 lower record values is (L_θ, U_θ) , where

$$L_\theta = \frac{\chi^2_{(2m+4, 1-\alpha/2)}}{2A} \quad \text{and} \quad U_\theta = \frac{\chi^2_{(2m+4, \alpha/2)}}{2A} \tag{3.6}$$

4-Bayesian Prediction

In the context of prediction of the future records, the prediction intervals provide bounds to contain the results of a future record, based upon the results of the previous record observed from the same distribution. Prediction problems coms up natuarlly in several real life situations, for example, Ahsanullah (1980), Nagraja (1984) and Doganakosoy and Balakrishnan (1997). Using generalized model, Baysian prediction interval for interval for future generalized order statistics (including record values as aspecial cases) was studied by Al-Hussaini and Ahmed (2003). Madi and Raqab (2004) considered the problem of Bayesian prediction of temperature record using Pareto model. Assuming that $R_0, R_1, ..., R_m$ are the lower record values from the inverse Rayleigh distribution, Balakrishnan and Chan (1994) obtained the best linear unbiased predictor (BLUP) of the $(m+1)$ future record based on the first m records.

This section is devoted to deriving Bayes predictive density function, which is necessary to obtain bounds for the predictive interval of future records. Let $R_0, R_1, ..., R_m$ be the first $(m+1)$ observed lower record values from inverse Rayleigh distribution. Based on such a record sample, Bayesian prediction is nedded for the sth future record R_s, $1 < m < s$. The conditional pdf of R_s given R_m is given by Ahsanulla (1995) in the form

$$f_{R_s|R_m}(r_s|r_m) = \frac{[\ln F(r_m) - \ln F(r_s)]^{s-m-1}}{s-m-1!} \frac{f(r_s)}{F(r_m)} \quad -\infty < r_s < r_m < \infty \qquad (4.1)$$

For the inverse Rayleigh distribution with probability density function (1.1) and cumulative density function (1.2), the function $f_{R_s|R_m}(r_s|r_m)$ becomes

$$f_{R_s|R_m}(r_s|r_m) = \frac{2\theta^{s-m}}{s-m-1!\, r_s^3}[\xi(r_s)]^{s-m-1} e^{-\theta\,\xi(r_s)} \quad , \quad -\infty < r_s < r_m < \infty \qquad (4.2)$$

where $\xi(r_s) = \left(\frac{1}{r_s^2} - \frac{1}{r_m^2}\right)$.

Using the fact that the record values form the Makove property, the conditional density function of R_s given $\underline{R} = (R_0, R_1, \ldots, R_m)$ is just the probability density function of R_s given R_m. The predective density function of R_s given $\underline{R}$ is

$$f^*(r_s|\underline{r}) = \int f_{R_s|R_m}(r_s|r_m).\pi(\theta)d\theta \cdot \qquad (4.3)$$

From (3.1), (4.2) and (4.3), the predictive density function can simplified as

$$f^*(r_s|\underline{r}) = \frac{2A^{m+2}}{\beta(s-m, m+2)\, r_s^3}[\xi(r_s)]^{s-m-1}[A + \xi(r_s)]^{-(s+2)} \qquad r_s < r_m, \qquad (4.4)$$

where $\beta(.,.)$ is the beta function. Prediction bounds on R_s is given from the following equation

$$P(r_s > \lambda \mid \underline{r}) = \int_{\lambda}^{r_m} f^*(r_s \mid \underline{r}) dr_s = \frac{InBet(s-m, s+2; \eta)}{\beta(s-n, s-2)} \quad (4.5)$$

Where $\eta = \frac{r_m^2 - \lambda^2}{\lambda^2 r_m^2 A}$, and $InBet(z_1, z_2; \eta)$ is the incomplete beta function defined by $InBet(z_1, z_2; \eta) = \int_0^{\eta} t^{z_1 - 1} (1-t)^{-(z_1 + z_2)} dt$. The predictive bounds of a two sided interval with cover $(1-\alpha)$, for the future lower record values R_s may be obtained by numerical solving for the following two equations, for the lower bound L and the upper bound U :

$$P(r_s > L \mid \underline{r}) = 1 - \alpha/2 ; \quad P(r_s > U \mid \underline{r}) = \alpha/2 . \quad (4.6)$$

Special case, when $s = m+1$ then the lower and upper bounds for the future lower record values R_{m+1} with cover $(1-\alpha)$ will be

$$L = \left[(\alpha/2)^{-1/(m+2)} A - a \right]^{-1/2} \text{ and } U = \left[(1-\alpha/2)^{-1/(m+2)} A - a \right]^{-1/2}$$

5 Numerical Illustrations

In this section numerical results of the Bayesian predictive interval for several values of different prior parameters will be obtained. In this case, the parameter θ has the gamma prior given by (3.1) with known parameter *a*. The calculations are carried out according to the following steps:

(1) For given values of the inverse Rayleigh parameters θ generate a random variable X from the inverse Rayleigh distribution (1.1) and selected the first 10 records .

(2) Consider the first 6 records as the observed upper records (*m*=5), while the last six records as the unobserved records, which are to be predicted.

(3) Using Mathcad (2001) program applying equations (4.6) to obtain the 95% equal tail Bayesian prediction interval for the s^{th} upper record values for m=5 and s=6 and for several different values of the prior parameters (a=0.5,1,2,3,5,7) and Tables 1 and 2 contained the results which show

(i) the values of the prior parameters a

(ii) the 95% Bayesian prediction interval for the s^{th} records,

(iii) the lengths of the prediction interval.

Steps 2 and 3 are repeated for the value m=4, 3, 2 and the results are presented in tables 3 to 8.

Table (1)

m=5				
Exact	prior	L	U	length
	0.5	0.845	1.276	0.431
	1	0.754	1.274	0.52
	2	0.637	1.27	0.633
	3	0.562	1.266	0.704
	5	0.468	1.258	0.79
X6=1.154	7	0.41	1.25	0.84

Baysian prediction interval for the s=6 future record

Table (2)

m=4				
Exact	prior	L	U	length
	0.5	0.859	1.428	0.569
	1	0.747	1.425	0.678
	2	0.614	1.418	0.804
	3	0.534	1.411	0.877
	5	0.438	1.398	0.96
X5=1.28	7	0.38	1.385	1.005

Baysian prediction interval for the s=5 future record

Table (3)

m=3				
Exact	prior	L	U	length
	0.5	0.856	1.639	0.783
	1	0.72	1.633	0.913
	2	0.574	1.62	1.046
	3	0.492	1.607	1.115
	5	0.398	1.583	1.185
X4=1.435	7	0.343	1.56	1.217

Baysian prediction interval for the s=4 future record

Table (4)

m=2				
Exact	prior	L	U	length
	0.5	0.825	1.983	1.158
	1	0.667	1.966	1.299
	2	0.514	1.935	1.421
	3	0.434	1.906	1.472
	5	0.346	1.853	1.507
X3=1.65	7	0.297	1.806	1.509

Baysian prediction interval for the s=3 future record

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