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Bayesian Estimation of Power Transmuted Inverse Rayleigh Distribution

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Abstract

In this article, Bayesian estimators of the population parameters of the power transmuted inverse Rayleigh (PTIR) distribution are discussed. The posteriors distribution of the PTIR distribution based on informative and non-informative priors represented by gamma and Jeffery's priors, respectively, are derived. Four loss functions, namely minimum expected, squared error, precautionary and linear exponential are considered. The highest posterior density credible interval is constructed by using the Markov Chain Monte Carlo (MCMC) method. Simulation study is performed to examine and compare the Bayes estimates using MCMC method based on Random Walk Metropolis-Hastings (RWMH) sampling algorithms. The results of the study show that the Bayes estimates under minimum expected loss function in case of non-informative prior are preferable than the other estimates in approximately most of the situations. While, the Bayes estimates under squared error loss function in case of informative prior are superior to the other estimates in approximately most of the situations.

Keywords: Power transformation, informative prior, squared error loss function, precautionary loss function, Markov Chain Monte Carlo.

1. Introduction

One of the widely-used statistical distributions in the context of reliability studies is the inverse Rayleigh (IR) distribution as introduced by Trayer (1964). Different works have been used for the IR distribution for various purposes. For example, Howlader et al. (2008) used a Bayesian approach to predict the bounds for Rayleigh and IR lifetime models. Aslam et al. (2009) designed an acceptance sampling plan from a truncated life test when the lifetime of an item followed either an IR or a log-logistic distribution. Soliman et al. (2010) discussed Bayesian and non-Bayesian estimators of parameter for the IR distribution based on record values. Sindhu et al. (2013) obtained a Bayesian estimator of the IR parameter in left censored data under different loss functions. Bayesian estimators of parameter and reliability function for the IR distribution using informative prior (IP) and non-informative prior (NIP) have been provided by Rasheed and Aref (2017). Recently Ahmed et al. (2014) studied the transmuted IR (TIR) distribution and discussed its theoretical properties. The cumulative distribution function (cdf) of the TIR distribution is given by

$$F(y;\theta,\lambda)=e^{-\theta y^{-2}}\left(1+\lambda-\lambda e^{-\theta y^{-2}}\right);\ \theta>0,|\lambda|\leq 1,\ y>0. \tag{1}$$

where θ and λ are the scale parameters. The probability density function (pdf) corresponding to (1) is given by

$$f(y;\theta,\lambda)=2\theta y^{-3}e^{-\theta/y^2}\left(1+\lambda-2\lambda e^{-\theta/y^2}\right);\ \theta>0,|\lambda|\leq 1,\ y>0. \tag{2}$$

Several generalizations and extended forms of the IR distribution have been provided by several authors. For example, modified IR distribution (Khan 2014), transmuted modified IR distribution (Khan and King 2015), transmuted exponentiated IR distribution (Haq 2015), Kumaraswamy exponentiated IR distribution (Haq 2016), weighted IR distribution (Fatima and Ahmad 2017) and odd Fréchet IR distribution (Elgarhy and Alrajhi 2018). More recently, Hassan et al. (2019) introduced power TIR (PTIR) with an extra shape parameter as a new generalized form of the TIR distribution. The PTIR is obtained depending on the transformation, $X=Y^{1/\beta}$, where the random variable Y follows the TIR distribution (2). The cdf of a random variable X has the PTIR distribution is defined as

$$F(x;\theta,\lambda,\beta)\ =\left(e^{\frac{-\theta}{x^{2\beta}}}+\lambda e^{\frac{-\theta}{x^{2\beta}}}-\lambda e^{\frac{-2\theta}{x^{2\beta}}}\right);\ \theta,\beta>0,\ |\lambda|\leq 1,\ x>0, \tag{3}$$

where θ and λ are scale parameters and β is the shape parameter. The pdf of the PTIR distribution corresponding to (3) is given by

$$f(x;\theta,\lambda,\beta)=\frac{2\theta\beta}{x^{2\beta+1}}e^{\frac{-\theta}{x^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x^{2\beta}}}\right);\ \theta,\beta>0,\ |\lambda|\leq 1, x>0. \tag{4}$$

The pdf (4) generalizes both the IR and TIR models. They discussed several properties of the PTIR distribution and estimated the model parameters through maximum likelihood, least squares and percentiles methods.

This paper concerns with Bayesian estimators of the unknown parameters θ and β of the PTIR distribution while assuming λ to be known. The Bayesian estimators and credible intervals are derived by considering informative priors (independent gamma prior) and non-informative priors (Jeffrey's prior). The Bayesian estimators are motivated by four loss functions which are minimum expected loss (MEL) function, squared error loss (SEL) function, precautionary loss (PL) function and linear exponential (LINEX) loss function. The Markov Chain Monte Carlo (MCMC) method is implemented for investigating the accuracy of estimates for different sample sizes. Simulation study is performed based on relative absolute biases, estimated risk and the width of credible intervals in order to examine and compare the behavior of the parameters' Bayesian estimates.

The rest of the paper is organized as follows; Bayesian estimators of θ and β based on non-informative priors under MEL function, SEL function, PL function and LINEX loss function are derived in Section 2. In Section 3, Bayesian estimators of θ and β based on informative priors under four loss functions are derived. Credible intervals of the Bayesian estimators regarding non-informative and informative priors are investigated in Section 4. In Section 5, the MCMC method is conducted based on Random Walk Metropolis-Hastings (RWMH) algorithm to compare the efficiency of the resulting estimates. Finally, the simulation results are provided in Section 6.

2. Bayesian Estimators in Case of Non-informative Priors

In this section Bayesian estimators are obtained assuming the scale parameter θ and the shape parameter β have uniform distribution while considering the transmuted parameter λ is a known under MEL function, SEL function, PL function and LINEX loss function.

A non-informative prior represented in Jeffrey's prior is proposed for parameters θ and β. Assuming independence of parameters, hence the joint prior distribution for θ and β is given by

$$\pi_1(\theta,\beta|\underline{x}) = \frac{1}{\theta\beta}. \tag{5}$$

The expression for the joint posterior can be written as

$$h_{1,2}(\theta,\beta|\underline{x}) = \frac{L(\theta,\beta|\underline{x})\,\pi_1(\theta,\beta|\underline{x})}{\int_\theta\int_\beta L(\theta,\beta|\underline{x})\,\pi_1(\theta,\beta|\underline{x})\;d\theta d\beta},$$

where the likelihood function of the PTIR is given by

$$L = (2\theta\beta)^n \prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right). \tag{6}$$

Hence, the joint posterior $h_{1,2}(\theta,\beta|\underline{x})$ of the parameters θ and β is obtained by using likelihood function (6) and joint prior density (5) as follows:

$$h_{1,2}(\theta,\beta|\underline{x}) \propto \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right).$$

Thus, the marginal posterior distributions of θ and β take the following forms

$$h_1(\theta|\underline{x}) = K\theta^{n-1}\int_0^\infty \beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\beta,$$

and

$$h_2(\beta|\underline{x}) = K\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)}\int_0^\infty \theta^{n-1}\prod_{i=1}^{n} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta,$$

where

$$K^{-1} = \int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta.$$

2.1. Bayesian estimators under MEL function

The MEL function is suggested by Tummala and Sathe (1978) and it is considered to be a special case of the widely used quadratic loss function which is given by

$$\ell_{MEL}(\vartheta,\hat{\vartheta}) = \varphi(\vartheta-\hat{\vartheta})^2. \tag{7}$$

If $\varphi = 1$ in (7), then it reduces to SEL function and for $\varphi = \vartheta^{-2}$ it becomes

$$\ell_{MEL}(\vartheta,\hat{\vartheta}) = \vartheta^{-2}(\vartheta-\hat{\vartheta})^2.$$

Based on MEL function, the Bayesian estimator of the unknown parameters is given by

$$\hat{\vartheta}_{MEL} = \frac{E(\vartheta^{-1}|\underline{x})}{E(\vartheta^{-2}|\underline{x})} = \frac{\int_0^\infty \vartheta^{-1}\;h(\vartheta|\underline{x})\;d\vartheta}{\int_0^\infty \vartheta^{-2}\;h(\vartheta|\underline{x})\;d\vartheta}, \tag{8}$$

where $\hat{\vartheta}_{MEL}$ is the Bayesian estimator for ϑ under MEL function. Considering (8), the Bayesian estimator of θ under MEL function, say $\hat{\theta}_{MEL}$, is obtained as follows:

$$\hat{\theta}_{MEL}=\frac{E(\theta^{-1}|\underline{x})}{E(\theta^{-2}|\underline{x})}=\frac{\int_0^\infty \theta^{-1}h_1(\theta|\underline{x})\,d\theta}{\int_0^\infty \theta^{-2}h_1(\theta|\underline{x})\,d\theta}.$$

Hence,

$$\hat{\theta}_{MEL}=\frac{\int_0^\infty\int_0^\infty \theta^{n-2}\beta^{n-1}\prod_{i=1}^{n}x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-3}\beta^{n-1}\prod_{i=1}^{n}x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}. \tag{9}$$

Similarly, the Bayesian estimator of β under MEL function, say $\hat{\beta}_{MEL}$, is given by

$$\hat{\beta}_{MEL}=\frac{E(\beta^{-1}|\underline{x})}{E(\beta^{-2}|\underline{x})}=\frac{\int_0^\infty \beta^{-1}h_2(\beta|\underline{x})\,d\beta}{\int_0^\infty \beta^{-2}h_2(\beta|\underline{x})\,d\beta}.$$

Hence $\hat{\beta}_{MEL}$ is obtained as

$$\hat{\beta}_{MEL}=\frac{\int_0^\infty\int_0^\infty \beta^{n-2}\theta^{n-1}\prod_{i=1}^{n}x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \beta^{n-3}\theta^{n-1}\prod_{i=1}^{n}x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}. \tag{10}$$

Integrals (9) and (10) can't be solved analytically, since they have not a closed form. So, the RWMH algorithm will be used to obtain the Bayesian estimator of θ and β under MEL function.

2.2. Bayesian estimators under SEL function

The SEL is considered to be one of the most useful symmetric loss functions, it is defined by

$$\ell_{SEL}(\vartheta,\hat{\vartheta})=(\vartheta-\hat{\vartheta})^2.$$

The Bayesian estimator of the ϑ under SEL function is given by

$$\hat{\vartheta}_{SEL}=E(\vartheta|\underline{x})=\int_0^\infty \vartheta h(\vartheta|\underline{x})\,d\vartheta, \tag{11}$$

where $\hat{\vartheta}_{SEL}$ is the Bayesian estimator for ϑ under SEL function. Regarding (11), the Bayesian estimator of θ under SEL function, denoted by $\hat{\theta}_{SEL}$ can be obtained as posterior mean as follows:

$$\hat{\theta}_{SEL}=\frac{\int_0^\infty\int_0^\infty \theta^{n}\beta^{n-1}\prod_{i=1}^{n}x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta\, d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n}x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta\, d\beta}. \tag{12}$$

By similar way, the Bayesian estimator of β under SEL function, denoted by $\hat{\beta}_{SEL}$, can be obtained as posterior mean as follows

$$\hat{\beta}_{SEL}=\frac{\int_0^\infty\int_0^\infty \beta^{n}\theta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)}\; e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta\, d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)}\; e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta\, d\beta}. \tag{13}$$

Integrals (12) and (13) are difficult to obtain, so RWMH is used to compute the Bayes' estimators.

2.3. Bayesian estimators under PL function

A very useful and simple asymmetric PL function is defined as follows

$$\ell_{PL}(\hat{\vartheta},\vartheta)=\frac{(\vartheta-\hat{\vartheta})^2}{\hat{\vartheta}}.$$

The Bayesian estimators of the unknown parameters under PL function is given by

$$\hat{\vartheta}_{PL}=\left\{E(\vartheta^2|\underline{x})\right\}^{1/2}=\sqrt{\int_0^\infty \vartheta^2 h(\vartheta|\underline{x})d\vartheta}, \tag{14}$$

where $\hat{\vartheta}_{PL}$ is the Bayesian estimator for ϑ under PL function. The Bayesian estimator of θ and β under PL function, say $\hat{\theta}_{PL}$ and $\hat{\beta}_{PL}$, are obtained as follows

$$\hat{\theta}_{PL}=\sqrt{\frac{\int_0^\infty\int_0^\infty \theta^{n+1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}}, \tag{15}$$

and

$$\hat{\beta}_{PL}=\sqrt{\frac{\int_0^\infty\int_0^\infty \beta^{n+1}\theta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)} e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}}. \tag{16}$$

The integrals involved in (15) and (16) are not solvable analytically and therefore RWMH algorithm is applied to obtain $\hat{\theta}_{PL}$ and $\hat{\beta}_{PL}$.

2.4. Bayesian estimators under LINEX loss function

Klebanov (1972) introduced the LINEX loss function as asymmetric loss function. The LINEX loss function with parameters v and w is defined by

$$\ell_{LINEX}(\vartheta,\hat{\vartheta})=w\left[e^{v(\hat{\vartheta}-\vartheta)}-v(\hat{\vartheta}-\vartheta)-1\right],$$

where, $w>0$ and $v\neq 0.$ The constant v determines the direction and shape (degree of symmetry) of loss function. The posterior risk corresponding to this loss function is given by

$$\hat{\vartheta}_{LINEX}=-\frac{1}{v}\ln E\left[e^{-v\vartheta}\right]=-\frac{1}{v}\ln\left[\int_0^\infty e^{-v\vartheta}h(\vartheta|\underline{x})\right]d\vartheta, \tag{17}$$

where $\hat{\vartheta}_{LINEX}$ is the Bayesian estimator for ϑ under LINEX loss function. Based on (17), the Bayesian estimator of θ under LINEX loss function, denoted by $\hat{\theta}_{LINEX}$, can be obtained as follows

$$\hat{\theta}_{LINEX} = -\frac{1}{v}\ln\left[\frac{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}e^{-v\theta}\prod_{i=1}^{n} x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}\right]. \quad (18)$$

Similarly, the Bayesian estimator of β under LINEX loss function, denoted by $\hat{\beta}_{LINEX}$, can be obtained as follows

$$\hat{\beta}_{LINEX} = -\frac{1}{v}\ln\left[\frac{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\ e^{-v\beta}\prod_{i=1}^{n} x_i^{-(2\beta+1)}\ e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^{n} x_i^{-(2\beta+1)}\ e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}\right]. \quad (19)$$

The integrals (18) and (19) are very complicated to obtain, so the RWMH algorithm will be utilized to obtain $\hat{\theta}_{LINEX}$ and $\hat{\beta}_{LINEX}$.

3. Bayesian Estimator in Case of Informative Priors

In this section Bayesian estimators will be obtained assuming the scale parameter θ and the shape parameter β have gamma priors while considering the transmuted parameter λ is known.

Following Rasheed and Aref (2017) and Prakash (2013), the gamma priors for θ and β are suggested with the following pdfs

$$\pi_2(\theta|a_1,b_1) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta},\ \ \theta>0,\ a_1>0, b_1>0, \quad (20)$$

and

$$\pi_3(\beta|a_2,b_2) = \frac{b_2^{a_2}}{\Gamma(a_2)}\beta^{a_2-1}e^{-b_2\beta},\ \ \beta>0,\ a_2>0, b_2>0, \quad (21)$$

where a_1, a_2, b_1 and b_2 are the hyper parameters. Assuming independence of parameters, the joint prior distribution of parameters θ and β can be obtained by combining (20) and (21) to be

$$\pi_4(\theta,\beta|\,\underline{x}) = \frac{b_1^{a_1}\ b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\theta^{a_1-1}\beta^{a_2-1}e^{-b_1\theta}e^{-b_2\beta}, \quad (22)$$

where a_i and b_i are assumed to be known for $i=1,2.$ The joint posterior distribution of parameters θ and β is defined as follows

$$h_{3,4}(\theta,\beta|\,\underline{x}) = \frac{L(\theta,\beta|\,\underline{x})\pi_4(\theta,\beta|\underline{x})}{\int_\theta\int_\beta L(\theta,\beta|\,\underline{x})\pi_4(\theta,\beta|\,\underline{x})\ d\theta d\beta}.$$

Hence, $h_{3,4}(\theta,\beta|\underline{x})$ can be obtained by using likelihood function (6) and joint prior (22) as follows

$$h_{3,4}(\theta,\beta|\,\underline{x}) \propto \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^{n} x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right).$$

Hence, the marginal posterior distributions of θ and β take the following forms

$$h_3(\theta|\underline{x}) = C\theta^{a_1+n-1}e^{-b_1\theta}\int_0^\infty \beta^{a_2+n-1}e^{-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\beta,$$

and

$$h_4(\beta|\underline{x}) = C\beta^{a_2+n-1}e^{-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}\int_0^\infty \theta^{a_1+n-1}e^{-b_1\theta}\prod_{i=1}^n e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta,$$

where

$$C^{-1}=\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta.$$

3.1. Bayesian estimators under MEL function

Here, the Bayesian estimator of θ and β under MEL function are derived. Considering (8), the Bayesian estimator of θ denoted by $\overline{\theta}_{MEL}$ is obtained as follows

$$\overline{\theta}_{MEL} = \frac{E(\theta^{-1}|\underline{x})}{E(\theta^{-2}|\underline{x})} = \frac{\int_0^\infty\int_0^\infty \theta^{a_1+n-2}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-3}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}. \quad (23)$$

By similar way, the Bayesian estimator of β denoted by $\overline{\beta}_{MEL}$ is obtained as follows

$$\overline{\beta}_{MEL} \frac{E(\beta^{-1}|\underline{x})}{E(\beta^{-2}|\underline{x})} = \frac{\int_0^\infty\int_0^\infty \beta^{a_2+n-2}\theta^{a_1+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \beta^{a_2+n-3}\theta^{a_1+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}. \quad (24)$$

The integrals (23) and (24) are very hard to obtain, so the RWMH algorithm is employed to get $\overline{\theta}_{MEL}$ and $\overline{\beta}_{MEL}$.

3.2. Bayesian estimators under SEL function

Here, the Bayesian estimator of θ and β under SEL function are derived. Hence, based on (11), the Bayesian estimator of θ under the SEL function, say $\overline{\theta}_{SEL}$ is obtained as follows

$$\overline{\theta}_{SEL} = \int_0^\infty \theta h_3(\theta|\underline{x})\, d\theta = \frac{\int_0^\infty\int_0^\infty \theta^{a_1+n}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}. \quad (25)$$

Similarly, the Bayesian estimator of β under the SEL function, say $\overline{\beta}_{SEL}$ is given by

$$\overline{\beta}_{SEL} = \int_0^\infty \beta\, h_4(\beta|\underline{x})\, d\beta = \frac{\int_0^\infty\int_0^\infty \beta^{a_2+n}\theta^{a_1+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}. \quad (26)$$

The integrals (25) and (26) are very hard to be solved analytically, so the RWMH algorithm will be used.

3.3. Bayesian estimators under PL function

Based on (14), the Bayesian estimators of θ and β under the PL function, denoted by $\overline{\theta}_{PL}$ and $\overline{\beta}_{PL}$ are obtained as follows

$$\overline{\theta}_{PL}=\sqrt{\frac{\int_0^\infty\int_0^\infty \theta^{a_1+n+1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}}. \tag{27}$$

By similar way, $\overline{\beta}_{PL}$ is as follows:

$$\overline{\beta}_{PL}=\sqrt{\frac{\int_0^\infty\int_0^\infty \beta^{a_2+n+1}\theta^{a_1+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}}. \tag{28}$$

Integrals (27) and (28) are obtained via RWMH algorithm.

3.4. Bayesian estimators under LINEX loss function

The Bayesian estimators of θ and β are obtained under LINEX loss function therefore, depending on (17), the Bayesian estimator of θ, denoted by $\overline{\theta}_{LINEX}$ is obtained as follows

$$\overline{\theta}_{LINEX}=-\frac{1}{v}\ln\left[\frac{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta-v\theta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}\right]. \tag{29}$$

By similar way, the Bayesian estimator of β, denoted by $\overline{\beta}_{LINEX}$ is given by

$$\overline{\beta}_{LINEX}=-\frac{1}{v}\ln\left[\frac{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta-v\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}\right]. \tag{30}$$

The integrals (29) and (30) can't be solved analytically, so the RWMH algorithm will be used.

4. Credible Intervals

Credible interval (CI) is an interval within which an unobserved parameter value falls with a particular subjective probability. It is an interval in the domain of a posterior probability distribution or a predictive distribution. In the following sub-sections, the CI of θ and β is obtained under informative and non-informative priors.

4.1. Credible interval under non-informative prior

The CI of θ and β is obtained under non-informative. The CI of θ, denoted by $\hat{\theta}_{CI}$ is obtained as follows

$$\hat{\theta}_{CI}=\int_L^U \theta h_1(\theta|\underline{x})d\theta=0.95$$

$$=\frac{\int_L^U\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}=0.95. \quad (31)$$

Similarly, the CI of β under non-informative prior, denoted by $\hat{\beta}_{CI}$ can be obtained as follows

$$\hat{\beta}_{CI}=\int_L^U \beta h_2(\beta|\underline{x})d\beta=0.95$$

$$=\frac{\int_L^U\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}\ e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{a_1+n-1}\beta^{a_2+n-1}e^{-b_1\theta-b_2\beta}\prod_{i=1}^n x_i^{-(2\beta+1)}\ e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}=0.95. \quad (32)$$

The integrals (31) and (32) are obtained via RWMH algorithm.

4.2. Credible interval under informative prior

The CI estimators of θ and β under informative prior are obtained. The CI of θ denoted by $\bar{\theta}_{CI}$ is obtained as follows

$$\bar{\theta}_{CI}=\int_L^U \theta h_3(\theta|\underline{x})d\theta=0.95$$

$$=\frac{\int_L^U\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}=0.95. \quad (33)$$

Similarly, the CI of β under informative prior, denoted by $\bar{\beta}_{CI}$ can be obtained as follows

$$\bar{\beta}_{CI}=\int_L^U \beta h_4(\beta|\underline{x})d\beta=0.95$$

$$=\frac{\int_L^U\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}{\int_0^\infty\int_0^\infty \theta^{n-1}\beta^{n-1}\prod_{i=1}^n x_i^{-(2\beta+1)}e^{\frac{-\theta}{x_i^{2\beta}}}\left(1+\lambda-2\lambda e^{\frac{-\theta}{x_i^{2\beta}}}\right)d\theta d\beta}=0.95. \quad (34)$$

The integrals (33) and (34) are very hard to be solved analytically, so the RWMH algorithm will be used.

5. Simulation Study

A numerical study is done to examine and compare the behavior of the different Bayesian estimates for the PTIR distribution. The Bayesian estimators are obtained using Jeffery's and gamma priors under MEL function, SEL function, PL function and LINEX loss function. The major difficulty in the implementation of the Bayesian procedure is that of obtaining the posterior distribution. The RWMH algorithm is one of the most famous subclasses of MCMC method in Bayesian literature to simulate the deviates from the posterior density and produce the good approximate results.

The following steps are designed as: The MCMC simulations are performed for different sample sizes n = 10, 20, 30, 50 and 100 under MEL function, SEL function, PL function and LINEX loss function. Let $\lambda = 0.7$ and select β and θ as (1, 0.5), (0.5, 1), (1, 1), (1, 1.5), (1.5, 1) and (1.5, 1.5). The hyper-parameters for gamma priors are selected as a_1=0.1, a_2=0.3, b_1=0.2, b_2=0.4 and $v = -2$. Each chain is performed for NR = 5000, where NR = number of replications. The performance of the estimates is evaluated through the relative absolute bias (RAB), estimated risks (ERs) and width for CI of the Bayesian estimates where these measures are computed as follows:

$$\text{ER(estimator)} = \frac{\sum_{i=1}^{NR}\left(\text{average} - \text{population parameter}\right)^2}{\text{NR}},$$

$$\text{RAB (estimator)} = \frac{\left|\text{estimator} - \text{true value}\right|}{\text{true value}},$$

and Width (CI)= Upper credible interval bound –Lower credible interval bound.

RWMH algorithm will be implemented via R 3.4.3 program as follows: Let $\text{g}(\theta^*|\theta)$ be defined as

$$\theta^* = \theta + \varepsilon,$$

where θ^* is the initial value given to start the program, θ is a random value for the estimator given θ^*, $\varepsilon \sim q$ where q is a probability density symmetric about zero. According to that definition,

$$\text{g}(\theta^*|\theta) = q(\varepsilon),$$

and

$$\text{g}(\theta|\theta^*) = q(\varepsilon) = q(-\varepsilon).$$

Because $\text{g}(\theta^*|\theta)$ is symmetric in θ and θ^* the RWMH acceptance ratio $\psi(\theta^*|\theta)$ simplifies to

$$\psi(\theta^*|\theta) = \min\left\{\frac{\pi(\theta^*)\,\text{g}(\theta|\theta^*)}{\pi(\theta)\,\text{g}(\theta^*|\theta)}, 1\right\}$$

$$= \min\left\{\frac{\pi(\theta^*)}{\pi(\theta)}, 1\right\}.$$

The RWMH algorithm proceeds as follows:

Step 1 Initialize a starting parameter value $\theta^o = 0.3$, $\beta^o = 0.5$ and determine the number of samples N.

Step 2 Simulate $\varepsilon \sim q$ and draw a candidate parameter θ^* from $g(\cdot, \theta)$ which is the proposal density considering $\theta^* = \theta + \varepsilon$.

Step 3 Compute $\psi(\theta^*|\theta) = \min\left\{\frac{\pi(\theta^*)}{\pi(\theta)}, 1\right\}$.

Step 4 Generate u from uniform (0, 1).

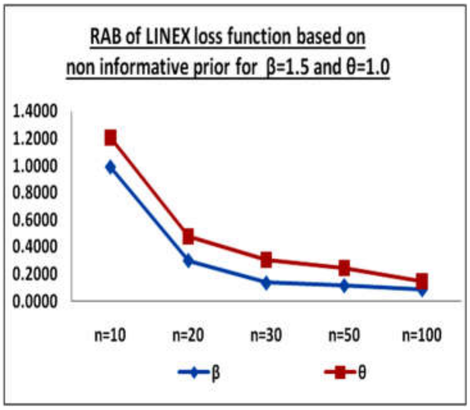
Step 5 If $u \le \psi(\theta^* | \theta)$ then, set $\theta^i = \theta^*$ otherwise, set $\theta^i = \theta$.

Step 6 Set $i = i+1$ then return to Step 2 and repeat the previous steps N times.

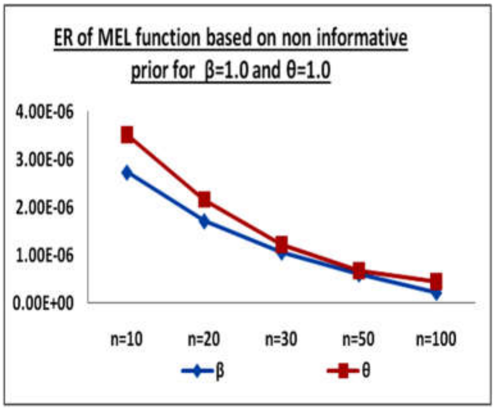
Parts of simulated results are listed in Tables 1 to 4 and represented through Figures 1 to 4.

From the Tables 1 and 2 and Figures 1 and 2, the following observations can be detected about the behavior of Bayes estimates under different loss functions in case of informative prior as follows:

The RAB and ER for Bayesian estimates of β, θ under non-informative decrease as the sample sizes increase for all sets of parameters under MEL function, SEL function, PL function and LINEX loss function (see Figure1).



(a) RAB of Bayes estimates under LINEX loss function



(b) ER of Bayes estimates under MEL loss function

Figure 1 Bayes estimates based on non-informative prior

The width of the Bayesian CI under non-informative priors for estimates of β is shorter than the corresponding Bayesian CI for estimates of θ (see Tables 1 and 2).

The RABs and ERs of $\hat{\beta}_{MEL}$ and $\hat{\theta}_{MEL}$ have the smallest values followed by $\hat{\beta}_{SEL}$ and $\hat{\theta}_{SEL}$, then by $\hat{\beta}_{PL}$ and $\hat{\theta}_{PL}$ and finally by $\hat{\beta}_{LINEX}$ and $\hat{\theta}_{LINEX}$ in approximately most of the cases (see Tables 1 and 2).

History plots for different estimates of β and θ are represented in case of non- informative priors (see for example; Figure 2 shows the case of $\hat{\beta}_{MEL}$ and $\hat{\theta}_{MEL}$ at $n = 10$ for $\beta = 1, \theta = 0.5$). The plots of chains for parameters β and θ look like a horizontal band with no long upward or downward trends which are an indicators to convergence.

From Tables 3 and 4 and Figures 3 and 4, we observe the following about the behavior of Bayes estimates under different loss functions in case of informative prior.

The RAB and ER for Bayesian estimates of β, θ under MEL function, SEL function, PL function and LINEX loss function decrease as the sample sizes increase for all sets of parameters (see for example; Figure 3).

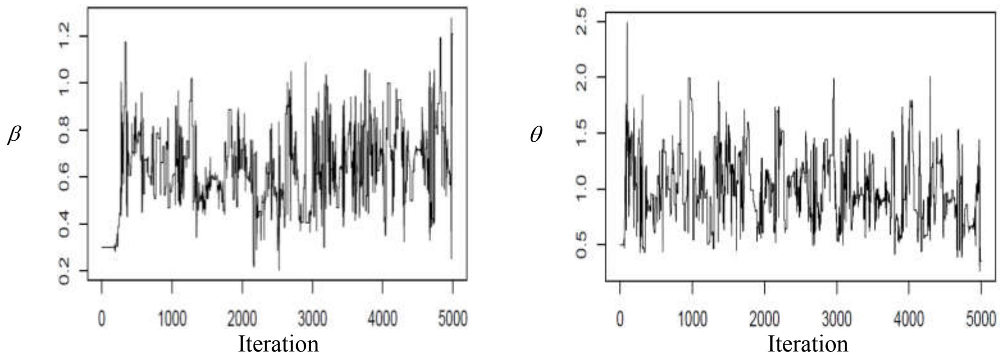
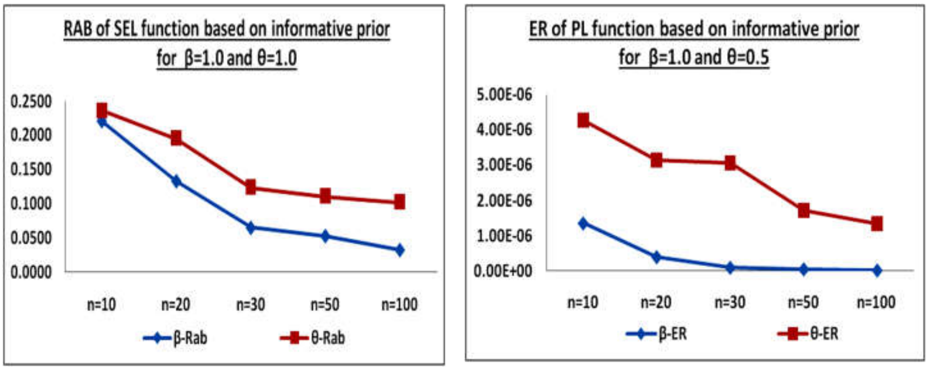


Figure 2 $\hat{\beta}_{MEL}$ and $\hat{\theta}_{MEL}$ at $n=10$ for $\beta=1,\ \theta=0.5$ in case of non- informative priors



(a) RAB of Bayes estimates under SEL loss function

(b) ER of Bayes estimates under PL loss function

Figure 3 Bayes estimates based on informative prior

The width of the Bayesian credible intervals under informative priors for estimates of β is shorter than the corresponding Bayesian credible intervals for estimates of θ (see Tables 3 and 4).

The RABs and ERs of $\bar{\beta}_{SEL}$ and $\bar{\theta}_{SEL}$ take the smallest values followed by $\bar{\beta}_{MEL}$ and $\bar{\theta}_{MEL}$, then by $\bar{\beta}_{PL}$ and $\bar{\theta}_{PL}$ and finally by $\bar{\beta}_{LINEX}$ and $\bar{\theta}_{LINEX}$ in approximately most of the cases (see Tables 3 and 4).

History plots of different estimates of β and θ are represented in case of informative priors (see for example; Figure 4 shows the case of $\bar{\beta}_{PL}$ and $\bar{\theta}_{PL}$ at $n=20$ for $\beta=1.5, \theta=1$) with no long upward or downward trends which are an indicators to convergence.

6. Conclusions

In this paper, the Bayesian estimators of the PTIR distribution concerning parameters θ and β based on informative and non-informative priors are considered. The MEL function, PL function and LINEX are employed as asymmetric loss functions, while we use the SEL function as symmetric loss function. Based on numerical study, it is observed that the RAB and ER for Bayesian estimates of β take the smallest values compared to the RAB and ER for Bayesian estimates of θ under informative and non-informative priors in approximately most of the situations. Moreover the width of the

Bayesian credible intervals under informative and non-informative priors for estimates of β is shorter than the corresponding Bayesian credible intervals for estimates of θ under informative and non-informative priors in approximately most of the situations.

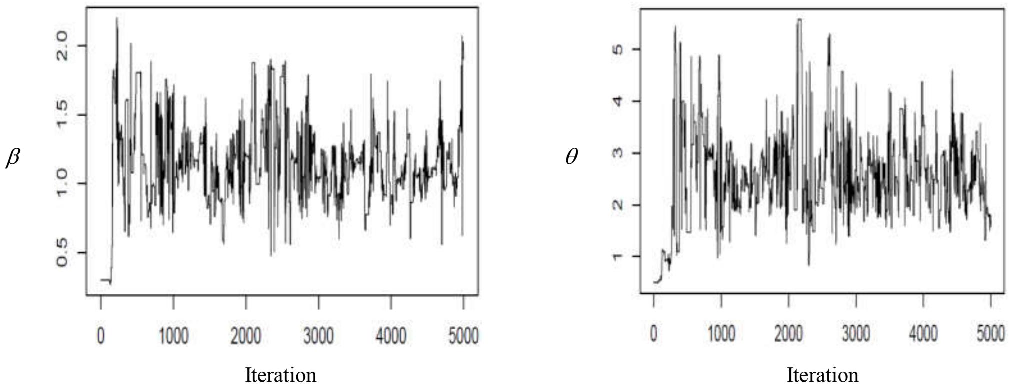


Figure 4 $\overline{\beta}_{PL}$ and $\overline{\theta}_{PL}$ at $n = 20$ for $\beta = 1.5, \theta = 1$ in case of informative priors

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Table 1 Bayes estimates, RAB, ER and width of credible intervals based on Jeffery's prior for $\beta = 1.5,\ \theta = 1$

Parameters	Sample size	10		20		30		50		100	
		β	θ	β	θ	β	θ	β	θ	β	θ
MEL	Estimates	2.1059	2.1188	1.0414	1.4075	1.4000	1.2701	1.5928	0.8895	1.4790	0.9232
	RAB	0.4040	1.1188	0.3057	0.4075	0.0667	0.2701	0.0619	0.1105	0.0140	0.0768
	ER	3.68E-06	1.25E-05	2.11E-06	1.66E-06	1.00E-07	7.30E-07	8.63E-08	1.22E-07	4.41E-09	5.90E-08
	Width	2.7828	2.9123	1.2175	1.6939	1.5456	1.2729	1.8123	0.6627	1.4713	0.5829
SEL	Estimates	2.2139	1.8669	1.1722	1.3302	1.2828	1.3114	1.6686	0.7747	1.4013	0.8037
	RAB	0.4760	0.8669	0.2185	0.3302	0.1448	0.3114	0.1124	0.2253	0.0658	0.1963
	ER	5.10E-06	7.52E-06	1.08E-06	1.09E-06	4.72E-07	9.71E-07	2.85E-07	5.08E-07	9.76E-08	3.86E-07
	Width	3.2139	2.2730	1.4007	1.6499	1.5043	1.4328	1.9464	0.6750	1.6125	0.5783
PL	Estimates	2.4098	2.3567	1.1385	2.0621	1.8062	1.9025	1.6688	1.8361	1.5494	1.5664
	RAB	0.6066	1.3567	0.2410	1.0621	0.2041	0.9025	0.1125	0.8361	0.0329	0.5664
	ER	8.29E-06	1.84E-05	1.31E-06	1.13E-05	9.38E-07	8.15E-06	2.85E-07	7.00E-06	2.44E-08	3.21E-06
	Width	3.5304	3.5528	1.6010	2.8601	2.3927	2.4387	2.4326	2.2222	1.8245	1.7799
LINEX	Estimates	2.9872	2.2047	1.0495	1.4787	1.7025	0.6946	1.3274	1.2443	1.3716	0.8545
	RAB	0.9915	1.2047	0.3004	0.4787	0.1350	0.3054	0.1151	0.2443	0.0856	0.1455
	ER	2.21E-05	1.45E-05	2.03E-06	2.29E-06	4.10E-07	9.33E-07	2.98E-07	5.97E-07	1.65E-07	2.12E-07
	Width	4.3784	3.0078	1.4576	1.4107	2.0892	0.4905	1.4381	1.0934	1.5516	0.6257

Table 2 Bayes estimates, RAB, ER and width of credible intervals based on Jeffery's prior for $\beta = 1,\ \theta = 1.5$

Parameters	Sample size	10		20		30		50		100	
		β	θ	β	θ	β	θ	β	θ	β	θ
MEL	Estimates	1.5210	2.3318	1.2877	1.0139	1.2556	1.2320	0.8770	1.6885	0.9210	1.4018
	RAB	0.5210	0.5545	0.2877	0.3241	0.2556	0.1787	0.1230	0.1257	0.0790	0.0655
	ER	2.72E-06	6.92E-06	8.29E-07	2.37E-06	6.54E-07	7.19E-07	1.51E-07	3.56E-07	6.25E-08	9.65E-08
	Width	2.3452	2.7302	1.5709	1.2212	1.5915	1.2279	0.9059	1.9315	0.8729	1.3230
SEL	Estimates	1.5648	3.3525	1.4929	0.9641	1.2989	1.0083	0.7875	1.2061	1.0994	1.3687
	RAB	0.5648	1.2350	0.4929	0.3573	0.2989	0.3278	0.2125	0.1960	0.0994	0.0875
	ER	3.19E-06	3.44E-05	2.43E-06	2.88E-06	8.94E-07	2.42E-06	4.52E-07	8.65E-07	9.89E-08	1.73E-07
	Width	2.6305	6.3393	1.8875	0.9233	1.5620	0.7796	0.8445	1.1560	1.2111	1.4754
PL	Estimates	1.8008	4.0122	1.7575	3.3731	1.4360	2.9344	1.2414	2.3327	1.1590	2.0298
	RAB	0.8008	1.6748	0.7575	1.2487	0.4360	0.9562	0.2414	0.5551	0.1590	0.3532
	ER	6.42E-06	6.32E-05	5.74E-06	3.51E-05	1.90E-06	2.06E-05	5.83E-07	6.94E-06	2.53E-07	2.81E-06
	Width	3.1285	8.2775	3.2468	5.0852	1.8768	3.9595	1.5358	3.0313	1.5192	2.5778
LINEX	Estimates	3.0080	7.3248	1.4456	2.4289	1.3742	1.0723	0.7747	1.7893	1.1444	1.3017
	RAB	2.0080	3.8832	0.4456	0.6192	0.3742	0.2851	0.2253	0.1928	0.1444	0.1322
	ER	4.04E-05	3.40E-04	1.99E-06	8.64E-06	1.40E-06	1.83E-06	5.08E-07	8.38E-07	2.09E-07	3.94E-07
	Width	4.6589	14.5955	1.8282	3.1534	1.8883	1.2149	0.7187	1.7776	1.2718	1.1696

Table 3 Bayes estimates, RAB, ER and width of credible intervals based on gamma prior for $\beta = 1,\ \theta = 1.5$

Parameters	Sample size	10		20		30		50		100	
		β	θ	β	θ	β	θ	β	θ	β	θ
MEL	Estimates	1.4461	0.6928	0.7985	1.1985	1.1319	1.3123	1.3284	0.0798	0.9542	1.4228
	RAB	0.4461	0.5381	0.2015	0.2010	0.1319	0.1251	0.0798	0.1144	0.0458	0.0515
	ER	1.99E-06	6.52E-06	4.06E-07	9.10E-07	1.74E-07	3.53E-07	6.37E-08	2.95E-07	2.10E-08	5.97E-08
	Width	1.6600	0.7166	0.8232	1.1482	1.4732	1.4222	0.8576	1.1633	0.9520	1.3429
SEL	Estimates	1.3596	3.2065	1.2411	1.2358	1.1107	1.2588	0.9085	1.3655	1.4626	0.9445
	RAB	0.3596	1.1376	0.2411	0.1761	0.1107	0.1608	0.0915	0.0897	0.0249	0.0555
	ER	1.29E-06	2.91E-05	5.82E-07	6.99E-07	1.23E-07	5.82E-07	8.38E-08	1.81E-07	1.40E-08	3.09E-08
	Width	1.5360	4.1606	1.4789	1.7802	1.2649	1.1934	0.9899	1.2845	1.3281	0.6239
PL	Estimates	1.4782	3.3590	1.1581	2.7410	1.1078	2.2983	1.0646	2.0457	1.0403	2.0191
	RAB	0.4782	1.2393	0.1581	0.8273	0.1078	0.5322	0.0646	0.3638	0.0403	0.3461
	ER	2.29E-06	3.46E-05	2.50E-07	1.54E-05	1.16E-07	6.38E-06	4.18E-08	2.98E-06	1.63E-08	2.70E-06
	Width	2.1255	6.0800	1.4452	3.6384	1.3679	3.4077	1.0529	2.4579	1.6146	2.4984
LINEX	Estimates	1.5483	0.9437	0.7428	2.0350	1.2205	1.1806	1.1636	1.7199	0.8891	1.3186
	RAB	0.5483	0.3709	0.2572	0.3566	0.2205	0.2130	0.1636	0.1466	0.1109	0.1209
	ER	3.01E-06	3.10E-06	6.62E-07	2.86E-06	4.87E-07	1.02E-06	2.68E-07	4.84E-07	1.23E-07	3.29E-07
	Width	1.8221	0.9502	0.7725	2.6919	1.2713	1.1663	1.2526	1.6411	0.7567	1.1209

Table 4 Bayes estimates, RAB, ER and width of credible intervals based on gamma prior for $\beta = 1.5,\ \theta = 1.5$

Parameters	Sample size	10		20		30		50		100	
		β	θ	β	θ	β	θ	β	θ	β	θ
MEL	Estimates	1.8731	2.2299	1.8203	1.1069	1.2596	1.2575	1.6613	1.2848	1.3701	1.2889
	RAB	0.2488	0.4866	0.2135	0.2621	0.1603	0.1617	0.1076	0.1435	0.0866	0.1408
	ER	1.39E-06	5.33E-06	1.03E-06	1.55E-06	5.79E-07	5.89E-07	2.61E-07	4.64E-07	1.69E-07	4.46E-07
	Width	2.8469	2.9482	2.5629	1.0800	1.5197	1.3014	2.0691	1.2291	1.4065	1.1533
SEL	Estimates	1.8312	1.1454	1.3009	1.2069	1.4033	1.3142	1.4214	1.3339	1.4522	1.3474
	RAB	0.2208	0.2364	0.1328	0.1954	0.0645	0.1238	0.0524	0.1107	0.0319	0.1017
	ER	1.10E-06	1.26E-06	3.97E-07	8.60E-07	9.37E-08	3.45E-07	6.18E-08	2.76E-07	2.29E-08	2.33E-07
	Width	2.4876	1.3677	1.5095	1.4238	1.7072	1.3410	1.6593	1.2647	1.5593	1.2081
PL	Estimates	1.9266	3.5632	1.7552	2.5262	1.6731	2.4807	1.5625	2.3559	1.5575	2.2627
	RAB	0.2844	1.3754	0.1701	0.6842	0.1154	0.6538	0.0417	0.5706	0.0383	0.5084
	ER	1.82E-06	4.26E-05	6.52E-07	1.05E-05	3.00E-07	9.63E-06	3.91E-08	7.33E-06	3.31E-08	5.82E-06
	Width	3.0007	6.7965	2.4589	4.0046	1.8976	2.7728	1.8460	2.7747	1.6293	2.5734
LINEX	Estimates	3.1724	3.4663	2.2971	2.3576	1.2180	1.1939	1.2397	1.2158	1.3470	1.2618
	RAB	1.1149	1.3109	0.5314	0.5717	0.1880	0.2041	0.1735	0.1894	0.1020	0.1588
	ER	2.80E-05	3.87E-05	6.36E-06	7.36E-06	7.96E-07	9.38E-07	6.78E-07	8.08E-07	2.34E-07	5.68E-07
	Width	4.8664	5.3489	3.2854	3.6969	1.5452	1.13702	1.4693	1.2272	1.3936	1.1652