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Statistical Properties and Estimation of Type II Half Logistic Lomax Distribution

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Abstract

In this study we present a new distribution named as Type II Half Logistic Lomax (TIIHLL) which offers more flexibility in modeling lifetime data. We derive some fundamental properties including; ordinary and incomplete moments, Rényi entropy, quantiles, probability weighted moments, and stochastic ordering. The maximum likelihood, least squares, weighted least squares and Anderson-Darling methods are used to estimate the model parameters. We present simulation results to assess the performance of the estimates. We prove empirically the importance and flexibility of the new distribution in modeling data set.

Keywords: Type II half logistic-G class, Maximum likelihood, Least squares, Anderson-Darling method.

1. Introduction

In recent times, many approaches have been introduced for modeling lifetime data. Generated family of distributions is one of the techniques used in modeling lifetime issues which can be formed by introducing one or more additional shape parameter(s) to the baseline distribution. Several generated distributions have been proposed by several authors. A general method of adding a shape parameter to a family of distributions has been pioneered by Marshall and Olkin (1997). The new parameter gives more flexibility and extends several well-known distributions. This family is called the Marshall-Olkin-G (**MO-G**) class. The probability density function (**pdf**) of the MO-G is defined as follows

$$f_{MO-G}(x;\gamma)=\frac{\gamma g(x)}{\left[1-\bar{\gamma}(\bar{G}(x))\right]^{2}}, \tag{1}$$

where $\bar{\gamma}=1-\gamma$ and $\bar{G}(x)=1-G(x)$ is the survival function (**sf**). It is obvious that many new families related to MO-G have been proposed by several authors. Our interest here, with univariate family of distributions generated by half logistic random variable as discussed by Hassan et al. (2017). For any continuous baseline cumulative distribution function (**cdf**); say $G(\cdot)$, of the Type II half logistic-G (**TIIHL-G**) distribution is defined by:

$$F_{TIIHL-G}(x;\lambda,\zeta)=\frac{2\left[G(x;\zeta)\right]^{\lambda}}{1+\left[G(x;\zeta)\right]^{\lambda}}, \tag{2}$$

where λ is the shape parameter and $G(x;\zeta)$ is a baseline cdf, which relies on a parameter vector ζ. The pdf corresponding to (2) is given by

$$f_{TIIHL-G}\left(x;\lambda,\zeta\right)=\frac{2\lambda g(x;\zeta)\left[G(x;\zeta)\right]^{\lambda-1}}{\left[1+\left[G(x;\zeta)\right]^{\lambda}\right]^{2}}, \quad \lambda>0. \tag{3}$$

For $\lambda=1$ in (3) and $\gamma=0.5$ in (1) the pdf (3) provides MO-G family. The pdf (3) provides a number of known distributions as particular cases with more flexibility in their skewness and kurtosis.

The Lomax (Pareto II) distribution was originally proposed by Lomax (1954) to model business failure data. The Lomax (**L**) distribution originates as a limit distribution of residual lifetime at a great age as mentioned by Balkema and de Haan (1974). Properties and moments of record values for L distribution were considered by Ahsanullah (1991) and Balakrishnan and Ahsanullah (1994). Order statistics from non-identical right-truncated L distribution were discussed by Childs et al. (2001). Abd-Elfattah et al. (2007) discussed the Bayesian and non-Bayesian estimators of the sample size in censored samples. The optimal times of changing stress level for simple step stress accelerated life testing for L distribution were considered by Hassan and Al-Ghamdi (2009). The generalized probability weighted moments estimators were discussed by Abd-Elfattah and Alharby (2010). Bayesian estimators of the L distribution were discussed by Ahmad et al. (2015) under different loss functions. Hassan et al. (2016) discussed the optimal times in step-stress accelerated life tests of the L distribution via adaptive Type-II progressive hybrid censoring. The cdf of L distribution with shape parameter α and scale parameter β is defined by

$$G(x;\alpha,\beta)=1-\left(1+\frac{x}{\beta}\right)^{-\alpha}, \quad x,\alpha,\beta>0. \tag{4}$$

The pdf corresponding to (4) is given by

$$g(x;\alpha,\beta)=\frac{\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha}, \quad x,\alpha,\beta>0. \tag{5}$$

In the literature, modified and extended forms of the L distribution were considered by several authors; examples include MO extended L (Ghitany et al. 2007), exponentiated L (**EL**) (Abdul-Moniem and Abdel-Hameed 2012), beta L, Kumaraswamy L and McDonald L (Lemonte and Cordeiro 2013), transmuted EL (Ashour and Eltehiwy, 2013), gamma L (Cordeiro et al., 2015), Gumbel-L (Tahir et al. 2016), power L (**PL**) (Rady et al. 2016), EL geometric (Hassan and Abd-Allah 2017), exponentiated Weibull L (Hassan and Abd-Alla 2018), PL Poisson (Hassan and Nassr 2018), inverse PL (Hassan and Abd-Allah 2019), Weibull inverse L (Hassan and Mohamed 2019a), inverted EL (Hassan and Mohamed 2019b), Type II Topp Leone PL (Al-Marzouki et al. 2020) and Marshall-Olkin PL(Haq et al. 2020) distributions among others.

The main objectives of this paper is providing a more flexible model by induce just one extra shape parameter to L model for improving its goodness-of-fit to real data. The basic motivations of the TIIHLL distribution in practice are: (i) to obtain more flexible pdf with right skewed, uni-modal and symmetric shapes; (ii) it is capable of modeling decreasing, reversed-J and upside-down hazard rates shapes; and (iii) to provide significant improvement in data modelling.

In this study, the TIIHLL distribution with an extra shape parameter is introduced as a modification of L distribution to provide a more flexible model. Statistical properties, estimation of the model parameters and its applications are considered. Sections 2 and 3 provide the pdf, the cdf and some statistical properties of the TIIHLL model. In Section 4, maximum likelihood (**ML**), least squares (**LS**), weighted least squares (**WLS**) and Anderson-Darling (**AD**) estimators are derived. Also, in Section 4, the behavior of the different estimates is studied via simulation study. Application to real data is provided in Section 5 and the article ends with conclusions.

2. Type II Half Logistic Lomax Distribution

Here we obtain the pdf of the TIIHLL distribution by subsituting (4) and (5) in (3), as

$$f_{TIIHLL}(x;\pi)=\frac{2\lambda\alpha(1+\frac{x}{\beta})^{-\alpha-1}\left[1-(1+\frac{x}{\beta})^{-\alpha}\right]^{\lambda-1}}{\beta\left[1+\left[1-(1+\frac{x}{\beta})^{-\alpha}\right]^{\lambda}\right]^{2}},\quad x,\lambda,\alpha,\beta>0, \tag{6}$$

where $\pi\equiv(\alpha,\beta,\lambda)$ is a set of parameters. A random variable X with pdf (6) shall be denoted by $X\sim(\alpha,\beta,\lambda),$ and

- For $\lambda=1$ in (6), we obtain the MO extended L distribution with $\gamma=0.5$ (see Ghitany et al. 2007) as special model.
- For $\beta=1,$ in (6), we obtain the TIIHL beta prime distribution with parameters α and λ as new model.
- For $\lambda=1$ and $\beta=1$ in (6), we obtain the MO beta prime distribution with $\gamma=0.5.$

The cdf corresponding to (6) is given as

$$F_{TIIHLL}(x;\pi)=\frac{2\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda}}{1+\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda}},\quad x,\lambda,\alpha,\beta>0. \tag{7}$$

Further, the sf and the hazard rate function (**hrf**) are given, respectively, by

$$R_{TIIHLL}(x;\pi)=\left\{1-\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda}\right\}\left\{1+\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda}\right\}^{-1},$$

and

$$h_{TIIHLL}(x;\pi)=\frac{2\lambda\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda-1}\left\{1-\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{2\lambda}\right\}^{-1}.$$

The pdf and hrf plots for the TIIHLL are displayed in Figures 1 and 2 for some choices of parameters.

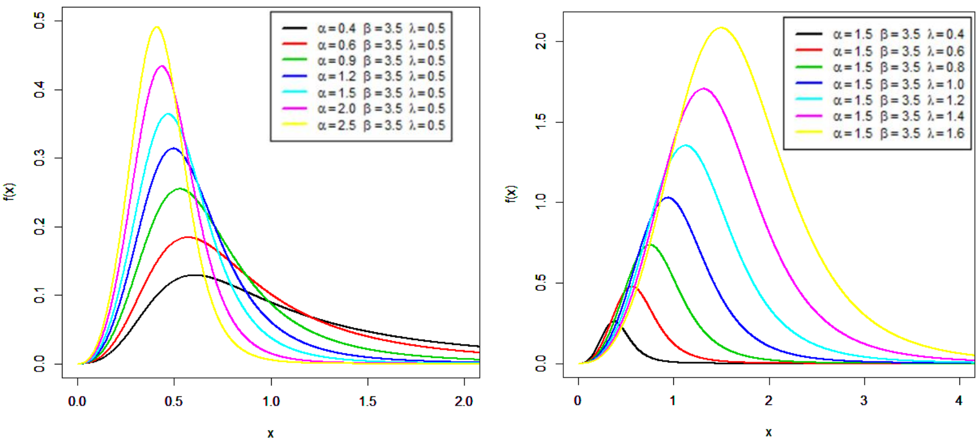

Figure 1 The pdf plots for different values of parameters

Figure 1 shows that the pdf of the TIIHLL distribution can be right skewed, uni-modal and symmetric, its more flexible than the L distribution.

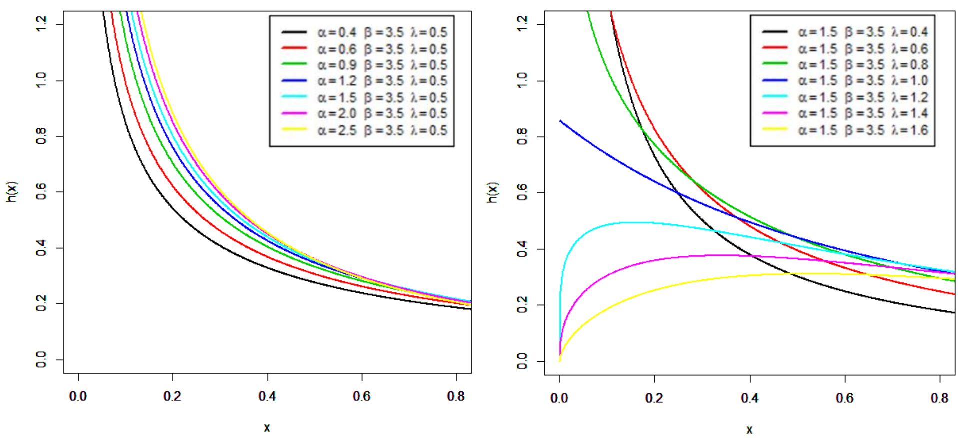

Figure 2 The hrf plots for different values of parameters

Figure 2 shows that the hrf of the TIIHLL distribution can be up-side down, decreasing and reversed J-shaped. One of the advantages of the TIIHLL distribution over the Lomax distribution is that the latter cannot model phenomenon showing up-side down or uni-modal shape failure rates.

3. Basic Properties

In this section, we give some important statistical properties of the TIIHLL distribution such as ordinary and incomplete moments, Rényi entropy, probability weighted moments, stochastic ordering, mean residual life and mean waiting time.

3.1. Ordinary and incomplete moments

In several applied works, it is necessary to emphasize the importance of calculating the moments of a random variable. If the random variable X has the TIIHLL distribution and by using the following generalized binomial series

$$(1+z)^{-k}=\sum_{i=0}^{\infty}(-1)^i\binom{k+i-1}{i}z^i,\quad |z|<1,\ k>0, \tag{8}$$

in the pdf (6), then its s^{th} moment is obtained as follows:

$$\mu_s'=\sum_{i=0}^{\infty}(-1)^i(i+1)\frac{2\lambda\alpha}{\beta}\int_0^{\infty}x^s\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda(i+1)-1}dx. \tag{9}$$

We employ the binomial expansion in (9), then we have

$$\mu_s'=\sum_{i,j=0}^{\infty}\frac{\varsigma_{i,j}}{\beta}\int_0^{\infty}x^s\left(1+\frac{x}{\beta}\right)^{-\alpha(j+1)-1}dx,$$

where $\varsigma_{i,j}=(-1)^{i+j}2\lambda\alpha(i+1)\binom{\lambda(i+1)-1}{j}$.

Hence, the s^{th} moment of the TIIHLL distribution is

$$\mu_s'=\sum_{i,j=0}^{\infty}\varsigma_{i,j}\beta^sB\left(s+1,\alpha(j+1)-s\right),\quad s=1,2,...,$$

where $B(\cdot,\cdot)$ is the beta function. Further, the s^{th} central moment of a given random variable X, is defined by:

$$\mu_s=E(X-\mu_1')^s=\sum_{i=0}^{s}(-1)^i\binom{s}{i}(\mu_1')^i\mu_{s-i}'.$$

The coefficient of skewness (**CS**) and coefficient of kurtosis (**CK**) are defined by:

$$\text{CS}=\frac{\mu_3}{\mu_2^{3/2}},\quad \text{CK}=\frac{\mu_4}{\mu_2^2}.$$

Thus, numerical values for some certain values of parameters of the first four ordinary moments; $\mu_1',\ \mu_2',\ \mu_3',\ \mu_4'$, variance (σ^2), CS and CK of the TIIHLL distribution are listed in Table 1.

From Table 1, it can be observed that the values of the mean and variance of the TIIHLL model are increasing as the value of λ increases for fixed value of α and β. While the mean and variance values are decreasing as the value of α increases for fixed values of λ and β. Further, the values of CS and CK of the TIIHLL model are decreasing as the value of λ increases for fixed values of α and β. Also, CS and CK values are decreasing as the value of α increases for fixed values of λ and β.

Table 1 Moments and related statistics of the TIIHLL distribution

μ_s'	$\lambda=0.5$, $\beta=0.5$, $\alpha=5$	$\lambda=1$, $\beta=0.5$, $\alpha=5$	$\lambda=2.5$, $\beta=0.5$, $\alpha=5$	$\lambda=0.5$, $\beta=0.5$, $\alpha=7$	$\lambda=0.5$, $\beta=0.5$, $\alpha=9$
μ_1'	0.046	0.084	0.163	0.031	0.0240
μ_2'	0.012	0.023	0.055	0.004	0.0030
μ_3'	0.008	0.016	0.040	0.002	0.0005
μ_4'	0.016	0.031	0.078	0.001	0.0002
σ^2	0.009	0.016	0.028	0.004	0.0020
CS	7.077	5.623	4.624	5.340	4.7450
CK	156.801	104.107	74.874	60.972	43.9230

Additionally, the s^{th} lower incomplete moment, say $\kappa_s(t)$, of the TIIHLL distribution is given by:

$$\kappa_s(t)=\frac{2\lambda\alpha}{\beta}\int_0^t x^s\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda-1}\left[1+\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda}\right]^{-2}dx.$$

Using the binomial expansion more than one time, we obtain

$$\kappa_s(t)=\sum_{i,j=0}^{\infty}\frac{\varsigma_{i,j}}{\beta}\int_0^t x^s\left(1+\frac{x}{\beta}\right)^{-\alpha(j+1)-1}dx,$$

which leads to

$$\kappa_s(t)=\sum_{i,j=0}^{\infty}\varsigma_{i,j}\beta^s B\left(\frac{t}{\beta+t},s+1,\alpha(j+1)-s\right),\quad s=1,2,... \tag{10}$$

where $B(\cdot,\cdot,z)$ is the incomplete beta function. Setting $s=1$ in (10) we obtain the first incomplete moment, $\kappa_1(t)$. The main applications of the first incomplete moment are the Lorenz and Bonferroni curves. The Lorenz curve; say $L_F(t)$, and Bonferroni curve; say $B_F(t)$, of the TIIHLL distribution are obtained as follows:

$$L_F(t)=\frac{\kappa_1(t)}{E(T)}=\frac{\sum_{i,j=0}^{\infty}\varsigma_{i,j}B\left(\frac{t}{t+\beta},2,\alpha(j+1)-1\right)}{\sum_{i,j=0}^{\infty}\varsigma_{i,j}B\left(2,\alpha(j+1)-1\right)},$$

and

$$B_F(t)=\frac{L_F(t)}{F_{TIIHLL}(t;\pi)}=\frac{\sum_{i,j=0}^{\infty}\varsigma_{i,j}B\left(\frac{t}{t+\beta},2,\alpha(j+1)-1\right)}{F_{TIIHLL}(t;\pi)\sum_{i,j=0}^{\infty}\varsigma_{i,j}B\left(2,\alpha(j+1)-1\right)}.$$

3.2. Mean residual life and mean waiting time

The mean residual life (**MRL**) function at age t measures the expected remaining lifetime of an individual of age t. The MRL of X is defined by:

$$m(t)=\frac{1}{R(t)}\left[E(T)-\kappa_1(t)\right]-t,$$

where $R(t)$ is the sf and $\kappa_1(t)$ is the first incomplete moment. Hence the MRL of the TIIHLL distribution is obtained as:

$$m(t)=\frac{1}{R_{TIIHLL}(t;\pi)}\left[\sum_{i,j=0}^{\infty}\varsigma_{i,j}\beta B\left(2,\alpha(j+1)-1\right)-\sum_{i,j=0}^{\infty}\varsigma_{i,j}\beta B\left(\frac{t}{t+\beta},2,\alpha(j+1)-1\right)\right]-t.$$

The mean waiting time (**MWT**) of an item failed in an interval $[0,t]$ is defined as:

$$\bar{m}(t)=t-\frac{\kappa_1(t)}{F(t)},$$

where $F(t)$ is the cdf of a distribution. The MWT of TIIHLL distribution is determined as

$$\bar{m}(t)=t-\frac{1}{F_{TIIHLL}(t;\pi)}\left[\sum_{i,j=0}^{\infty}\varsigma_{i,j}\beta B\left(\frac{t}{t+\beta},2,\alpha(j+1)-1\right)\right].$$

3.3. The probability weighted moments

The probability weighted moments (**PWMs**) are the expectations of multiplication of two certain functions of a random variable X which can be defined as:

$$\varphi_{r,s}=E[X^rF(x)^s]=\int_{-\infty}^{\infty}x^rf(x)(F(x))^s\,dx, \qquad (11)$$

where r and s are positive integers. Substituting (6) and (7) in (11) and using binomial expansion (8), we obtain

$$\varphi_{r,s}=\sum_{k=0}^{s+2}\frac{(-1)^k2^{s+1}\lambda\alpha}{\beta}\binom{s+k+1}{k}\int_0^{\infty}x^r\left(1+\frac{x}{\beta}\right)^{-\alpha-1}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\lambda(1+k+s)-1}dx. \qquad (12)$$

Hence, using the binomial expansion in (12), then

$$\varphi_{r,s}=\sum_{k=0}^{s+2}\sum_{i=0}^{\infty}\frac{(-1)^{k+i}2^{s+1}\lambda\alpha}{\beta}\binom{s+k+1}{k}\binom{\lambda(1+k+s)-1}{i}\int_0^{\infty}x^r\left(1+\frac{x}{\beta}\right)^{-\alpha(i+1)-1}dx.$$

Therefore, the PWM of TIIHLL model is given by

$$\varphi_{r,s}=\sum_{k=0}^{s+2}\sum_{i=0}^{\infty}\varpi_{k,i}\beta^rB(r+1,\alpha(i+1)-r),$$

where $\varpi_{k,i} = (-1)^{k+i} 2^{s+1} \lambda\alpha \binom{s+k+1}{k}\binom{\lambda(1+k+s)-1}{i}.$

3.4. Quantile function

The quantile function of X is obtained by inverting (7) as

$$Q(u) = \beta\left[1-\left(\frac{u}{2-u}\right)^{1/\lambda}\right]^{-1/\alpha} - \beta, \quad 0<u<1, \tag{13}$$

where u is the uniform random variable. Equation (13) can be used to simulate the TIIHLL distribution.

3.5. Rényi entropy

Entropy has been used in various situations in science and engineering. The entropy of a random variable X is a measure of variation of the uncertainty. The Rényi entropy is defined by:

$$R_\delta(X) = \frac{1}{1-\delta}\log\int_{-\infty}^{\infty} f(x)^\delta\, dx, \quad \delta>0 \text{ and } \delta\neq 1. \tag{14}$$

Substituting (6) in (14) and using expansion (8), the Rényi entropy of the TIIHLL distribution is

$$R_\delta(X) = \frac{1}{1-\delta}\log\left(\frac{2\lambda\alpha}{\beta}\right)^\delta \sum_{i=0}^{\infty}(-1)^i\binom{2\delta+i-1}{i}\int_0^{\infty}\left(1+\frac{x}{\beta}\right)^{-\delta(\alpha+1)}\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^{\delta(\lambda-1)+\lambda i} dx. \tag{15}$$

Therefore, the Rényi entropy of the TIIHLL distribution is given by

$$R_\delta(X) = \frac{1}{1-\delta}\log\left\{\sum_{i=0}^{\infty}\upsilon_i B\left(\frac{\delta(\alpha+1)-1}{\alpha}, \delta(\lambda-1)+\lambda i+1\right)\right\},$$

where $\upsilon_i = (-1)^i\binom{2\delta+i-1}{i}\left(\frac{(2\lambda)^\delta\alpha^{\delta-1}}{\beta^{\delta-1}}\right).$

Table 2 gives $I_R(X)$ of the TIIHLL distribution for different choices of parameters $\delta, \alpha, \beta,$ and $\lambda.$

Table 2 Entropy for several arbitrary parameter values

λ	$\delta=0.5$, $\beta=0.5$, $\alpha=2$	$\delta=0.5$, $\beta=1$, $\alpha=3$	$\delta=3$, $\beta=0.5$, $\alpha=0.5$	$\delta=3$, $\beta=2$, $\alpha=1.5$	$\delta=3$, $\beta=2$, $\alpha=0.8$
	Entropy	Entropy	Entropy	Entropy	Entropy
1	0.418	0.321	0.091	0.157	0.342
2	0.616	0.491	1.420	0.584	0.890
3	0.720	0.574	1.023	0.746	1.124
5	0.844	0.668	1.420	0.923	1.402

From Table 2, we notice that the entropy values are increasing as the values of λ increase for all values of parameters. The entropy values are decreasing as the values of α and β increase at $\delta = 0.5$. While, values of entropy are increasing as the values of α and λ increase for $\delta = 3$.

3.6. Stochastic ordering

Let X and Y are independent random variables with cdfs F_X and F_Y respectively, then X is said to be smaller than Y in the following ordering (see Shaked and Shanthikumar 2007); if the following holds;

- Stochastic order $(X \leq_{sr} Y)$ if $F_X(x) \geq F_Y(x)$ for all x.
- Likelihood ratio order $(X \leq_{lr} Y)$ if $f_X(x)/f_Y(x)$ is decreasing in x.
- Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x.
- Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x)$ for all x.

We have the following chain of implications among the various partial orderings discussed above:

$$\begin{array}{ccc} X \leq_{lr} Y \Rightarrow & X \leq_{hr} Y & \Rightarrow X \leq_{mrl} Y \\ & \Downarrow & \\ & X \leq_{sr} Y & \end{array}$$

Theorem 1. *Let* $X \sim \text{TIIHLL}(\alpha_1, \beta_1, \lambda_1)$ *and* $Y \sim \text{TIIHLL}(\alpha_2, \beta_2, \lambda_2)$. *If* $\beta_1 = \beta_2 = \beta$, $\alpha_1 \geq \alpha_2$ *and* $\lambda_1 \geq \lambda_2$, *then* $X \leq_{lr} Y, X \leq_{hr} Y, X \leq_{mrl} Y$, *and* $X \leq_{sr} Y$.

Proof:

It is sufficient to show $f_X(x)/f_Y(x)$ is a decreasing function of x; the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{2\lambda_1\alpha_1\beta_2\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1-1}\left(1-\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1}\right)^{\lambda_1-1}\left\{1+\left(1-\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2}\right)^{\lambda_2}\right\}^2}{2\lambda_2\alpha_2\beta_1\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2-1}\left(1-\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2}\right)^{\lambda_2-1}\left\{1+\left(1-\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1}\right)^{\lambda_1}\right\}^2},$$

therefore,

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} = -\frac{(\alpha_1+1)}{\beta_1\left(1+\frac{x}{\beta_1}\right)} - \frac{\alpha_1(\lambda_1-1)\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1-1}}{\beta_1\left(1-\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1}\right)} - \frac{2\alpha_2\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2-1}}{\beta_2\left\{1+\left(1-\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2}\right)^{\lambda_2}\right\}}$$

$$+\frac{2\alpha_1\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1-1}}{\beta_1\left\{1+\left(1-\left(1+\frac{x}{\beta_1}\right)^{-\alpha_1}\right)^{\lambda_1}\right\}}+\frac{(\alpha_2+1)}{\beta_2\left(1+\frac{x}{\beta_2}\right)}+\frac{\alpha_2(\lambda_2-1)\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2-1}}{\beta_2\left(1-\left(1+\frac{x}{\beta_2}\right)^{-\alpha_2}\right)}.$$

If $\beta_1=\beta_2=\beta$ and let $Z=1+\frac{x}{\beta}$, then the previous will be simplified as

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}=\frac{\alpha_2-\alpha_1}{(\beta+x)}+\frac{\alpha_1 Z^{-\alpha_1-1}\left[3-\lambda_1-2Z^{-\alpha_1}-(\lambda_1-1)(1-Z^{-\alpha_1})^{\lambda_1}\right]}{\beta(1-Z^{-\alpha_1})\left\{1+(1-Z^{-\alpha_1})^{\lambda_1}\right\}}$$
$$+\frac{\alpha_2 Z^{-\alpha_2-1}\left[3-\lambda_2-2Z^{-\alpha_2}-(\lambda_2-1)(1-Z^{-\alpha_2})^{\lambda_2}\right]}{\beta(1-Z^{-\alpha_2})\left\{1+(1-Z^{-\alpha_2})^{\lambda_2}\right\}}.$$

Now if $\alpha_1\geq\alpha_2$ and $\lambda_1\geq\lambda_2$, then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}\leq 0$, which implies that Y is stochastically greater than X with respect to likelihood ratio order i.e., $X\leq_{lr} Y$. Similarly, we can conclude for $X\leq_{hr} Y, X\leq_{mrl} Y$, and $X\leq_{sr} Y$.

4. Methods of Estimation

The estimators of the population parameters of the TIIHLL distribution are obtained using ML, LS, WLS and AD methods.

4.1. Maximum likelihood estimators

Let $X_1, X_2, \ldots, X_n$ be a random sample from TIIHLL distribution with unknown parameters λ,α and β. The log-likelihood function, denoted by $\ln l$, is given by

$$\ln l=n\ln 2\lambda+n\ln\alpha-(\alpha+1)\sum_{i=1}^{n}\ln H_i+(\lambda-1)\sum_{i=1}^{n}\ln\left[1-H_i^{-\alpha}\right]-2\beta\sum_{i=1}^{n}\ln\left[1+\left[1-H_i^{-\alpha}\right]^{\lambda}\right],$$

where $H_i=1+\frac{x_i}{\beta}$. Taking partial derivatives of $\ln l$ with respect to λ,α and β, we obtain

$$\frac{\partial\ln l}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln H_i+(\lambda-1)\sum_{i=1}^{n}(H_i^{\alpha}-1)^{-1}\ln H_i-2\beta\lambda\sum_{i=1}^{n}\left[1+\left[1-H_i^{-\alpha}\right]^{\lambda}\right]^{-1}(1-H_i^{-\alpha})^{\lambda-1}H_i^{-\alpha}\ln H_i,$$

$$\frac{\partial\ln l}{\partial\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln(1-H_i^{-\alpha})-2\beta\sum_{i=1}^{n}\ln(1-H_i^{-\alpha})\left[(1-H_i^{-\alpha})^{-\lambda}+1\right]^{-1}$$

and

$$\frac{\partial\ln l}{\partial\beta}=-(\alpha+1)\sum_{i=1}^{n}\frac{1}{H_i}\frac{\partial H_i}{\partial\beta}+(\lambda-1)\sum_{i=1}^{n}\frac{\alpha H_i^{-\alpha-1}}{1-H_i^{-\alpha}}\frac{\partial H_i}{\partial\beta}-2\beta\sum_{i=1}^{n}\frac{\lambda\alpha\left[1-H_i^{-\alpha}\right]^{\lambda-1}H_i^{-\alpha-1}}{1+\left[1-H_i^{-\alpha}\right]^{\lambda}}\frac{\partial H_i}{\partial\beta}$$
$$-2\sum_{i=1}^{n}\ln\left[1+\left[1-H_i^{-\alpha}\right]^{\lambda}\right],$$

where $\partial H_i/\partial\beta = -x_i/\beta^2$. The non-linear equations $\partial \ln l/\partial\alpha = 0$, $\partial \ln l/\partial\lambda = 0$, and $\partial \ln l/\partial\beta = 0$, have no explicit form solutions, so numerical technique is employed to determine the ML estimators.

4.2. Ordinary and weighted least squares estimators

Let $X_{(1)},\ldots,X_{(n)}$ denote the order statistics of the random sample of size n from a distribution function $F_{TIIHLL}(x;\pi)$. The LS and WLS estimators of α,β and λ can be obtained by minimizing, respectively, the following:

$$\text{LS} = \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{i}{n+1}\right]^2,$$

and $\text{WLS} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F(x_{(i)}) - E\left(F(x_{(i)})\right)\right]^2.$

Equivalently, the LS and WLS estimators of TIIHLL model can be obtained by minimizing the following

$$\text{LS} = \sum_{i=1}^{n}\left[2\left[1-\left(1+\frac{x_{(i)}}{\beta}\right)^{-\alpha}\right]^{\lambda}\left\{1+\left[1-\left(1+\frac{x_{(i)}}{\beta}\right)^{-\alpha}\right]^{\lambda}\right\}^{-1} - \frac{i}{n+1}\right]^2,$$

and $$\text{WLS} = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[2\left[1-\left(1+\frac{x_{(i)}}{\beta}\right)^{-\alpha}\right]^{\lambda}\left\{1+\left[1-\left(1+\frac{x_{(i)}}{\beta}\right)^{-\alpha}\right]^{\lambda}\right\}^{-1} - \frac{i}{n+1}\right]^2,$$

with respect to $(\alpha,\beta,\lambda)^T$.

4.3. The Anderson-Darling estimators

The AD is a type of minimum distance estimators (also called maximum goodness-of-fit estimators) which is based on the difference between the estimate of the cdf and the empirical distribution function. The AD estimators are obtained by minimizing the following

$$\text{A}(\alpha,\beta,\lambda) = -n - \frac{1}{n} + \sum_{i=1}^{n}(2i-1)\left\{\log F\left(x_{(i)};\alpha,\beta,\lambda\right) + \log R\left(x_{(n+1-i)};\alpha,\beta,\lambda\right)\right\},$$

with respect to $(\alpha,\beta,\lambda)^T$.

4.4. Simulation study

Here, a simulation study is implemented to compare the performance of different estimates of the TIIHLL distribution. We generate random data from TIIHLL distribution for different sample sizes and different parameter values. The simulation study is repeated $N = 1000$ times each with sample size $n = 10$, 30, 50, 100 and the selected parameter values. The ML, LS, WLS and AD estimates of α,β and λ are computed. Mean square errors (**MSEs**) and relative biases (**RBs**) measures are computed based on the resulting estimates. All the computations are done via Mathcad (14). Four set parameter values are selected as, Set 1 ≡ $(\alpha=1.5,\beta=1.5,\lambda=1.5)$, Set 2 ≡ $(\alpha=2,\beta=1.5,\lambda=1.5)$, Set 3 ≡ $(\alpha=2,\beta=1.5,\lambda=2)$ and Set 4 ≡ $(\alpha=1.5,\beta=1.5,\ \lambda=1)$. Table 3 contains the outcomes of this simulation. From Table 3, we can conclude the following:

- The MSEs decrease as sample sizes increase for all estimates.
- Based on ML method; as the values of λ and α increase for fixed value of β, the MSEs are increasing for α and β estimates and are decreasing for λ estimate. Also, the RBs are decreasing for α and β estimates, whereas the values of RBs of λ estimate are increasing.
- For fixed values of α,β and as the value of λ increases, the RBs of α and β estimates are decreasing, whereas RBs values of λ estimate increase, in approximately most of the situations.
- The MSEs of the ML estimate are smaller than the corresponding for LS, WLS and AD estimates.
- The MSEs of ML for λ estimates are smaller than the corresponding for α and β.

Table 3 The MSEs and RBs of the ML, LS, WLS and AD estimates of TIIHLL distribution

n	Methods	Properties	Set 1			Set 2			Set 3			Set 4		
			α	β	λ	α	β	λ	α	β	λ	α	β	λ
			1.5	1.5	1.5	2	1.5	1.5	2	1.5	2	1.5	1.5	1
10	ML	MSE	0.5120	0.9480	0.4890	0.9046	0.7032	0.4394	0.7341	1.4578	0.3800	0.5397	0.4551	0.4431
		RB	0.4740	0.2850	0.0400	0.4742	0.3487	0.0230	0.4268	0.2204	0.0613	0.4870	0.3020	0.0438
	LS	MSE	1.4060	1.6550	1.7230	2.5900	1.6237	0.9716	2.6466	2.6766	1.9226	1.5871	1.1217	0.9810
		RB	0.5848	0.3874	0.4182	0.6588	0.4304	0.1238	0.6737	0.5699	0.4627	0.5276	0.0995	0.0437
	WLS	MSE	2.1100	0.9830	2.2500	2.8239	1.1493	1.9282	2.9177	1.8455	1.9069	1.4949	1.8364	1.7254
		RB	0.9684	0.6610	1.0000	0.7592	0.3724	0.6598	0.7820	0.3609	0.6915	0.6583	0.9923	0.6459
	AD	MSE	0.5842	1.2601	1.9961	2.0253	1.4714	1.5480	1.2354	2.9296	1.4902	0.5683	1.3875	0.6404
		RB	0.0799	0.6462	0.6220	0.4004	0.7191	0.7849	0.4777	0.8255	0.7532	0.4032	0.7704	0.6583
30	ML	MSE	0.4210	0.5310	0.1980	0.8766	0.5400	0.1866	0.8641	0.8983	0.2037	0.4971	0.2716	0.1926
		RB	0.4320	0.2870	0.1170	0.4679	0.3579	0.1187	0.4646	0.3137	0.1410	0.4693	0.3318	0.1273
	LS	MSE	1.0399	0.9406	1.0264	1.9951	0.9970	0.7806	1.9629	1.6464	1.4335	0.8897	0.6512	0.7854
		RB	0.6798	0.6466	0.6754	0.5970	0.3377	0.0235	0.5562	0.4660	0.3028	0.3973	0.1139	0.1127
	WLS	MSE	2.1082	0.9695	2.2424	1.6462	0.9751	1.1518	1.7155	1.4935	1.1394	0.6767	1.1830	0.8667
		RB	0.9680	0.6560	0.9973	0.5065	0.2049	0.2834	0.5308	0.1856	0.3168	0.2855	0.5657	0.2685
	AD	MSE	0.4627	1.4416	1.3859	1.1969	1.4373	1.5198	1.1856	2.8524	1.4659	0.4986	0.6675	0.5731
		RB	0.3183	0.7870	0.7709	0.5288	0.7960	0.8169	0.4728	0.8404	0.8009	0.1125	0.5151	0.6776

Table 3 (Continued)

n	Methods	Properties	Set 1			Set 2			Set 3			Set 4		
			α	β	λ	α	β	λ	α	β	λ	α	β	λ
			1.5	1.5	1.5	2	1.5	1.5	2	1.5	2	1.5	1.5	1
50	ML	MSE	0.3730	0.4787	0.1598	0.8662	0.4830	0.1603	0.7707	0.7129	0.1777	0.4463	0.2235	0.1695
		RB	0.4067	0.2806	0.1155	0.4652	0.3472	0.1384	0.4388	0.2920	0.1483	0.4448	0.3317	0.1221
	LS	MSE	0.9383	0.8885	0.7629	1.7956	0.8913	0.7272	1.8836	1.6432	1.4157	0.6981	0.6272	0.7814
		RB	0.4639	0.2419	0.4267	0.5452	0.2691	0.0101	0.5677	0.4707	0.3177	0.3521	0.2253	0.1720
	WLS	MSE	2.1072	0.9646	2.2111	0.9478	0.8132	0.7550	1.0154	1.4012	0.7600	0.6374	0.9400	0.5434
		RB	0.9677	0.6548	0.9776	0.3443	0.0586	0.0507	0.3662	0.0703	0.0811	0.1549	0.4124	0.1626
	AD	MSE	0.4493	1.0059	1.3087	1.1029	1.3451	1.4557	0.8903	2.2777	1.3907	0.4564	0.6494	0.5553
		RB	0.0216	0.6158	0.6889	0.4849	0.7687	0.7966	0.2349	0.7289	0.7551	0.0690	0.4954	0.6582
100	ML	MSE	0.2841	0.4177	0.1350	0.8653	0.4209	0.1475	0.6851	0.5469	0.1475	0.3502	0.1583	0.1409
		RB	0.3550	0.1907	0.1360	0.4650	0.3400	0.1657	0.4138	0.2618	0.1721	0.3941	0.2693	0.1314
	LS	MSE	0.8164	0.7431	0.6323	1.6580	0.8161	0.7133	1.7727	1.5291	1.3978	0.6346	0.5645	0.7475
		RB	0.4298	0.1983	0.4133	0.5260	0.2354	0.0308	0.5442	0.4504	0.3087	0.3282	0.2606	0.2044
	WLS	MSE	2.1056	0.9593	2.1668	0.9014	0.7728	0.7482	0.7198	0.9622	0.5076	0.4475	0.5749	0.3501
		RB	0.9674	0.6529	0.9587	0.3605	0.1260	0.0231	0.3036	0.0448	0.0122	0.0824	0.2389	0.0843
	AD	MSE	0.4121	0.9568	1.1445	0.9765	1.2540	1.3987	0.8235	2.2728	1.2839	0.4231	0.6429	0.5422
		RB	0.2807	0.7083	0.6496	0.4533	0.7403	0.7790	0.2809	0.7555	0.7785	0.0279	0.5114	0.6849

5. Data Analysis

We employ a real data set to compare the fits of the TIIHLL distribution with five models. We consider the models, namely; half-logistic L (**HLL**) (Anwar and Zahoor 2018), inverse L (**IL**), EL, McDonald Weibull (**McW**) (Cordeiro et al. 2014), and beta Weibull (**BW**) (Lee et al. 2007) distributions.

The data set consists of the relief times of twenty patients receiving an analgesic (Gross and Clark 1975). The data are 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

Table 4 gives the ML estimates (corresponding standard error (**SE**)) of the model parameters. The following criterions are used to select the distribution with the best fit; minus double log-likelihood $(-2\log L)$, Anderson Darling (A^*) statistic, Cramér-von Mises (W^*) statistic, Akaike information criterion (**AIC**), corrected AIC (**AICc**), Bayesian information criterion (**BIC**) and Hannan-Quinn information criterion (**HQIC**) are presented. Values of these criterions are outlined in Table 5.

Table 4 ML estimates and SEs for the data set

Model	ML estimates and SEs				
TIIHLL(α,β,λ)	5.7900 (6.081)	1.1530 (2.931)	217.3870 (742.820)	-	-
HLL(α,β)	41.9010 (55.802)	0.0190 (0.026)	-	-	-
IL(α,β)	73.4350 (147.356)	0.0240 (0.048)	-	-	-
EL(α,β,θ)	22.5860 (50.211)	8.9610 (23.438)	40.8900 (38.299)	-	-
McW(α,β,a,b,c)	2.7738 (6.380)	0.3802 (0.188)	79.1080 (119.131)	17.8976 (39.511)	3.0063 (13.968)
BW(α,β,a,b)	0.8314 (0.954)	0.6126 (0.340)	29.9468 (40.413)	11.6319 (21.900)	-

Table 5 Goodness-of-fit statistics for the data set

Model	$-2\log L$	AIC	AICc	BIC	HQIC	A^*	W^*
TIIHLL	30.808	36.808	38.308	34.711	37.391	0.15934	0.02882
HLL	59.859	63.859	64.565	62.461	64.247	8.24537	0.47749
IL	65.588	69.588	70.293	68.190	69.976	16.9461	0.50852
EL	32.091	38.091	39.591	35.994	38.674	0.28152	0.04825
McW	33.907	43.907	48.193	40.412	44.879	0.46927	0.08021
BW	34.396	42.396	45.063	39.600	43.174	0.51316	0.08730

Based on the results of Table 5, it is noted that the TIIHLL model has the lowest values of the considered statistics, so it could be chosen as the best model for this data. Figure 3 displays the histogram and PP plots of the data in order to assess if the TIIHLL model is an appropriate model. Also, Figure 4 represents the empirical cdf and their estimated survival functions.

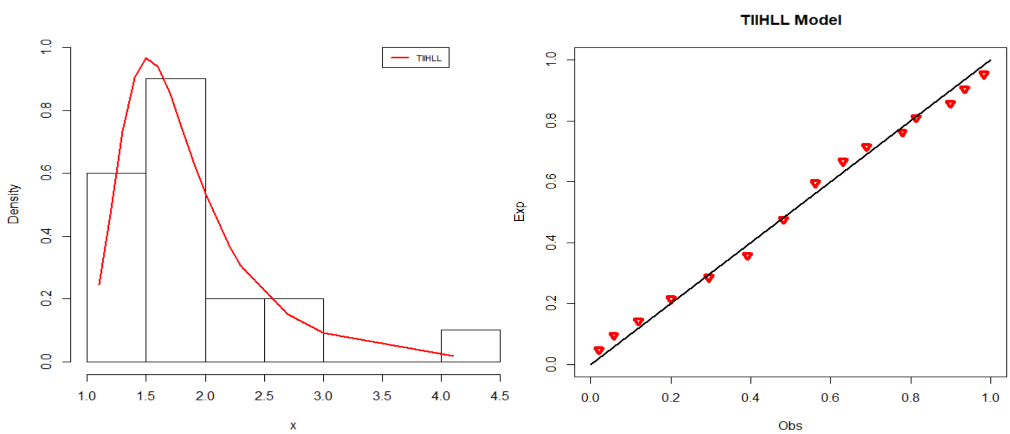

Figure 3 The empirical pdf and PP plots of the TIIHLL model

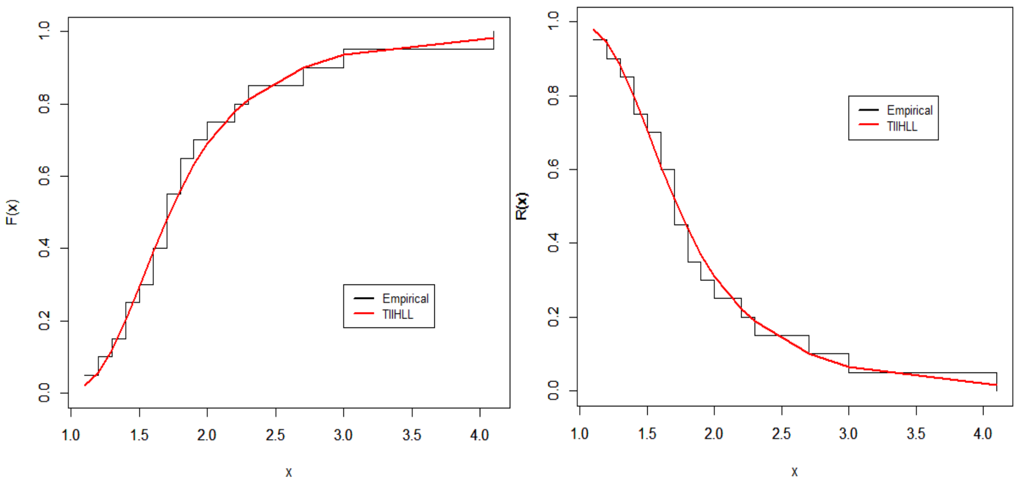

Figure 4 The empirical cdf and sf of the TIIHLL model

The plots in Figures 3 and 4 affirm the results of the analysis that the TIIHLL model is more suitable for these data than the other competing distributions.

6. Conclusions

A new three-parameter lifetime distribution, referred to Type II Half Logistic Lomax, is introduced. The new model provides more flexible distribution than the Lomax distribution. Structural properties are discussed. The model parameters of the TIIHLL distribution are estimated by using the maximum likelihood, least squares, weighted least squares and Anderson-Darling methods. A simulation study is provided to assess and compare the performance of the parameters. We prove empirically via a real lifetime data set that the TIIHLL model reveals its superiority over other competitive models.

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