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A New Family of Upper-Truncated Distributions: Properties and Estimation

Amal S. Hassan, Mohamed A. H. Sabry and Ahmed M. Elsehetry*
Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research (FGSSR), Cairo University, Egypt
*Corresponding author; e-mail: ah_sehetry@hotmail.com

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Abstract

In this paper, a new truncated distribution related to Lomax distribution is introduced. The proposed distribution is referred to as upper-truncated Lomax distribution. Our purpose in this study includes introducing a new family of probability distributions based on the new [0,1] truncated Lomax distribution. Statistical properties of the [0,1] truncated Lomax-G family like; moments, moment generating function, probability weighted moments, quantile function, order statistics and Rényi entropy are derived. Some sub-models of the family like; truncated Lomax-uniform, truncated Lomax-linear failure rate, truncated Lomax-Frèchet and truncated Lomax-power function distributions are discussed. We discuss the estimation of the model parameters via maximum likelihood method in case of complete and censored samples. Furthermore, a simulation study is provided to evaluate the validity of maximum likelihood estimates for one sub-model. Finally, analysis of real data set, representing the breaking stress of carbon fibers, is conducted to demonstrate the usefulness of truncated Lomax-Frèchet distribution compared with some competitor distributions.

Keywords: Truncated distributions, Lomax distribution, orders statistics, maximum likelihood method, censored samples.

1. Introduction

Recently, generated families of distributions have attracted the attention of several authors. Some of the generators are the beta-G (Eugene et al. 2002), Kumaraswamy-G (Cordeiro and de Castro 2011), exponentiated generalized-G (Cordeiro et al. 2013), transformed–transformer (Alzaatreh et al. 2013), Weibull-G (Bourguignon et al. 2014), exponentiated half-logistic-G (Cordeiro et al. 2014a), Lomax-G (Cordeiro et al. 2014b), the beta odd log-logistic generalized-G (Cordeiro et al. 2016), exponentiated Weibull-G (Hassan and Elgarhy 2016a), Kumaraswamy Weibull-G (Hassan and Elgarhy 2016b), additive Weibull-G (Hassan and Hemeda 2016), exponentiated extended-G (Elgarhy et al. 2017), Type II half logistic-G (Hassan et al. 2017a), generalized additive Weibull-G (Hassan et al. 2017b), [0,1] truncated Fréchet-G (Abid and

Abdulrazak 2017), Lomax-R{Y} (Mansoor et al. 2017), odd Frèchet-G (Haq and Elgarhy 2018), inverse Weibull-G (Hassan and Nassr 2018), and power Lindley-G (Hassan and Nassr 2019) among others.

The truncated distributions have been extensively applied, essentially in life-testing and reliability studies. Truncated form of a distribution is results from applying bound on the range of the distribution so that it is defined properly on a subset of the original range. Hence, truncated distributions are utilized in ways where occurrences are limited to values which lie above or below a given threshold or within a specified range. The lower (left) truncated distribution is obtained if occurrences are limited to values which lie below a given threshold. On the other hand, the upper (right) truncated distribution arises if the occurrences are limited to values which lie above a given threshold.

The Lomax distribution has been widely applied in some areas, such as, analysis of income and wealth data, modeling business failure data, biological sciences, model firm size and queuing problems (see for example Harris 1968, Atkinson and Harrison 1978, and Hassan and Al-Ghamdi 2009). The cumulative distribution function (cdf) and probability density function (pdf) of the Lomax distribution are given, respectively, by

$$F_L(t;\alpha,\lambda) = 1-\lambda^{\alpha}(\lambda+t)^{-\alpha}, \qquad t,\alpha,\lambda>0,$$

and

$$f_L(t;\alpha,\lambda) = \alpha\lambda^{\alpha}(\lambda+t)^{-(\alpha+1)},$$

where α and λ are the shape and scale parameters respectively. Let $\lambda=1$ then, we can write the pdf and cdf of the Lomax distribution with one parameter α as follows:

$$G_L(t;\alpha) = 1-(1+t)^{-\alpha}, \qquad t,\alpha>0,$$

and

$$g_L(t;\alpha) = \alpha(1+t)^{-(\alpha+1)}.$$

Three motivations are considered here. Firstly, a new upper ([0,1]) truncated Lomax distribution is introduced. Secondly, we propose a new upper ([0,1]) truncated family of probability distribution based on ([0,1]) truncated Lomax distribution. Thirdly, we derive some of its statistical properties besides estimating the model parameters based on complete and censored samples. This paper is organized as follows: in Section 2, [0,1] truncated Lomax distribution is introduced. Sections 3 and 4 define [0,1] Lomax-G and investigate some of its general statistical properties respectively. In Section 5, some new sub- models of proposed family are considered. In Sections 6 and 7 the maximum likelihood (ML) estimators and simulation issues of the truncated Lomax Fréchet parameters via complete and censored samples are discussed, respectively. Real data example is presented in Section 8 and article ends with concluding remarks.

2. Truncated Lomax Distribution

In this section, we introduce the [0,1] truncated Lomax (TL) random variable.

Definition *A random variable T has the [0,1] TL distribution with parameter* α, *say TL*(α), *if its pdf has the following form*

$$r_{TL}(t;\alpha)=\frac{g_L(t;\alpha)}{G_L(1;\alpha)-G_L(0;\alpha)}=\frac{\alpha(1+t)^{-(\alpha+1)}}{1-2^{-\alpha}},\qquad 0<t<1,\alpha>0, \tag{1}$$

where $g_L(t;\alpha)$ *and* $G_L(t;\alpha)$ *are, respectively, the pdf and the cdf of the Lomax random variable* T *with parameter* α.

The cdf corresponding to (1) is as follows

$$R_{TL}(t;\alpha)=\frac{\int_0^t g_L(t;\alpha)dt}{G_L(1;\alpha)-G_L(0;\alpha)}=\frac{G_L(t;\alpha)-G_L(0;\alpha)}{G_L(1;\alpha)-G_L(0;\alpha)}=\frac{1-(1+t)^{-\alpha}}{1-2^{-\alpha}}. \tag{2}$$

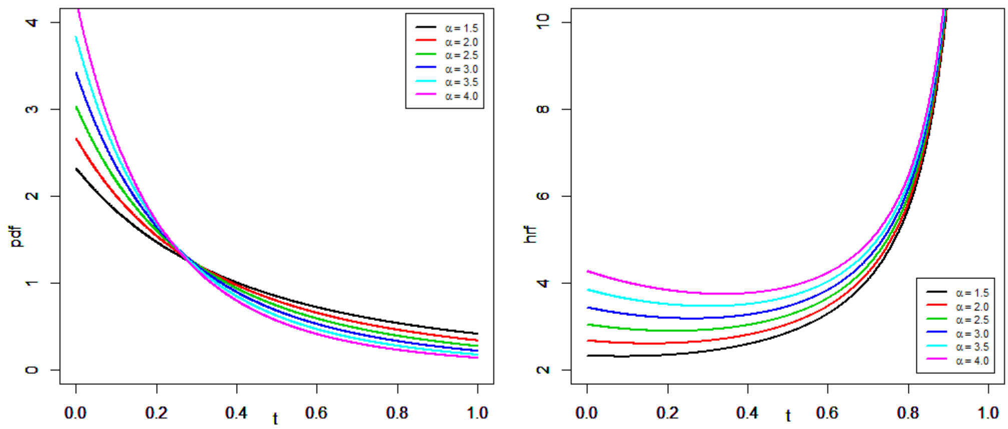
The survival function and hazard rate function (hrf) of the [0,1] TL are, respectively, given by,

$$\bar{R}_{TL}(t;\alpha)=\frac{(1+t)^{-\alpha}-2^{-\alpha}}{1-2^{-\alpha}},$$

and

$$h_{TL}(t;\alpha)=\frac{\alpha(1+t)^{-(\alpha+1)}}{(1+t)^{-\alpha}-2^{-\alpha}}.$$

A variety of possible shapes of the pdf and the hrf of the [0,1] TL distribution for some choices of values of parameters are represented in Figure 1.



(a) (b)

Figure 1 Plots of: (a) $r_{TL}(t;\alpha)$ and (b) $h_{TL}(t;\alpha)$ of the [0,1] TL distribution

It can be detected from Figure 1 that the pdf shape can be uni-model, reversed J-shaped and right skewed. Also, the shape of the hrf of the [0,1] TL distribution could be increasing, and J-shaped. Furthermore; reversed hrf and cumulative hrf are, respectively, given by

$$\tau_{TL}(t;\alpha)=\frac{\alpha(1+t)^{-(\alpha+1)}}{1-(1+t)^{-\alpha}},$$

and

$$H_{TL}(t;\alpha)=-\ln\left(\frac{(1+t)^{-\alpha}-2^{-\alpha}}{1-2^{-\alpha}}\right).$$

3. [0,1] Lomax-G Family

In this section, a new truncated family of distribution is introduced based on [0,1] TL distribution. Expansions of its pdf and cdf are obtained. Also, the quantile function is derived. The cdf of the truncated Lomax-G (TL-G) family is defined as

$$F_{TL-G}(x;\alpha,\xi)=\int_{0}^{G(x;\xi)}\frac{\alpha(1+t)^{-(\alpha+1)}}{1-2^{-\alpha}}dt=A\left(1-\left(1+G(x;\xi)\right)^{-\alpha}\right),\tag{3}$$

where $\alpha>0,\ A=\dfrac{1}{1-2^{-\alpha}},\xi$ is the parameter vector and $G(x;\xi)$ is the cdf of any distribution.

A random variable X that has cdf (3) will be denoted by $X\sim TL-G.$ The pdf corresponding to (3) is as

$$f_{TL-G}(x;\alpha,\xi)=\alpha A g(x;\xi)\left(1+G(x;\xi)\right)^{-\alpha-1},\tag{4}$$

where $g(x;\xi)$ is pdf corresponding to cdf $G(x;\xi)$ The survival function; say $\overline{F}_{TL-G}(x;\alpha,\xi)$ and hrf, say $h_{TL-G}(x;\alpha,\xi),$ are respectively, given by

$$\overline{F}_{TL-G}(x;\alpha,\xi)=1-A\left(1-\left(1+G(x;\xi)\right)^{-\alpha}\right),$$

and

$$h_{TL-G}(x;\alpha,\xi)=\frac{\alpha A g(x;\xi)\left(1+G(x;\xi)\right)^{-\alpha-1}}{1-A\left(1-\left(1+G(x;\xi)\right)^{-\alpha}\right)},$$

respectively. Expansion of the pdf (4) is obtained by using the following generalized binomial series

$$(1+Z)^{-\alpha}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\alpha+i-1}{i}Z^{i},\quad \alpha>0 \text{ and } |Z|<1.\tag{5}$$

Employing (5) in (4), the pdf of TL-G distribution, where α is real, becomes

$$f_{TL-G}\left(x;\alpha,\xi\right)=\sum_{i=0}^{\infty}\eta_{i}g(x;\xi)G(x;\xi)^{i},\tag{6}$$

where $\eta_{i}=A\alpha(-1)^{i}\binom{\alpha+1}{i}$. Also, an expansion for $\left[F_{TL-G}(x;\alpha,\xi)\right]^{h}$ is obtained, when h is an integer as follows

$$\left[F_{TL-G}(x;\alpha,\xi)\right]^{h}=\sum_{k=0}^{\infty}S_{k}G(x;\xi)^{k},\tag{7}$$

where

$$S_{k}=A^{h}\sum_{j=0}^{h}(-1)^{j+k}\binom{h}{j}\binom{\alpha j+k-1}{k}.$$

The quantile function, say $Q_{(u)}$ of X is obtained by inverting (3), as follows

$$Q_{(u)}=G^{-1}\left\{\left[1-(1-2^{-\alpha})u\right]^{\frac{-1}{\alpha}}-1\right\},$$

where u is a uniform random variable on the interval (0,1) and $G^{-1}(x;\xi)$ is the inverse cdf of

$G(x;\xi)$.

4. Main Properties of the TL-G Family

This section provides some statistical properties of the TL-G family of distributions.

4.1. Probability weighted moments

The class of probability weighted moments (PWM) is primarily used in estimating the parameters of a distribution whose inverse cannot be expressed explicitly. For a random variable X its PWM, denoted by $\tau_{r,h}$ is defined as

$$\tau_{r,h} = E\left(X^r \left[F_{TL-G}(x;\alpha,\xi)\right]^h\right) = \int_{-\infty}^{\infty} x^r f_{TL-G}(x;\alpha,\xi)\left[F_{TL-G}(x;\alpha,\xi)\right]^h dx. \tag{8}$$

The PWM of the TL-G is obtained by inserting (6) and (7) in (8), as follows

$$\tau_{r,h} = \sum_{i,k=0}^{\infty} \eta_i S_k \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+k} dx.$$

Then,

$$\tau_{r,h} = \sum_{i,k=0}^{\infty} \eta_i S_k \tau_{r,i+k},$$

where

$$\tau_{r,i+k} = \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^{i+k} dx.$$

4.2. Moments and moment generating function

The r^{th} moment of a random variable X having TL-G distribution is obtained as follows

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f_{TL-G}(x;\alpha,\xi) dx = \sum_{i=0}^{\infty} \eta_i \int_{-\infty}^{\infty} x^r g(x;\xi) G(x;\xi)^i dx.$$

Then,

$$\mu'_r = \sum_{i=0}^{\infty} \eta_i \tau_{r,i}.$$

For a random variable X it is known that, the moment generating function (MGF) is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r.$$

So, the MGF of the TL-G distributions is as follows

$$M_X(t) = \sum_{i,r=0}^{\infty} \frac{t^r}{r!} \eta_i \tau_{r.i}.$$

4.3. Order statistics

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with continuous distribution function $F(x)$. Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the corresponding ordered random sample from a population of size n. The pdf of the r^{th} order statistic is defined as

$$f_{x_{(r)}}(x;\alpha,\xi)=\frac{f_{TL-G}(x;\alpha,\xi)}{B(r,n-r+1)}\sum_{m=0}^{n-r}(-1)^m\binom{n-r}{m}F_{TL-G}(x;\alpha,\xi)^{m+r-1}, \tag{9}$$

where $B(.,.)$ is the beta function. The pdf of the r^{th} order statistic of TL-G family is derived by substituting (6) and (7) in (9), replacing h with $m+r-1$

$$f_{x_{(r)}}(x;\alpha,\xi)=\frac{g(x;\xi)}{B(r,n-r+1)}\sum_{m=0}^{n-r}\sum_{i,k=0}^{\infty}C^{*}G(x;\xi)^{i+k}, \tag{10}$$

where $C^{*}=(-1)^m\binom{n-r}{m}\eta_i S_k$, $g(x;\xi)$, and $G(x;\xi)$ are the pdf and cdf of any baseline distribution.

Further, the m^{th} moment of the r^{th} order statistic for the TL-G distribution is defined by

$$E(X_{(r)}^{m})=\int_{-\infty}^{\infty}x^m f_{x(r)}(x;\alpha,\xi)dx. \tag{11}$$

By substituting (10) in (11), then

$$E(X_{(r)}^{m})=\frac{1}{B(r,m-r+1)}\sum_{j=0}^{n-r}\sum_{i,k=0}^{\infty}C^{*}\int_{-\infty}^{\infty}x^m g(x;\xi)G(x;\xi)^{i+k}\,dx.$$

Then,

$$E(X_{(r)}^{m})=\frac{1}{B(r,m-r+1)}\sum_{j=0}^{n-r}\sum_{i,t,k,l=0}^{\infty}C^{*}\tau_{m,i+k}.$$

4.4. Rényi entropy

An entropy is a measure of variation or uncertainty of a random variable X. The Rényi entropy of X with pdf (4) is defined by

$$I_\delta(X)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}\left[f_{TL-G}(x;\alpha,\xi)\right]^{\delta}dx,\quad \delta>0 \ \text{ and } \ \delta\neq 1.$$

Now, we are considering the generalized binomial theory in the pdf (4), then the pdf $f_{TL-G}(x;\alpha,\xi)^{\delta}$ can be expressed as follows:

$$f_{TL-G}(x;\alpha,\xi)^{\delta}=\sum_{i=0}^{\infty}C_i g(x;\xi)^{\delta}G(x;\xi)^{i},$$

where $C_i=(-1)^i(A\alpha)^{\delta}\binom{\delta(\alpha+1)+i-1}{i}$. Therefore, the Rényi entropy of the TL-G family of distributions is given by

$$I_\delta(X)=\frac{1}{1-\delta}\log\sum_{i=0}^{\infty}C_i\int_{-\infty}^{\infty}g(x;\xi)^{\delta}G(x;\xi)^{i}\,dx.$$

5. Sub-Models

In this section, we define and describe four sub-models of the TL-G namely, TL-uniform, TL-linear failure rate, TL-Frèchet, and TL-power function distributions.

5.1. TL-uniform distribution

For $g(x;\theta)=\dfrac{1}{\theta}, 0<x<\theta,$ and $G(x;\theta)=\dfrac{x}{\theta}$ the pdf of the TL-uniform (TLU) is derived from (4) as follows

$$f_{\mathrm{TLU}}(x;\alpha,\theta)=\frac{A\alpha}{\theta}\left(1+\frac{x}{\theta}\right)^{-\alpha-1}, \quad 0<x<\theta.$$

The corresponding cdf takes the following form

$$F_{\mathrm{TLU}}(x;\alpha,\theta)=A\left(1-\left(1+\frac{x}{\theta}\right)^{-\alpha}\right).$$

The hrf of the TLU is given by

$$h_{\mathrm{TLU}}(x;\alpha,\theta)=\frac{A\alpha}{\theta}\left(1+\frac{x}{\theta}\right)^{-\alpha-1}\left[1-A\left(1-\left(1+\frac{x}{\theta}\right)^{-\alpha}\right)\right]^{-1}.$$

Plots of the pdf and hrf for the TLU are displayed in Figure 2.

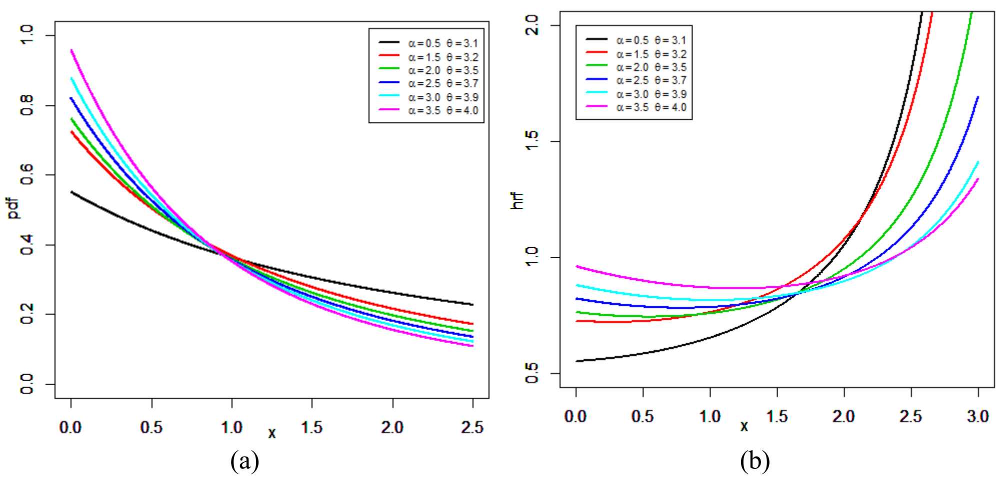


Figure 2 Plots of (a) $f_{\mathrm{TLU}}(x;\alpha,\theta)$ and (b) hrf $h_{\mathrm{TLU}}(x;\alpha,\theta)$ of the TLU distribution

5.2. TL-linear failure rate distribution

Let us consider the linear failure rate distribution with pdf; $g(x;a,b)=(a+bx)e^{-ax-\frac{b}{2}x^2}$, $x,a,b>0$ and cdf; $G(x;a,b)=1-e^{-ax-\frac{b}{2}x^2}$, hence we obtain the TL-linear failure rate (TLLFR) density function as

$$f_{\mathrm{TLLFR}}(x;\alpha,a,b)=A\alpha(a+bx)e^{-ax-\frac{b}{2}x^2}\left(2-e^{-ax-\frac{b}{2}x^2}\right)^{-\alpha-1}, x,a,b>0.$$

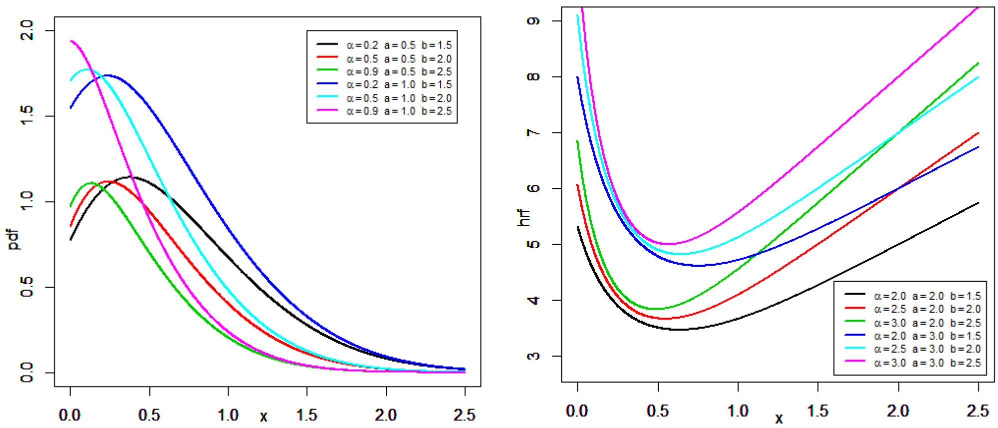
The cdf and hrf of the TLLFR distribution are given, respectively, by

$$F_{\text{TLLFR}}(x;\alpha,a,b) = A\left(1-\left(2-e^{-ax-\frac{b}{2}x^{2}}\right)^{-\alpha}\right),$$

and

$$h_{\text{TLLFR}}\left(x;\alpha,a,b\right) = A\alpha(a+bx)e^{-ax-\frac{b}{2}x^{2}}\left(2-e^{-ax-\frac{b}{2}x^{2}}\right)^{-\alpha-1}\left\{1-A\left(1-\left(2-e^{-ax-\frac{b}{2}x^{2}}\right)^{-\alpha}\right)\right\}^{-1}.$$

Plots of the pdf and hrf for the TLLFR are displayed in Figure 3.



(a) (b)

Figure 3 Plots of (a) $f_{\text{TLLFR}}(x;\alpha,a,b)$ and (b) $h_{\text{TLLFR}}(x;\alpha,a,b)$ of the TLLFR distribution

5.3. TL-Frèchet distribution

We consider the Frèchet distribution with pdf; $g(x;\mu,\delta)=\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}, x,\mu,\delta>0$ and cdf $G(x;\mu,\delta)=e^{-\left(\frac{\mu}{x}\right)^{\delta}}$, hence the TL-Frèchet (TLFr) density function is as

$$f_{\text{TLFr}}(x;\alpha,\mu,\delta) = A\alpha\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}\left(1+e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-\alpha-1}, x,\mu,\delta>0.$$

The cdf and hrf of the TLFr distribution are given, respectively, by

$$F_{\text{TLFr}}\left(x;\alpha,\mu,\delta\right) = A\left[1-\left(1+e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-\alpha}\right],$$

and

$$h_{\text{TLFr}}(x;\alpha,\mu,\delta) = A\alpha\delta\mu^{\delta}x^{-\delta-1}e^{-\left(\frac{\mu}{x}\right)^{\delta}}\left(1+e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-\alpha-1}\left\{1-A\left(1-\left(1+e^{-\left(\frac{\mu}{x}\right)^{\delta}}\right)^{-\alpha}\right)\right\}^{-1}.$$

Plots of pdf and hrf for the TLFr are displayed in Figure 4.

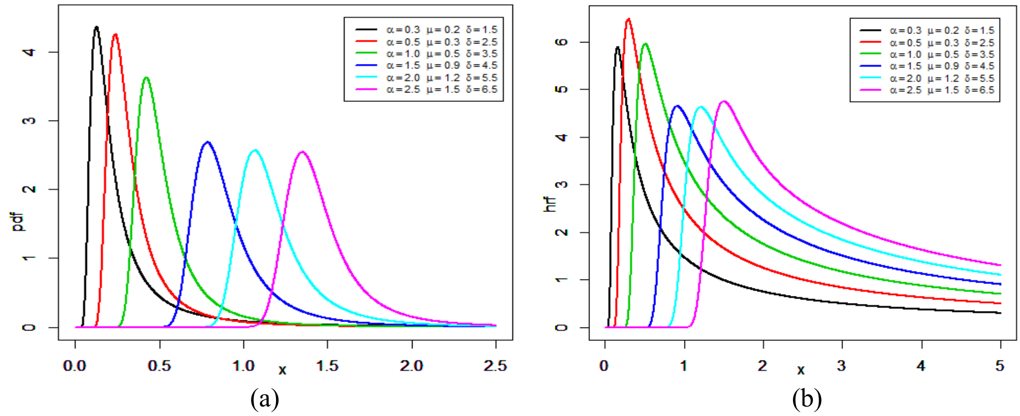


Figure 4 Plots of (a) $f_{\text{TLFr}}(x;\alpha,\mu,\delta)$ and (b) $h_{TLFr}(x;\alpha,\mu,\delta)$ of the TLFr distribution

5.4. TL-power function distribution

For $g(x;\beta,\gamma)=\frac{\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1}, 0<x<\gamma,\ \ \beta,\gamma>0$ and $G(x;\beta,\gamma)=\left(\frac{x}{\gamma}\right)^{\beta}$, we obtain the TL-power function (TLPF) density function as

$$f_{\text{TLPF}}(x;\alpha,\beta,\gamma)=\frac{A\alpha\beta}{\gamma}\left(\frac{x}{\gamma}\right)^{\beta-1}\left(1+\left(\frac{x}{\gamma}\right)^{\beta}\right)^{-\alpha-1}, 0<x<\gamma.$$

The cdf and hrf of the TLPF distribution are given, respectively, by

$$F_{\text{TLPF}}(x;\alpha,\beta,\gamma)=A\left(1-\left(1+\left(\frac{x}{\gamma}\right)^{\beta}\right)^{-\alpha}\right),$$

and

$$h_{\text{TLPF}}(x;\alpha,\beta,\gamma)=\frac{A}{\gamma}\alpha\beta\left(\frac{x}{\gamma}\right)^{\beta-1}\left(1+\left(\frac{x}{\gamma}\right)^{\beta}\right)^{-\alpha-1}\left[1-A\left(1-\left(1+\left(\frac{x}{\gamma}\right)^{\beta}\right)^{-\alpha}\right)\right]^{-1}.$$

Plots of the pdf and hrf for the TLPF are displayed in Figure 5.

6. Parameter Estimation Based on Complete Samples

In this section, we obtain the maximum likelihood (ML) estimators of the TL-G family in case of complete samples and simulation study is implemented to examine the performance of the ML estimates.

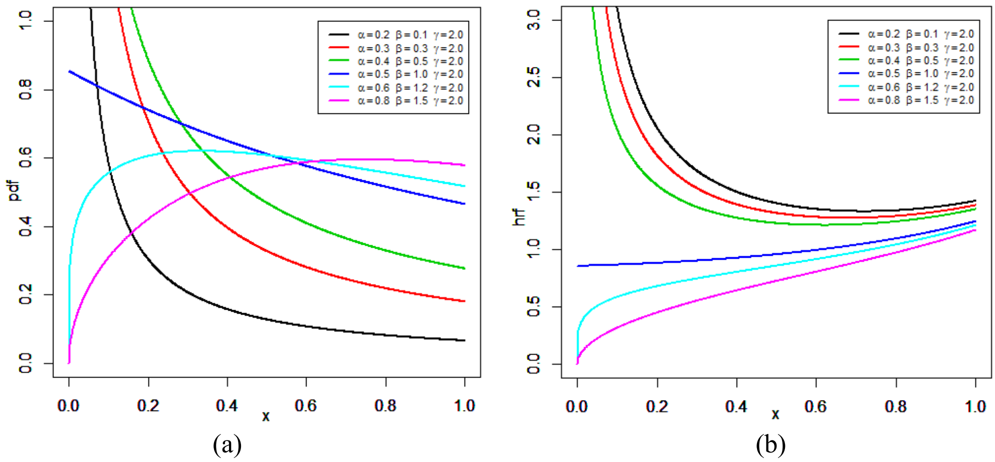


Figure 5 (a) $f_{\text{TLPF}}(x;\alpha,\beta,\gamma)$ and (b) $h_{\text{TLPF}}(x;\alpha,\beta,\gamma)$ of the TLPF distribution

6.1. Maximum likelihood estimators

Let $X_1, X_2, \ldots, X_n$ be the observed values from the TL-G family with set of parameter $\Phi = (\alpha, \xi)^T$. The log-likelihood function for parameter vector $\Phi = (\alpha, \xi)^T$ is obtained as follows

$$\ln(L, \Phi) = n \ln \alpha - n \ln(1 - 2^{-\alpha}) + \sum_{i=1}^{n} \ln g(x_i; \xi) - (\alpha + 1) \sum_{i=1}^{n} \ln\left[1 + G(x_i; \xi)\right].$$

The partial derivatives of the log-likelihood function with respect to α and ξ components of the score vector $U_L = (U_\alpha, U_{\xi_k})^T$ can be obtained as follows:

$$U_\alpha = \frac{n}{\alpha} - \frac{n 2^{-\alpha} \ln 2}{1 - 2^{-\alpha}} - \sum_{i=1}^{n} \ln\left[1 + G(x_i; \xi)\right],$$

and

$$U_{\xi_k} = \sum_{i=1}^{n} \frac{g_k'(x_i; \xi)}{g(x_i; \xi)} - (\alpha + 1) \sum_{i=1}^{n} \frac{G_k'(x_i; \xi)}{1 + G(x_i; \xi)}.$$

where $g_k'(x_i; \xi) = \partial g(x_i; \xi) / \partial \xi_k$ and $G_k'(x_i; \xi) = \partial G(x_i; \xi) / \partial \xi_k$. Setting U_α and U_{ξ_k} equal to zeros and solving these equations simultaneously yield the ML estimators $\hat{\Phi} = (\hat{\alpha}, \hat{\xi})^T$ of $\Phi = (\alpha, \xi)^T$. Unfortunately these equations cannot be solved analytically and numerical iterative methods can be employed to solve them.

6.2. Simulation study

The performance of the ML estimates is assessed in terms of the sample size n. A numerical evaluation is carried out to examine the performance of the ML estimates for TLFr model. The ML estimates are evaluated based on biases and mean square errors (MSEs). The simulation procedure is achieved via the MATHEMATICA package (Wolfram Research 2014). The simulation algorithm for generating random samples from TLFr distribution and ML estimates from those samples are shown as below:

- A random sample $X_1, X_2, \ldots, X_n$ of sizes; $n = 20, 30, 50$ and 100 are considered, these random samples are generated from the TLFr distribution by using inversion method.
- The values of parameters are considered as $(\alpha = 0.7, \mu = 0.3, \delta = 1.5)$ and $(\alpha = 1.2, \mu = 0.5, \delta = 1.2)$. The ML estimate of the TLFr model is evaluated based on parameters value and sample sizes.
- The process is repeated 10,000 times and then we obtain the means, biases and MSEs of the ML estimates for values of model parameters. Empirical results are reported in Table 1.

Table 1 ML estimates, Bias and MSE of the TLFr model parameters

n	Parameter	$\alpha = 0.7, \mu = 0.3, \delta = 1.5$			$\alpha = 1.2, \mu = 0.5, \delta = 1.2$		
		ML	Bias	MSE	ML	Bias	MSE
20	α	0.7004	0.0004	0.0004	1.2028	0.0028	0.0041
	μ	0.3232	0.0232	0.0091	0.5423	0.0423	0.0295
	δ	1.6792	0.1792	0.4033	1.3520	0.1520	0.2758
30	α	0.7008	0.0008	0.0003	1.2019	0.0019	0.0026
	μ	0.3122	0.0122	0.0052	0.5264	0.0264	0.0160
	δ	1.6034	0.1034	0.2209	1.3008	0.1008	0.1638
50	α	0.7002	0.0002	0.0001	1.2012	0.0012	0.0015
	μ	0.3068	0.0068	0.0025	0.5142	0.0142	0.0083
	δ	1.5577	0.0577	0.1141	1.2560	0.0560	0.0824
100	α	0.7001	0.0001	0.0001	1.2002	0.0001	0.0008
	μ	0.3047	0.0047	0.0012	0.5083	0.0083	0.0041
	δ	1.5419	0.0419	0.0560	1.2288	0.0288	0.0386

We can detect from Table 1 that the estimates are quite stable and are close to the true value of the parameters as the sample sizes increase.

7. Parameter Estimation Based on Censoring Samples

In reliability or lifetime testing experiments, most of the encountered data are censored due to various reasons such as time limitation, cost or other resources. Here, we discuss estimation of population parameters of the TL-G distributions based on two censoring schemes; namely, Type I and Type II. In Type-I censoring (TIC), we have a fixed time say; ω, but the number of items fail during the experiment is random. Whereas, in Type-II censoring (TIIC) scheme, the experiment is continued (i.e., time varies) until the specified number of failures c occurs.

7.1. ML estimators in case of TIC

Suppose that n items, whose lifetime's follow TL-G are placed on a life test, and the test is terminated at specified time ω before all n items have failed. The number of failures c and all failure times are random variables. The log-likelihood function, based on TIC, is given by

$$\ln(L_1,\Phi)=\ln\frac{n!}{(n-c)!}+(n-c)\ln\left(\left(1+G(\omega;\xi)\right)^{-\alpha}-2^{-\alpha}\right)-(n-c)\ln(1-2^{-\alpha})+c\ln\alpha$$

$$-c\ln(1-2^{-\alpha})+\sum_{i=1}^{c}\ln g(x_{(i)};\xi)-(\alpha+1)\sum_{i=1}^{c}\ln\left[1+G(x_{(i)};\xi)\right].$$

Then, the first partial derivatives of the log-likelihood are given by

$$U_{\alpha}=\frac{c}{\alpha}-\frac{c2^{-\alpha}\ln 2}{1-2^{-\alpha}}-\sum_{i=1}^{c}\ln\left[1+G(x_{(i)};\xi)\right]-\frac{(n-c)\left(1+G(\omega;\xi)\right)^{-\alpha}\ln\left(1+G(\omega;\xi)-2^{-\alpha}\ln 2\right)}{\left(1+G(\omega;\xi)\right)^{-\alpha}-2^{-\alpha}},$$

and

$$U_{\xi_k}=\sum_{i=1}^{c}\frac{g_k'(x_{(i)};\xi)}{g(x_{(i)};\xi)}-(\alpha+1)\sum_{i=1}^{c}\frac{G_k'(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}-\frac{(n-c)\alpha\left(1+G(\omega;\xi)\right)^{-\alpha-1}G_K'(\omega;\xi)}{\left(1+G(\omega;\xi)\right)^{-\alpha}-2^{-\alpha}},$$

and equating these partial derivatives to zeros and solving simultaneously yield the ML estimators of α and ξ based on TIC.

7.2. ML estimators in case of TIIC

Consider $X_{(1)}<X_{(2)}<\ldots<X_{(c)}$ be a TIIC sample of size n observed from lifetime testing experiment whose lifetime have the pdf (4). The log-likelihood based on TIIC, is given by

$$\ln(L_2,\Phi)=\ln\frac{n!}{(n-c)!}+(n-c)\ln\left(\left(1+G(x_{(c)};\xi)\right)^{-\alpha}-2^{-\alpha}\right)-(n-c)\ln\left(1-2^{-\alpha}\right)+c\ln\alpha$$

$$-c\ln(1-2^{-\alpha})+\sum_{i=1}^{c}\ln g(x_{(i)};\xi)-\left(\alpha+1\right)\sum_{i=1}^{c}\ln\left[1+G(x_{(i)};\xi)\right].$$

The first partial derivatives of $\ln(L_2,\Phi)$ are given by

$$U_{\alpha}=\frac{c}{\alpha}-\frac{c2^{-\alpha}\ln 2}{1-2^{-\alpha}}-\sum_{i=1}^{c}\ln\left[1+G(x_{(i)};\xi)\right]-\frac{(n-c)\left(1+G(x_{(c)};\xi)\right)^{-\alpha}\ln\left(1+G(x_{(c)};\xi)-2^{-\alpha}\ln 2\right)}{\left(1+G(x_{(c)};\xi)\right)^{-\alpha}-2^{-\alpha}},$$

and

$$U_{\xi_k}=\sum_{i=1}^{c}\frac{g_k'(x_{(i)};\xi)}{g(x_{(i)};\xi)}-(\alpha+1)\sum_{i=1}^{c}\frac{G_k'(x_{(i)};\xi)}{1+G(x_{(i)};\xi)}-\frac{(n-c)\alpha\left(1+G(x_{(c)};\xi)\right)^{-\alpha-1}G_K'(T;\xi)}{\left(1+G(x_{(c)};\xi)\right)^{-\alpha}-2^{-\alpha}}.$$

By solving $U_{\alpha}=0$ and $U_{\xi_K}=0$ numerically, the ML estimators of α and ξ are obtained.

7.3. Numerical studies

In this subsection, we provide numerical study to evaluate the performance of the ML estimates of the TLFr as sub model of the TL-G family based on TIC and TIIC schemes. The algorithm used here is outlined as follows:

- A random sample of sizes $n=20,30,50,$ and 100 are generated from the TLFr distribution under TIC and TIIC.

- Select initial values for parameters as; $(\alpha = 0.7, \mu = 0.3, \delta = 1.5)$ and $(\alpha = 1.2, \mu = 0.5, \delta = 1.2)$.
- Two termination times are selected as $\omega = 1.2$ and $\omega = 1.5$. The number of failure items; c, based on TIIC are selected as 60% and 80%.
- This process is repeated 10,000 times and then obtains the means, biases and MSEs of the ML estimates. Empirical results are reported in Tables 2 and 3.

Table 2 ML estimates, Biases and MSEs of TLFr distribution under TIC

n	Parameter	ω	$\alpha=0.7,\mu=0.3,\delta=1.5$			$\alpha=1.2,\mu=0.5,\delta=1.2$		
			ML	Bias	MSE	ML	Bias	MSE
20	α	1.2	0.7001	0.0001	0.0005	1.2006	0.0006	0.0058
		1.5	0.7008	0.0008	0.0005	1.1993	−0.0006	0.0052
	μ	1.2	0.3291	0.0291	0.0141	0.5390	0.0390	2.1941
		1.5	0.3249	0.0249	0.0116	0.5731	0.0731	0.10251
	δ	1.2	1.6659	0.1659	0.3756	1.3884	0.1884	0.3776
		1.5	1.6952	0.1952	0.4774	1.3552	0.1552	0.3269
30	α	1.2	0.6993	−0.0006	0.0003	1.1979	−0.0020	0.0037
		1.5	0.6989	−0.0010	0.0003	1.1983	−0.0016	0.0032
	μ	1.2	0.3207	0.0207	0.0081	0.5547	0.0547	0.1311
		1.5	0.3221	0.0221	0.0072	0.5425	0.0425	0.0274
	δ	1.2	1.6020	0.1020	0.2618	1.3168	0.1168	0.2001
		1.5	1.6175	0.1175	0.2266	1.3097	0.1097	0.1876
50	α	1.2	0.7001	0.0001	0.0001	1.2001	0.0001	0.0023
		1.5	0.6991	−0.0008	0.0002	1.2008	0.0008	0.0022
	μ	1.2	0.3084	0.0084	0.0032	0.5238	0.0238	0.0159
		1.5	0.3124	0.0124	0.0034	0.5259	0.0259	0.0162
	δ	1.2	1.5607	0.0607	0.1320	1.2613	0.0613	0.1072
		1.5	1.5614	0.0614	0.1242	1.2762	0.0762	0.1067
100	α	1.2	0.7003	0.0003	0.0001	1.2009	0.0009	0.0011
		1.5	0.7003	0.0003	0.0001	1.2000	0.0001	0.0010
	μ	1.2	0.3039	0.0039	0.0016	0.5095	0.0095	0.0062
		1.5	0.3039	0.0039	0.0015	0.5103	0.0103	0.0055
	δ	1.2	1.5273	0.0273	0.0592	1.2323	0.0323	0.0522
		1.5	1.5347	0.0347	0.0581	1.2309	0.0309	0.0497

From Table 2 we conclude that as the sample size n increases the MSE of ML estimates decrease. Also, as the termination time ω increases, the MSE of estimates decreases. Based on Table 3, we can see that as the sample size n increases the MSE of ML estimates decreases. Also, as the censoring level time $X_{(c)}$ increases, the MSE of ML estimates decreases.

Table 3 ML estimates, Biases and MSEs of TLFr distribution under TIIC

n	Parameter	$X_{(c)}$	$\alpha=0.7,\mu=0.3,\delta=1.5$			$\alpha=1.2,\mu=0.5,\delta=1.2$		
			ML	Bias	MSE	ML	Bias	MSE
20	α	60%	0.7003	0.0003	0.0012	1.2106	0.0106	0.0102
		80%	0.7008	0.0008	0.0007	1.2099	0.0099	0.0065
	μ	60%	0.3747	0.0747	0.6608	0.5435	0.0436	3.0435
		80%	0.3448	0.0448	0.0570	0.5615	0.0615	0.0695
	δ	60%	1.9949	0.4949	3.1561	1.5920	0.3920	1.4584
		80%	1.7739	0.2739	0.7319	1.4385	0.2385	0.5921
30	α	60%	0.7011	0.0011	0.0007	1.2046	0.0046	0.0066
		80%	0.7029	0.0029	0.0004	1.2045	0.0045	0.0041
	μ	60%	0.3522	0.0522	0.0429	0.6031	0.1031	0.1397
		80%	0.3167	0.0167	0.0084	0.5466	0.0466	0.0381
	δ	60%	1.8040	0.3040	0.8836	1.5184	0.3183	1.8815
		80%	1.7030	0.2030	0.4078	1.3574	0.1574	0.2922
50	α	60%	0.7009	0.0009	0.0005	1.2031	0.0031	0.0041
		80%	0.7009	0.0009	0.0002	1.2029	0.0029	0.0025
	μ	60%	0.3261	0.0261	0.0128	0.5438	0.0439	0.0345
		80%	0.3104	0.0104	0.0047	0.5269	0.0269	0.0168
	δ	60%	1.6538	0.1538	0.3402	1.3363	0.1363	0.3108
		80%	1.5901	0.0901	0.1932	1.3002	0.1002	0.1546
100	α	60%	0.7016	0.0016	0.0002	1.1994	−0.0005	0.0019
		80%	0.6998	−0.0001	0.0001	1.2017	0.0017	0.0013
	μ	60%	0.3076	0.0076	0.0037	0.5280	0.0280	0.0152
		80%	0.3075	0.0075	0.0021	0.5092	0.0092	0.0065
	δ	60%	1.5797	0.0797	0.1389	1.2742	0.0741	0.1006
		80%	1.5460	0.0460	0.0744	1.2322	0.0322	0.0519

8. Data Analysis

In this section, real data set is analyzed using MATHCAD package (Mathsoft 2010) to illustrate the merit of TLFr distribution compared to some models; namely, Frèchet (Fr), exponentiated Frèchet (EFr) (Nadarajah and Kotz 2003), Marshall-Olkin Frèchet (MOFr) (Krishna et al. 2013), transmuted Frèchet (TFr) (Mahmoud and Mandouh 2013), Kumaraswamy Frèchet (KFr) (Mead and Abd-Eltawab 2014), transmuted Marshall-Olkin Frèchet (TMOFr) (Afify et al. 2015) and the Weibull Frèchet (WFr) (Afify et al. 2016). Their density functions for $(x>0)$ are given in Table 4.

We obtain the ML estimates, and standard errors (SEs) of the model parameters. To compare the distribution models, we consider criteria like; minus two of log-likelihood function (-2lnL), Akaike information criterion (AIC), the corrected Akaike information criterion (AICc), the Bayesian information criterion (BIC), the Hannan-Quinn information criterion (HQIC). However, the better distribution corresponds to the smaller values of −2lnL, AIC, AICc, BIC and HQIC.

Table 4 The pdfs for some lifetime distributions

Model	The probability density function
Fr	$f_{Fr}\left(x;\alpha,\beta\right)=\beta\alpha^{\beta}x^{-(\beta+1)}e^{-\left(\frac{\alpha}{x}\right)^{\beta}};\quad x,\alpha,\beta>0.$
EFr	$f_{\mathrm{EFr}}\left(x;\alpha,\beta,a\right)=a\beta\alpha^{\beta}\left[1-e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right]^{a-1}x^{-(1+\beta)}e^{-\left(\frac{\alpha}{x}\right)^{\beta}};x,\alpha,\beta,a>0.$
MOFr	$f_{\mathrm{MOFr}}\left(x;\alpha,\beta,a\right)=a\beta\alpha^{\beta}x^{-\beta-1}e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\left\{a+\left(1-a\right)e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right\}^{-2};\qquad x,\alpha,\beta,a>0.$
TFr	$f_{\mathrm{TFr}}\left(x;\alpha,\beta,b\right)=\beta\alpha^{\beta}x^{-\beta-1}e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\left\{1+b-2be^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right\};\ x,\alpha,\beta,b>0.$
KFr	$f_{\mathrm{KFr}}\left(x;\alpha,\beta,a,b\right)=ab\beta\alpha^{\beta}x^{-\beta-1}e^{-a(\alpha/x)^{\beta}}\left[1-e^{-a(\alpha/x)^{\beta}}\right]^{b-1};\ x,\alpha,\beta,a,b>0.$
TMOFr	$f_{\mathrm{TMOFr}}\left(x;\alpha,\beta,a,b\right)=a\beta\alpha^{\beta}x^{-\beta-1}\left\{a+\left(1-a\right)e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\right\}^{-2}e^{-\left(\frac{\alpha}{x}\right)^{\beta}}\left(1+b-\frac{2be^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{a+\left(1-a\right)e^{-\left(\frac{\alpha}{x}\right)^{\beta}}}\right)$ $;x,\alpha,\beta,a,b>0.$
WFr	$f_{WFr}\left(x;\alpha,\beta,a,b\right)=ab\beta\alpha^{\beta}x^{-\beta-1}e^{-b(\alpha/x)^{\beta}}\left[1-e^{-(\alpha/x)^{\beta}}\right]^{-b-1}\exp\left(-a\left\{\mathrm{e}^{[\alpha/x]^{\beta}}-1\right\}^{-b}\right)$ $;x,\alpha,\beta,a,b>0.$

The data set consists of 100 observations of breaking stress of carbon fibers (in Gba) given by Nichols and Padgett (2006) are stated below:
0.39, 0.81, 0.85, 0.98, 1.08, 1.12, 1.17, 1.18, 1.22, 1.25, 1.36, 1.41, 1.47, 1.57, 1.57, 1.59, 1.59, 1.61, 1.61, 1.69, 1.69, 1.71, 1.73, 1.80, 1.84, 1.84, 1.87, 1.89, 1.92, 2.00, 2.03, 2.03, 2.05, 2.12, 2.17, 2.17, 2.17, 2.35, 2.38, 2.41, 2.43, 2.48, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.76, 2.77, 2.79, 2.81, 2.81, 2.82, 2.83, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.51, 3.56, 3.60, 3.65, 3.68, 3.68, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90, 4.91, 5.08, 5.56.

Table 5 gives the ML estimates of the eight models and their SEs. Values of, −2lnL, AIC, BIC, HQIC and AICc are recorded in Table 6.

Table 5 The ML estimates and SEs of the model parameters for the data set

Model	ML estimates and SEs					
	α	β	a	b	μ	δ
TLFr	62.7080 (43.6460)				146.9610 (103.9010)	0.5180 (0.0730)
Fr	1.8705 (0.1120)	1.7766 (0.1130)				
EFr	69.1489 (57.3490)	0.5019 (0.0800)	145.3275 (122.9240)			
MOFr	2.3066 (0.4980)	1.5796 (0.1600)	0.5988 (0.3091)			
TFr	1.9315 (0.0970)	1.7435 (0.0760)		0.0819 (0.1980)		
KFr	2.0556 (0.0710)	0.4654 (0.0070)	6.2815 (0.0630)	224.1800 (0.1640)		
TMOFr	0.6496 (0.0680)	3.3313 (0.2060)	101.923 (47.625)	0.2936 (0.2700)		
WFr	0.6942 (0.3630)	0.6178 (0.2840)	0.0947 (0.4560)	3.5178 (2.9420)		

Table 6 The values of −2lnL, AIC, BIC, HQIC and AICc for the data set

Model	Goodness of fit criteria				
	−2lnL	AIC	BIC	HQIC	AICc
TLFr	286.246	292.246	292.246	295.409	292.496
Fr	344.300	348.300	353.500	350.400	348.400
EFr	289.700	295.700	303.500	298.900	296.000
MOFr	345.300	351.300	359.100	354.500	351.600
TFr	344.500	350.500	358.300	353.600	350.700
KFr	289.100	297.100	307.500	301.300	297.500
TMOFr	302.000	310.000	320.400	314.200	310.400
WFr	286.600	294.600	305.000	298.800	295.000

We find that the TLFr distribution with three parameters provides a better fit than the proposed seven models. It has the smallest AIC, BIC, HQIC and AICc values among those considered here.

Moreover, the plots of empirical cdf of the data set and probability-probability (PP) plots of TLFr, WFr, EFr, KFr, TMOFr, Fr, TFr and MOFr models are displayed in Figures 6 and 7, respectively.

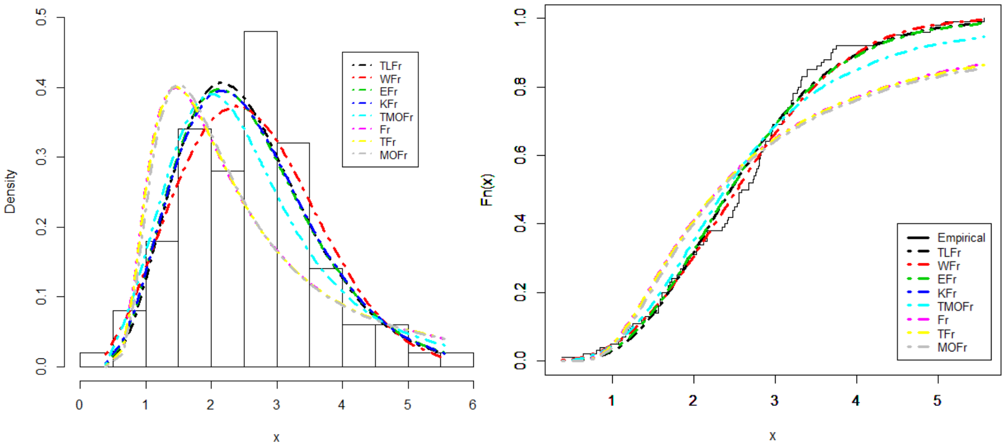


Figure 6 Estimated pdf and cdf for the data set of models for the data set

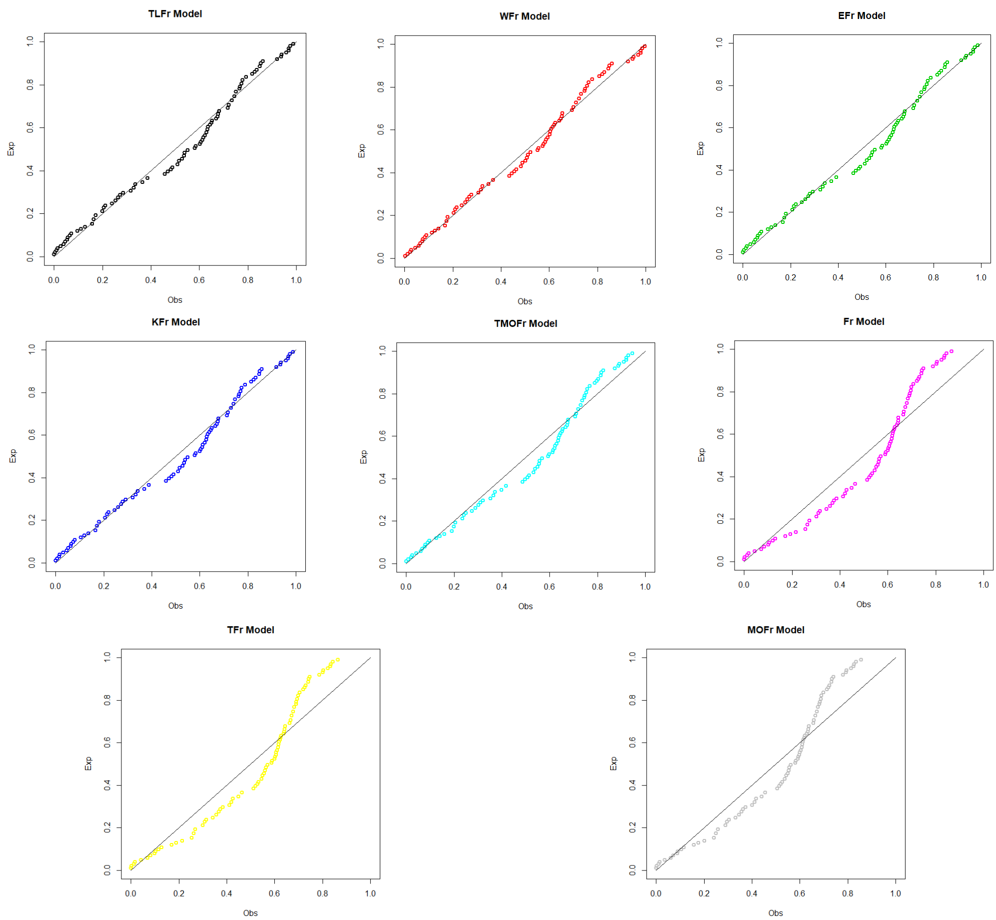


Figure 7 PP plots of the fitted models for the data set

Also from Figures 6 and 7, we can see the TLFr distribution provides a better fit than the competitive models.

9. Summary and Conclusions

In this paper, a new [0,1] truncated Lomax distribution is introduced. Based on the new truncated Lomax distribution, we propose a new truncated class of probability distributions called truncated Lomax-G family. Four special sub-models are presented. We investigate several structural properties of the family, such as, linear representations for the density function and cumulative distribution function, expressions for the ordinary moments, generating function and order statistics. The model parameters are estimated by the maximum likelihood method in case of complete and censored samples. A simulation study reveals that the estimates of one sub-model have desirable properties such as, (i) the maximum likelihood estimates are not too far from the true parameter values; (ii) the biases and mean square errors of estimates in case of complete sample are smaller than the corresponding in censored samples; and (iii) the biases and the mean square error values decreases as the sample size increases. An application to a real life data shows that the truncated Lomax Frèchet distribution is a strong and better competitor for the Frèchet distribution, exponentiated Frèchet distribution, Marshall-Olkin Frèchet distribution, transmuted Frèchet distribution, Kumaraswamy Frèchet distribution, transmuted Marshall-Olkin Frèchet distribution and the Weibull Frèchet distribution.

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