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Optimum Step-Stress Accelerated Life Test Plan for Lomax Distribution with an Adaptive Type-II Progressive Hybrid Censoring

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Abstract

The adaptive type-II progressive hybrid censoring has the advantage of saving both the total test time and the cost of the experiment; also it increases the efficiency of the statistical analysis. This article discusses k-level step stress accelerated life tests based on an adaptive type-II progressive hybrid censoring with product's life time following Lomax distribution. The scale parameter of the Lomax failure time distribution at constant levels is assumed to be a log linear function of the stress level. Maximum likelihood estimators of the model parameters are derived. Based on normal approximation to the asymptotic distribution of maximum likelihood estimators, the approximate confidence intervals for model parameters are obtained. The optimal times of changing stress levels are discussed under D-optimality and A-optimality criteria. Such methods maximize the determinant and the trace of Fisher's information matrix for the model parameters. Analysis of the numerical data has been presented for illustrative proposes.

Keywords: Adaptive type-II progressive censoring; k-level step stress accelerated life testing; Cumulative exposure model; Optimum test plan; Lomax distribution.

Acronyms and Notation

ALT *Accelerated life test.*
APHC *Adaptive type-II progressive hybrid censoring.*
CEM *Cumulative exposure model.*

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CIs	*Confidence intervals*
SSALT	*Step stress accelerated life test.*
MLEs	*Maximum likelihood estimates.*
MSEs	*Mean square errors.*
V_0	*Design stress.*
V_j	*High stress levels, where* $j = 1, \ldots, k$.
Υ	*Ideal total time.*
k	*Number of stress levels.*
N	*Number of test units (total sample size).*
$t_{i:m:N}$	*Observed failure times,* $i = 1,2, \ldots., m$.
m	*Predetermined number of failures.*
β_j, α	*Scale and shape parameters of Lomax distribution.*
τ_j	*Time of changing stress level* V_{j-1} *to* V_j, $1 \le j \le k, \tau_1 < \tau_2 < \cdots < \tau_k$.

1 Introduction

In life testing and reliability studies, the experimenter may not always observe the failure times of all components placed on the test. In such cases, data obtained from such experiments are called censored data. The most common censoring schemes are type-I (time) censoring, where the life testing experiment will be terminated at a predetermined time Y, and type-II (failure) censoring, where the life testing experiment will be terminated upon the rth (r is pre-fixed) failure. However, the conventional type-I and type-II censoring schemes do not have the flexibility of allowing removal of units at points other than the terminal point of the experiment. To allow for more flexibility in removing items from the test before termination of experiment, more general censoring approaches known as progressive censoring are desired.

According to [1], under progressive censoring, from a total of N units placed simultaneously on a life test, only m are completely observed until failure. Then, given a censoring plan $\mathcal{R} = (r_1, \ldots, r_m)$ at the time $t_{1:m:N}$ of the first failure, r_1 of the $N-1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. At the time $t_{2:m:N}$ of the second failure, r_2 of the $N-2-r_1$ surviving units are randomly withdrawn (or censored) from the life testing experiment and so on. Finally, at the time $t_{m:m:N}$ of the m failure, all the remaining $r_m = N - m - \sum_{i=1}^{m-1} r_i$ surviving units are removed from the life testing experiment. An integer $m < N$ is predetermined and the progressive type-II censoring scheme $(r_1, \ldots, r_m)$ with $r_i > 0$ and $\sum_{i=1}^{m} r_i + m = N$ is also specified.

Kundu and Joarder [2] proposed a censoring scheme called type-II progressive hybrid censoring scheme, in which a life testing experiment with progressive type-II right censoring scheme $\mathcal{R} = (r_1, \ldots, r_m)$ is terminated at a prefixed time $\Upsilon > 0$. However, the drawback of the type-II progressive hybrid censoring, similar to the conventional type-I censoring (time censoring), is that the effective sample size is random and it can turn out to be a very small number (even equal to zero) and therefore the standard statistical inference procedures may not be applicable or they will have low efficiency. For the purpose of increasing the efficiency of statistical analysis as well as saving the total test time, [3] introduced an adjustment of type-II progressive hybrid censoring scheme, so called adaptive type-II progressive hybrid censoring (APHC) scheme. Based on this scheme the number of observed failures m is fixed in advanced but the experimental time is allowed to run over a prefixed time $\Upsilon > 0$. If $t_{m:m:N} < \Upsilon$, the experiment stops at time $t_{m:m:N}$ and it will have a usual type-II progressive censoring scheme with the prefixed progressive censoring $(r_1, \ldots, r_m)$. If $t_{s:m:N} < \Upsilon < t_{s+1:m:N}$, where $s+1 < m$, then the number of items progressively removed from the experiment is adapted upon failure by setting $(r_{s+1} = 0, r_{s+2} = 0, \ldots, r_{m-1} = 0)$ and $r_m = N - m - \sum_{i=1}^{s} r_i$.

There are several studies concerned with the estimation problem based upon APHC, for example, [4] introduced APHC for exponential and extreme values distributions respectively. Mahmoud et al. [5] obtained the maximum likelihood estimates (MLEs) of unknown parameters for generalized Pareto distribution under APHC.

Accelerated life test (ALT) is often used for reliability analysis. In order to obtain failures quickly, test units are run at higher than usual stress conditions. The stress loading in an accelerated life testing can be applied in various ways, constant stress, step stress and random stress. There are mainly two types of stress accelerated life testing (SSALT), a simple SSALT and a multiple-step (k–level) SSALT. In simple SSALT, a test unit is subjected to successively higher levels of stress. A test unit starts at a specified low stress for a specified length of time. If it does not fail, the stress is raised and held a specified time. The stress is thus increased step by step until the test unit fails. Generally, all test units go through the same specified pattern of stress levels and test times. The simplest step stress ALT uses only two stress levels and it is called simple step stress ALT. Nelson [6] originally proposed the simple SSALT, in which only one change of stress occurs with a cumulative exposure model (CEM) for type-I and type-II censored data. Simple SSALT has been investigated by several authors such as [7], [8], [9], [10] and [11].

In k-level SSALT there are changes of stress more than once. Khamis and Higgins [12] considered the optimum three steps SSALT for the exponentially distributed type-I censored data. Khamis [13] proposed an optimal m level SSALT design with multiple stress. Wu et al. [14] considered k-level SSALT with an equal duration time for progressively type-I censored data for exponential distribution. Wu et al. [15] have discussed k-level SSALT under type-I progressive censoring with random removals for exponential grouped data. Balakrishnan and Han [16] considered k-level SSALT with an equal duration time for progressively type-I censored data for exponential distribution. Recently, based on progressive type-I interval censoring, [17] discussed k-level SSALT based on progressive type-I interval censoring when the inspection times and the proportions of removed units in the experiment are pre-fixed.

Lomax distribution is also known in literature as Pareto distribution of type-II. It is considered to be useful for modeling and analyzing the life time data in medical and biological sciences, engineering, etc. It also has been received the greatest attention from theoretical and applied statisticians primarily due to its use in reliability and life testing studies see for example [18]. Also, [19] used Lomax distribution as an alternative to the exponential distribution when the data are heavy tailed. In other hand, [20] used the Lomax distribution for applications in economics and biological sciences. Lomax distribution is also used in SSALT, for example, [21] determined the optimum test plan for simple SSALT using Lomax distribution. Also, [22] considered simple SSALT under type-I censoring using two-parameter Lomax distribution. The probability density function (pdf) and the cumulative distribution function (cdf) for Lomax distribution respectively are as follows:

$$f(t) = \alpha\beta^{\alpha}(t+\beta)^{-(\alpha+1)}, \qquad t > 0, \alpha, \beta > 0. \tag{1}$$

$$F(t) = 1 - \beta^{\alpha}(t+\beta)^{-\alpha}, \qquad t > 0, \alpha, \beta > 0, \tag{2}$$

In the literature, there were no studies that had been performed on the estimation and optimization problems about the step stress accelerated life testing models based on APHC scheme. Therefore, in this article, an attempt had been made on designing an optimum k-level step stress accelerated life tests for Lomax distribution based on APHC scheme. The scale parameter of the distribution is assumed to be log linear function of the stress level and cumulative exposure model holds. The model and assumptions are described in details in Section (2). In Section (3), the maximum likelihood method is applied to obtain the point estimators of the unknown parameters. In Section (4) the asymptotic Fisher information matrix and the confidence intervals of the model parameters that based on the asymptotic normality of the MLEs are obtained. In Section (5) the optimum test plan is presented using D optimality and an optimality. The numerical study is presented to demonstrate the theoretical results in Section (6). Finally conclusion is presented in Section (7).

2 Description of the Model

This section describes the model and presents some necessary assumptions for k-level SSALT with APHC. The model assumptions for k-level SSALT for Lomax distribution based on APHC will be described as follows:

I. There are multiple (k-level) of high stress, $V_j, j = 1,2, \dots, k$ in the experiment and V_0 is the design stress that is the stress level under normal use conditions, where $V_0 < V_1 < \cdots < V_k$.

II. A random sample of N units are simultaneously placed on the test at a stress V_1 and run until time τ_1. At the time $t_{1:m:N}$ of the first failure, r_1 of the $N-1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. At the time $t_{2:m:N}$ of the second failure, r_2 of the $N-2-r_1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. Starting at time τ_1, the remaining surviving units from the first step are put on the test under a stress V_2, where $V_2 > V_1$, these units are run until time τ_2. Starting at time τ_{k-1}, the remaining surviving units from the previous steps are put on the test under a stress V_k, where $V_k > V_{k-1}$.

III. The failure times $t_{i:m:N}, i = 1,2, \dots, m$; are independent and identically distributed at stress levels $V_j, j = 1,2, \dots, k.$ The life time of test unit is assumed to be Lomax distribution; with pdf (1) and cdf (2).

IV. Prior to the experiment, an integer $m < N$ is predetermined; where m is the number of failures and the progressive type-II censoring scheme $(r_1, \dots, r_m)$ with $r_i > 0$ and $\sum_{i=1}^{m} r_i + m = N$ is specified. At the time $t_{s:m:N}$ of the s^{th} failure, r_s of the remaining surviving units are randomly withdrawn (or censored) from the life testing experiment.

V. For given time Υ, allowing the experiment to run over time Υ, then there are two cases when $t_{m:m:N}$ is reached. If $t_{m:m:N} < \Upsilon$, the experiment stops at time $t_{m:m:N}$ and it will have a usual type-II progressive censoring scheme with the prefixed progressive censoring $(r_1, \dots, r_m)$. Otherwise, once the experimental time passes time Υ but the number of failures hasn't reached m failures ($t_{s:m:N} < \Upsilon < t_{s+1:m:N}$ where $s + 1 < m$), then the number of items progressively removed from the experiment is adapted upon failure by setting $(r_{s+1} = 0, r_{s+2} = 0, \dots, r_{m-1} = 0)$ and $r_m = N - m - \sum_{i=1}^{s} r_i$. In general, as long as the failures occur before time Υ, the initially planned progressive censoring scheme will be applied. After passing time Υ, no more items will be withdrawn except for the time of the m^{th} failure where all remaining surviving items are removed.

VI. The Lomax scale parameters $\beta_j, j = 1,2, \dots, k$ of the underlying lifetime distribution are assumed to be log linear function of stress levels $\log(\beta_j) = a + bV_j,\ j = 1,2, \dots, k\ ; a, b > 0,$ where a, b are unknown parameters. The Lomax shape parameter α is independent of stress.

Therefore, at stress level $V_j, j = 1,2, \dots, k$ and according to the CEM, the cdf of the lifetime of a test unit under k-level SSALT for Lomax distribution is given by:

$$G(t) = \begin{cases} F_1(t;\beta_1) & 0 \le t < \tau_1, \\ F_2(t-\tau_1+u_1;\beta_2) & \tau_1 \le t < \tau_2, \\ \vdots & \\ F_k(t-\tau_{k-1}+u_{k-1};\beta_k) & \tau_{k-1} \le t < \infty. \end{cases}$$

Where $F_j(t-\tau_{j-1}+u_{j-1};\beta_j) = 1 - {\beta_j}^{\alpha}\left[\left(t-\tau_{j-1}+u_{j-1}\right)+\beta_j\right]^{-\alpha}$ is the cumulative distribution function of the failure at stresses $V_j, j = 1,2, \dots, k; u_j$ is the solution of the equation

$F_{j+1}(u_j;\beta_{j+1}) = F_j(\tau_j - \tau_{j-1} + u_{j-1};\beta_j)$. Therefore, the general form of u_j is $u_j = \frac{\beta_{j+1}}{\beta_j}(\tau_j - \tau_{j-1} + u_{j-1})$, note that $u_0 = 0, \tau_0 = 0$ where τ_j is the time of changing stress level. Hence, the cumulative distribution function of a test unit under k-level SSALT for Lomax distribution is:

$$G(t) = \begin{cases} 1 - {\beta_1}^{\alpha}(t+\beta_1)^{-\alpha} & 0 \le t < \tau_1, \\ 1 - {\beta_2}^{\alpha}\left\{\left[t - \tau_1 + \left(\frac{\beta_2}{\beta_1}\tau_1\right)\right] + \beta_2\right\}^{-\alpha} & \tau_1 \le t < \tau_2, \\ \vdots & \\ 1 - {\beta_k}^{\alpha}\left\{\left[t - \tau_{k-1} + \left[\frac{\beta_k}{\beta_{k-1}}(\tau_{k-1} - \tau_{k-2} + u_{k-2})\right]\right] + \beta_k\right\}^{-\alpha} & \tau_{k-1} \le t < \infty. \end{cases}$$

Thus the associated pdf of a test unit is

$$g(t) = \begin{cases} \alpha{\beta_1}^{\alpha}(t+\beta_1)^{-(\alpha+1)} & 0 \le t < \tau_1, \\ \alpha{\beta_2}^{\alpha}\left\{\left[t - \tau_1 + \left(\frac{\beta_2}{\beta_1}\tau_1\right)\right] + \beta_2\right\}^{-(\alpha+1)} & \tau_1 \le t < \tau_2, \\ \vdots & \\ \alpha{\beta_k}^{\alpha}\left\{\left[t - \tau_{k-1} + \left[\frac{\beta_k}{\beta_{k-1}}(\tau_{k-1} - \tau_{k-2} + u_{k-2})\right]\right] + \beta_k\right\}^{-(\alpha+1)} & \tau_{k-1} \le t < \infty. \end{cases}$$

3 Maximum Likelihood Estimators Based on APHC

Let $t_{i:m:N}, i = 1,2,\dots,m$, be the m completely observed (ordered) lifetimes from Lomax distribution with censoring scheme $(r_1, r_2, \dots r_m)$ where m is the predetermined number of failures, s is the number of failures observed before time Υ and k is the number of stress levels. The likelihood function for k-level SSALT with APHC data is considered to have the following form: (See [3])

$$l \propto \prod_{j=1}^{k}\left[\prod_{i=1}^{m} f_j({t_{ij}}^*)\right]\left[\prod_{i=1}^{s}[1 - F_j({t_{ij}}^*)]^{r_i}\right][1 - F(t_m^*)]^{N-m-\sum_{i=1}^{s} r_i} \quad , \tag{3}$$

where ${t_{ij}}^* = t_{i:m:N} - \tau_{j-1} + u_{j-1}$ for $i = 1, \dots m$, $j = 1, \dots, k$and $t_m^* = t_{m:m:N} - \tau_{k-1} + u_{k-1}$.

The likelihood function for the two-parameter Lomax distribution in k-level SSALT based on an adaptive type-II progressive censoring data takes the following form:

$$l \propto \prod_{j=1}^{k}\left[\prod_{i=1}^{m} \alpha\beta_j^{\alpha}({t_{ij}}^* + \beta_j)^{-(\alpha+1)}\right]\left[\prod_{i=1}^{s}[\beta_j^{\alpha}({t_{ij}}^* + \beta_j)^{-\alpha}]^{r_i}\right][\beta_k^{\alpha}(t_m^* + \beta_k)^{-\alpha}]^{N-m-\sum_{i=1}^{s} r_i}. \tag{4}$$

The maximum likelihood estimators of the parameters for likelihood function (4) are obtained by maximizing the logarithm of the likelihood function will be expressed in the following form:

$$\log l \propto mk\log\alpha + m\alpha\sum_{j=1}^{k}\log\beta_j - (\alpha+1)\sum_{j=1}^{k}\sum_{i=1}^{m}\log({t_{ij}}^* + \beta_j) + \alpha\sum_{j=1}^{k}\sum_{i=1}^{s} r_i\log(\beta_j)$$

$$- \alpha \sum_{j=1}^{k} \sum_{i=1}^{s} r_i \log\left(t_{ij}{}^* + \beta_j\right) \; - \alpha \left(N - m - \sum_{i=1}^{s} r_i \right) \log(t_m^* + \beta_k)$$

$$+ \alpha \left(N - m - \sum_{i=1}^{s} r_i \right) \log(\beta_k). \tag{5}$$

Applying the log linear function relationship $\log(\beta_j) = a + bV_j$ then $\beta_j = e^{a + bV_j}$ and the logarithm of the likelihood function (5) will be:

$$\log l \propto mk \log \alpha + m\alpha \left(ka + \sum_{j=1}^{k} bV_j \right) - (\alpha + 1) \sum_{j=1}^{k} \sum_{i=1}^{m} \log\left(t_{ij}{}^* + e^{a + bV_j}\right) + \; \alpha \sum_{j=1}^{k} \sum_{i=1}^{s} r_i \left(a + bV_j\right)$$

$$- \alpha \sum_{j=1}^{k} \sum_{i=1}^{s} r_i \log\left(t_{ij}{}^* + e^{a + bV_j}\right) - \alpha \left(N - m - \sum_{i=1}^{s} r_i \right) \log(t_m^* + e^{a + bV_k})$$

$$+ \alpha \left(N - m \; - \sum_{i=1}^{s} r_i \right) (a + bV_k). \tag{6}$$

The first partial derivatives of the log-likelihood function (6) with respect to the parameters a, b and α respectively will be as follows:

$$\frac{\partial \log l}{\partial a} = m\alpha k - (\alpha + 1) \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{e^{a + bV_j}}{t_{ij}{}^* + e^{a + bV_j}} + \alpha k \sum_{i=1}^{s} r_i - \alpha \sum_{j=1}^{k} \sum_{i=1}^{s} \frac{r_i e^{a + bV_j}}{t_{ij}{}^* + e^{a + bV_j}}$$

$$- \alpha \left(N - m - \sum_{i=1}^{s} r_i \right) \frac{e^{a + bV_k}}{t_m^* + e^{a + bV_k}} \; + \; \alpha \left(N - m - \sum_{i=1}^{s} r_i \right), \tag{7}$$

$$\frac{\partial \log l}{\partial b} = m\alpha \sum_{j=1}^{k} V_j - (\alpha + 1) \sum_{j=1}^{k} \sum_{i=1}^{m} \frac{V_j e^{a + bV_j}}{t_{ij}{}^* + e^{a + bV_j}} + \alpha \sum_{j=1}^{k} \sum_{i=1}^{s} r_i V_j - \alpha \sum_{j=1}^{k} \sum_{i=1}^{s} \frac{r_i V_j e^{a + bV_j}}{t_{ij}{}^* + e^{a + bV_j}}$$

$$- \alpha \left(N - m - \sum_{i=1}^{s} r_i \right) \frac{V_k e^{a + bV_k}}{t_m^* + e^{a + bV_k}} + \alpha \left(N - m - \sum_{i=1}^{s} r_i \right) V_k \, , \tag{8}$$

and

$$\frac{\partial \log l}{\partial \alpha} = \frac{mk}{\alpha} + m \left(ka + \sum_{j=1}^{k} bV_j \right) + \sum_{j=1}^{k} \sum_{i=1}^{s} r_i \left(a + bV_j\right) \; - \sum_{j=1}^{k} \sum_{i=1}^{m} \log\left(t_{ij}{}^* + e^{a + bV_j}\right)$$

$$- \sum_{j=1}^{k} \sum_{i=1}^{s} r_i \log\left(t_{ij}{}^* + e^{a + bV_j}\right)$$

$$-\left(N-m-\sum_{i=1}^{s} r_i\right)\log(t_m^* + e^{a+bV_k}) + \left(N-m-\sum_{i=1}^{s} r_i\right)(a+bV_k), \quad (9)$$

$$\hat{\alpha} = mk \Big/ \left[-m\left(ka + \sum_{j=1}^{k} bV_j\right) - \sum_{j=1}^{k}\sum_{i=1}^{s} r_i(a+bV_j) + \sum_{j=1}^{k}\sum_{i=1}^{m}\log\left(t_{ij}{}^* + e^{a+bV_j}\right)\right.$$
$$+\sum_{j=1}^{k}\sum_{i=1}^{s} r_i \log\left(t_{ij}{}^* + e^{a+bV_j}\right)$$
$$\left. +\left(N-m-\sum_{i=1}^{s} r_i\right)\log(t_m^* + e^{a+bV_k}) - \left(N-m-\sum_{i=1}^{s} r_i\right)(a+bV_k)\right].$$

It is observed that the maximum likelihood estimates do not exist in closed form and the nonlinear Equations (7)-(8) should be solved numerically with respect to the unknown parameters.

4 Asymptotic Fisher Information Matrix

The asymptotic Fisher information matrix $\hat{F}$ of the maximum likelihood estimator of the model parameters can be approximated by numerically inverting the asymptotic Fisher-information matrix. It is composed of the negative second and mixed partial derivatives of the natural logarithm of the likelihood function evaluated at the MLE. It can be given according to the following matrix:

$$\hat{F} = -\begin{bmatrix} \frac{\partial^2 \log l}{\partial a^2} & \frac{\partial^2 \log l}{\partial a \partial b} & \frac{\partial^2 \log l}{\partial a \partial \alpha} \\ \frac{\partial^2 \log l}{\partial b \partial a} & \frac{\partial^2 \log l}{\partial b^2} & \frac{\partial^2 \log l}{\partial b \partial \alpha} \\ \frac{\partial^2 \log l}{\partial \alpha \partial a} & \frac{\partial^2 \log l}{\partial \alpha \partial b} & \frac{\partial^2 \log l}{\partial \alpha^2} \end{bmatrix} = -\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \downarrow (\hat{a}, \hat{b}, \hat{\alpha}),$$

where, the elements of the asymptotic Fisher information matrix $f_{12}, f_{13}, f_{11}, f_{22}, f_{23}$ and f_{33} are obtained as follows:

$$f_{12} = -(\alpha+1)\sum_{j=1}^{k}\sum_{i=1}^{m} \frac{\left(t_{ij}{}^* + e^{a+bV_j}\right)e^{a+bV_j}V_j - e^{a+bV_j}}{\left(t_{ij}{}^* + e^{a+bV_j}\right)^2} - \alpha\sum_{j=1}^{k}\sum_{i=1}^{s} r_i \frac{t_{ij}{}^* V_j e^{a+bV_j}}{\left(t_{ij}{}^* + e^{a+bV_j}\right)^2},$$

$$-\alpha\left(N-m-\sum_{i=1}^{s} r_i\right)\frac{t_m^* V_k e^{a+bV_k}}{(t_m^* + e^{a+bV_k})^2} \quad (10)$$

$$f_{13} = mk - \sum_{j=1}^{k}\sum_{i=1}^{m}\frac{e^{a+bV_j}}{t_{ij}{}^* + e^{a+bV_j}} + k\sum_{i=1}^{s} r_i - \sum_{j=1}^{k}\sum_{i=1}^{s}\frac{r_i e^{a+bV_j}}{t_{ij}{}^* + e^{a+bV_j}} - \left(N-m-\sum_{i=1}^{s} r_i\right)\frac{e^{a+bV_k}}{t_m^* + e^{a+bV_k}}$$

$$+\left(N-m-\sum_{i=1}^{s} r_i\right), \quad (11)$$

$$f_{11} = -(\alpha+1)\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{t_{ij}{}^{*}e^{a+bV_j}}{\left(t_{ij}{}^{*}+e^{a+bV_j}\right)^2} - \alpha\sum_{j=1}^{k}\sum_{i=1}^{s}\frac{r_i\left(t_{ij}{}^{*}e^{a+bV_j}\right)}{\left(t_{ij}{}^{*}+e^{a+bV_j}\right)^2}$$

$$-\alpha\left(N-m-\sum_{i=1}^{s}r_i\right)\frac{t_m^{*}e^{a+bV_k}}{(t_m^{*}+e^{a+bV_k})^2}, \quad (12)$$

$$f_{22} = -(\alpha+1)\sum_{j=1}^{k}\sum_{i=1}^{m}\frac{t_{ij}{}^{*}\left(V_j\right)^2e^{a+bV_j}}{\left(t_{ij}{}^{*}+e^{a+bV_j}\right)^2} - \alpha\sum_{j=1}^{k}\sum_{i=1}^{s}r_i\frac{t_{ij}{}^{*}\left(V_j\right)^2e^{a+bV_j}}{\left(t_{ij}{}^{*}+e^{a+bV_j}\right)^2}$$

$$-\alpha\left(N-m-\sum_{i=1}^{s}r_i\right)\frac{t_m^{*}(V_k)^2e^{a+bV_k}}{(t_m^{*}+e^{a+bV_k})^2} \quad (13)$$

$$f_{23} = m\sum_{j=1}^{k}V_j - \sum_{j=1}^{k}\sum_{i=1}^{m}\frac{V_je^{a+bV_j}}{t_{ij}{}^{*}+e^{a+bV_j}} + \sum_{j=1}^{k}\sum_{i=1}^{s}r_iV_j - \sum_{j=1}^{k}\sum_{i=1}^{s}\frac{r_iV_je^{a+bV_j}}{t_{ij}{}^{*}+e^{a+bV_j}}$$

$$-\left(N-m-\sum_{i=1}^{s}r_i\right)\frac{V_ke^{a+bV_k}}{t_m^{*}+e^{a+bV_k}} + \left(N-m-\sum_{i=1}^{s}r_i\right)V_k, \quad (14)$$

and

$$f_{33} = \frac{-mk}{\alpha^2}. \quad (15)$$

The determinant of the asymptotic Fisher information matrix can be derived from the following equation:

$$\left|\hat{F}\right| = f_{11}(f_{22}f_{33}-f_{23}f_{32}) - f_{12}(f_{21}f_{33}-f_{23}f_{31}) + f_{13}(f_{21}f_{32}-f_{22}f_{31}). \quad (16)$$

In addition, it can be said that the maximum likelihood estimators have an asymptotic variance-covariance matrix defined by the inverse of $\hat{F}$. The approximate confidence intervals (CIs) of the parameters are derived based on the asymptotic distribution of the maximum likelihood estimators for the unknown parameters. The asymptotic distribution of $\frac{\hat{\underline{\theta}}-E(\hat{\underline{\theta}})}{\sqrt{var(\hat{\underline{\theta}})}}$ can be approximated by a standard normal distribution, where $var(\hat{\underline{\theta}})$ is the asymptotic variance. Therefore, the two-sided approximate $\gamma\,100$ percent confidence limits for $\underline{\theta}$ (lower bound (LB), upper bound (UB) can be obtained, such that $LB(\underline{\theta}) = \hat{\underline{\theta}} - Z_{\gamma/2}\sqrt{var(\hat{\underline{\theta}})}$, $UB(\underline{\theta}) = \hat{\underline{\theta}} + Z_{\gamma/2}\sqrt{var(\hat{\underline{\theta}})}$ were $Z_{\gamma/2}$ is the $100(\gamma/2)\%$ standard normal percentile and $\hat{\underline{\theta}} \equiv (\hat{a}\,,\hat{\alpha},\hat{b})$.

5 Optimum Test Plan

The main objective of this section is to determine the optimal test plan which leads to the most accurate estimate. Optimum time of changing stress levels $\tau_{j-1}, j = 2,\dots,k$ will be introduced. Based on D-optimality criterion (which is based on maximizing the determinant of the Fisher information matrix of the maximum likelihood estimators for the model parameters), the optimum time of changing stress level $\tau_{j-1}, j = 2, \dots, k$ can be obtained by solving the following equation:

$$\frac{\partial|\hat{F}|}{\partial\tau_{j-1}} = 0, j = 2, \dots, k, \tag{17}$$

where $|\hat{F}|$ is the determinant of the asymptotic Fisher information matrix. In general, the first partial derivative for the determinant of the asymptotic Fisher information matrix is as follows:

$$|\hat{F}|' = f_{11}'(f_{22}f_{33} - f_{23}f_{32}) + f_{11}(f_{22}f_{33} - f_{23}f_{32})' - f_{12}'(f_{21}f_{33} - f_{23}f_{31}) - f_{12}(f_{21}f_{33} - f_{23}f_{31})' + f_{13}'(f_{21}f_{32} - f_{22}f_{31}) + f_{13}(f_{21}f_{32} - f_{22}f_{31})',$$

$$|\hat{F}|' = f_{11}'(f_{22}f_{33} - f_{23}f_{32}) + f_{11}(f_{22}'f_{33} + f_{22}f_{33}' - f_{23}'f_{32} - f_{23}f_{32}') - f_{12}'(f_{21}f_{33} - f_{23}f_{31}) - f_{12}(f_{21}'f_{33} + f_{21}f_{33}' - f_{23}'f_{31} - f_{23}f_{31}') + f_{13}'(f_{21}f_{32} - f_{22}f_{31}) + f_{13}(f_{21}'f_{32} + f_{21}f_{32}' - f_{22}'f_{31} - f_{22}f_{31}'). \tag{18}$$

To obtain the optimum time of changing stress levels, Equation (18) will be derived by taking the first partial derivatives of Equations from (10) to (15) with respect to $\tau_{j-1}, j = 2, \dots, k$. Furthermore, some other optimality criteria can also be used in this content, such as maximization of the trace of the asymptotic Fisher information matrix of the MLEs (A optimality). (See [23]).

6 Numerical Illustration

To obtain the optimum test plan and the maximum likelihood estimators for k-level SSALT with an adaptive type-II progressive hybrid censoring, random samples of sizes $N = 30,50,70$ and $100$are generated from two-parameter Lomax distribution. The MSEs for the MLEs are calculated. In addition, the optimum times for changing stress levels are calculated by using two optimality schemes (A and D optimality). Furthermore, following [24], three progressive censoring schemes are considered as follows:

Scheme 1: $r_1 = \cdots = r_{m-1} = 0$ and $r_m = N - m$.
Scheme 2: $r_1 = \cdots = r_{m-1} = 1$ and $r_m = N - 2m + 1$.
Scheme 3: $r_1 = \cdots = r_{m-1} = r_m = \frac{N-m}{m}$.

For each progressive scheme, the simulation procedures are described according to the following algorithm:

i. The value of the shape parameter of Lomax lifetime are selected as $\alpha = (0.2, 0.3, 0.4)$. The stress values are selected as $V_1 = 0.5$, $V_2 = 0.75$, $V_3 = 1$ and $V_4 = 2$ for each stress level k, where $(k = 2,3,4)$, then calculate $\beta_j = e^{a+bV_j}$ for $a = 0.5$, $b = 0.5$, $\Upsilon = 1.5$ and $m = 10$.
ii. Generate random samples of sizes $N = 30,50,70$ and 100 from uniform (0,1) distribution and then obtain the order statistics $(U_{1:N} \dots, U_{N:N})$.
iii. For a given value of the first time of changing stress level $\tau_1 = 1$, if $0 < U_{i:N} \leq F(\tau_1, \beta_1)$ where $F(\tau_1; \beta_1)$ is defined in Equation (2) and $i = 1, \dots, N$, obtain $T = \beta_1(1 - U_{i:N})^{\frac{-1}{\alpha}} - \beta_1$.

iv. For a given value of the second time of changing stress level $\tau_2 = 2$, if $F(\tau_1, \beta_1) < U_{i:N} \le F(\tau_2 - \tau_1 + u_1, \beta_2)$ obtain, $T = \beta_2(1 - U_{i:N})^{\frac{-1}{\alpha}} - \beta_2 - u_1 + \tau_1$.

v. For a given value of the third time of changing stress level $\tau_3 = 4$, if $F(\tau_2 - \tau_1 + u_1, \beta_2) < U_{i:N} \le F(\tau_3 - \tau_2 + u_2, \beta_3)$ obtain, $T = \beta_3(1 - U_{i:N})^{\frac{-1}{\alpha}} - \beta_3 - u_2 + \tau_2$.

vi. For a given value of the fourth time of changing stress level $\tau_4 = 8$, if $F(\tau_3 - \tau_2 + u_2, \beta_3) < U_{i:N} \le F(\tau_4 - \tau_3 + u_3, \beta_4)$ obtain, $T = \beta_4(1 - U_{i:N})^{\frac{-1}{\alpha}} - \beta_4 - u_3 + \tau_3$.

vii. Based on different values of $m, \Upsilon, \tau_j, j = 1,2,3,4$ and for given values of the parameters a, b and α, MLEs for the unknown parameters are calculated numerically Equations from (7) to (9).

viii. The MSEs for MLEs of the unknown parameters are obtained. Also, the approximate confidence intervals of the parameters are obtained assuming 95% confidence level.

ix. The optimum times of changing stress levels using the two different optimality schemes for the three different progressive schemes.

x. The previous steps are repeated for 100 replications.

The numerical results based on k-level SSALT for Lomax distribution under APHC data with $k = 2, 3,$ and 4 are summarized in Tables 1-6.

From Tables 1-3, the following conclusions can be observed:

I. In all cases, based on the three selected progressively censored schemes ($r_1 = 0 = r_{m-1} = 0$, $r_m = N - m$), ($r_1 = \cdots = r_{m-1} = 1$ and $r_m = N - 2m + 1$) and ($r_1 = \cdots = r_{m-1} = r_m = \frac{N-m}{m}$), the biases and the MSEs of $\hat{\alpha}$ have the smallest values for all different values of the parameters and for each value of k.

II. In almost all cases, for different values of the parameters and based on the three progressively censored schemes, the values of the biases and the MSEs decrease as sample size increases for each value of k.

III. In all cases, for different values of the parameters and based on the three progressively censored plans, it is observed that $\hat{b}$ has the shortest confidence intervals.

IV. In some cases, it is noticed that the asymptotic CIs of $\hat{a}$ and $\hat{b}$ can take negative values especially for $N = 70$ and 100.

V. The values of the MSEs of $\hat{\alpha}$ are the smallest for the third progressively censored scheme for almost all different values of the parameter and for each value of k.

VI. The MSEs of $\hat{a}$ have the smallest values for the first and the second progressively censored plan, for almost all different values of the parameter and for each value of k.

VII. The MSEs of $\hat{\alpha}$ have the smallest values as k increases for almost all different values of the parameter and for the three selected progressively censored schemes.

From Tables 4-6, the following conclusion can be observed:

Based on the three progressively censoring plan, the optimum times of changing stress levels using A optimality have greater values than D optimality for almost all different values of the parameters and for each value of k.

Table 1. The biases, the MSEs and asymptotic confidence intervals of the MLEs of a, b and α under various censoring schemes of the k-level SSALT with APHC scheme for $a = 0.5, b = 0.5, \alpha = 0.2$ and $\Upsilon = 1.5$ when $k = (2, 3, 4)$

N	Scheme	Bias			MSE			Confidence Interval		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
					$k = 2$					
	1	-0.320	-0.485	-0.077	0.102	0.235	0.007	(0.175,0.185)	(0.015,0.016)	(0.111,0.135)
30	2	-0.334	-0.485	-0.063	0.122	0.235	0.005	(0.129,0.203)	(0.014,0.015)	(0.123,0.152)
	3	-0.320	-0.485	-0.049	0.103	0.235	0.004	(0.177,0.182)	(0.015,0.016)	(0.133,0.168)
	1	-0.317	-0.485	-0.086	0.100	0.235	0.008	(0.180,0.187)	(0.015,0.016)	(0.106,0.122)
50	2	-0.315	-0.485	-0.071	0.100	0.235	0.006	(0.181,0.189)	(0.015,0.016)	(0.119,0.138)
	3	-0.325	-0.485	-0.057	0.108	0.235	0.004	(0.162,0.189)	(0.015,0.016)	(0.132,0.155)
	1	-0.323	-0.484	-0.100	0.106	0.234	0.011	(0.168,0.187)	(0.015,0.017)	(0.093,0.107)
70	2	-0.328	-0.484	-0.085	0.111	0.234	0.008	(0.159,0.185)	(0.016,0.017)	(0.107,0.123)
	3	-0.349	-0.484	-0.070	0.135	0.234	0.006	(0.124,0.179)	(0.015,0.017)	(0.120,0.139)
	1	-0.343	-0.483	-0.103	0.126	0.233	0.011	(0.139,0.176)	(0.016,0.018)	(0.092,0.102)
100	2	-0.337	-0.483	-0.100	0.122	0.233	0.011	(0.145,0.182)	(0.016,0.018)	(0.095,0.105)
	3	-0.344	-0.483	-0.088	0.135	0.234	0.008	(0.131,0.182)	(0.016,0.017)	(0.105,0.118)
					$k = 3$					
	1	-0.326	-0.485	-0.059	0.110	0.235	0.004	(0.152,0.196)	(0.015,0.016)	(0.127,0.155)
30	2	-0.325	-0.485	-0.034	0.107	0.235	0.003	(0.161,0.189)	(0.015,0.016)	(0.149,0.182)
	3	-0.352	-0.485	-0.030	0.140	0.235	0.003	(0.102,0.193)	(0.014,0.016)	(0.152,0.187)
	1	-0.331	-0.484	-0.070	0.112	0.234	0.006	(0.155,0.184)	(0.015,0.017)	(0.121,0.140)
50	2	-0.325	-0.484	-0.050	0.107	0.234	0.003	(0.163,0.188)	(0.015,0.017)	(0.140,0.160)
	3	-0.347	-0.484	-0.035	0.131	0.235	0.003	(0.125,0.180)	(0.015,0.016)	(0.152,0.178)
	1	-0.330	-0.484	-0.084	0.115	0.234	0.008	(0.152,0.188)	(0.015,0.017)	(0.108,0.123)
70	2	-0.337	-0.486	-0.070	0.119	0.237	0.005	(0.145,0.180)	(0.008,0.02)	(0.123,0.137)
	3	-0.581	-0.480	-0.022	0.368	0.23	0.001	(-0.122,-0.04)	(0.018,0.022)	(0.172,0.185)
	1	-0.364	-0.482	-0.090	0.154	0.232	0.008	(0.107,0.164)	(0.017,0.019)	(0.105,0.115)
100	2	-0.359	-0.482	-0.087	0.151	0.233	0.008	(0.112,0.169)	(0.017,0.018)	(0.108,0.119)
	3	-0.361	-0.485	-0.068	0.144	0.236	0.005	(0.116,0.163)	(0.009,0.021)	(0.126,0.137)

N	Scheme	Bias			MSE			Confidence Interval		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
					$k = 4$					
	1	-0.328	-0.485	-0.047	0.110	0.235	0.003	(0.154,0.189)	(0.015,0.016)	(0.138,0.168)
30	2	-0.335	-0.485	-0.015	0.113	0.235	0.002	(0.153,0.177)	(0.015,0.016)	(0.167,0.203)
	3	-0.440	-0.484	-0.008	0.123	0.234	0.003	(0.134,0.172)	(0.015,0.017)	(0.187,0.229)
	1	-0.340	-0.484	-0.056	0.121	0.234	0.004	(0.140,0.179)	(0.015,0.018)	(0.134,0.154)
50	2	-0.347	-0.483	-0.030	0.124	0.234	0.003	(0.136,0.170)	(0.015,0.018)	(0.156,0.184)
	3	-0.440	-0.482	-0.008	0.137	0.233	0.002	(0.117,0.155)	(0.016,0.019)	(0.194,0.223)
	1	-0.330	-0.484	-0.065	0.113	0.234	0.005	(0.155,0.186)	(0.015,0.017)	(0.127,0.143)
70	2	-0.350	-0.482	-0.052	0.128	0.233	0.004	(0.133,0.168)	(0.016,0.019)	(0.138,0.157)
	3	-0.370	-0.481	-0.016	0.143	0.232	0.003	(0.112,0.148)	(0.017,0.020)	(0.170,0.198)
	1	-0.367	-0.482	-0.075	0.149	0.232	0.006	(0.109,0.156)	(0.017,0.019)	(0.120,0.130)
100	2	-0.364	-0.482	-0.071	0.147	0.232	0.005	(0.113,0.160)	(0.017,0.019)	(0.123,0.134)
	3	-0.340	-0.480	-0.046	0.157	0.231	0.003	(0.098,0.141)	(0.018,0.021)	(0.146,0.162)

Table 2. The biases, the MSEs and asymptotic confidence intervals of the MLEs of a, b and α under various censoring schemes of the k-level SSALT with APHC scheme for $a = 0.5, b = 0.5, \alpha = 0.3$ and $\Upsilon = 1.5$ when $k = (2, 3, 4)$

N	Scheme	Bias			MSE			Confidence Interval		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
					$k = 2$					
	1	-0.320	-0.485	-0.157	0.102	0.235	0.027	(0.177,0.184)	(0.015,0.016)	(0.127,0.158)
30	2	-0.360	-0.485	-0.113	0.147	0.235	0.015	(0.093,0.187)	(0.014,0.016)	(0.171,0.203)
	3	-0.324	-0.485	-0.147	0.105	0.235	0.024	(0.169,0.182)	(0.015,0.016)	(0.135,0.170)
	1	-0.341	-0.484	-0.145	0.123	0.234	0.022	(0.137,0.181)	(0.015,0.017)	(0.145,0.165)
50	2	-0.355	-0.483	-0.126	0.134	0.233	0.017	(0.120,0.169)	(0.016,0.018)	(0.164,0.184)
	3	-0.380	-0.485	-0.111	0.159	0.235	0.014	(0.086,0.154)	(0.010,0.020)	(0.178,0.200)
	1	-0.372	-0.475	-0.170	0.210	0.231	0.030	(0.065,0.191)	(0.008,0.042)	(0.123,0.136)
70	2	-0.342	-0.483	-0.164	0.125	0.233	0.028	(0.137,0.178)	(0.016,0.018)	(0.129,0.143)
	3	-0.380	-0.483	-0.150	0.168	0.233	0.024	(0.084,0.156)	(0.016,0.019)	(0.142,0.158)

N	Scheme	Bias			MSE			Confidence Interval		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
	1	-0.404	-0.480	-0.180	0.186	0.230	0.033	(0.066,0.125)	(0.019,0.021)	(0.115,0.125)
100	2	-0.434	-0.472	-0.182	0.258	0.228	0.034	(0.014,0.117)	(0.014,0.041)	(0.113,0.124)
	3	-0.430	-0.482	-0.156	0.223	0.233	0.025	(0.032,0.108)	(0.014,0.022)	(0.139,0.149)
$k = 3$										
	1	-0.335	-0.485	-0.107	0.116	0.235	0.013	(0.143,0.188)	(0.015,0.016)	(0.178,0.209)
30	2	-0.325	-0.485	-0.134	0.107	0.235	0.020	(0.161,0.189)	(0.015,0.016)	(0.149,0.182)
	3	-0.455	-0.490	-0.094	0.256	0.241	0.010	(-0.03,0.125)	(-0.003,0.02)	(0.194,0.219)
	1	-0.331	-0.484	-0.170	0.112	0.234	0.030	(0.155,0.184)	(0.015,0.017)	(0.121,0.140)
50	2	-0.383	-0.483	-0.103	0.159	0.234	0.012	(0.086,0.148)	(0.011,0.022)	(0.187,0.207)
	3	-0.489	-0.482	-0.102	0.283	0.232	0.012	(-0.047,0.06)	(0.016,0.020)	(0.186,0.209)
	1	-0.372	-0.482	-0.152	0.154	0.233	0.024	(0.099,0.156)	(0.016,0.019)	(0.141,0.155)
70	2	-0.370	-0.482	-0.146	0.149	0.232	0.022	(0.105,0.156)	(0.017,0.02)	(0.147,0.161)
	3	-0.426	-0.482	-0.127	0.209	0.233	0.017	(0.035,0.113)	(0.016,0.020)	(0.165,0.181)
	1	-0.476	-0.478	-0.176	0.270	0.228	0.032	(-0.017,0.06)	(0.021,0.024)	(0.118,0.129)
100	2	-0.501	-0.478	-0.179	0.298	0.228	0.033	(-0.044,0.04)	(0.021,0.024)	(0.115,0.127)
	3	-0.489	-0.480	-0.146	0.270	0.231	0.022	(-0.024,0.04)	(0.015,0.026)	(0.150,0.159)
$k = 4$										
	1	-0.334	-0.485	-0.138	0.115	0.235	0.021	(0.145,0.186)	(0.015,0.016)	(0.146,0.178)
30	2	-0.378	-0.483	-0.059	0.150	0.233	0.006	(0.091,0.152)	(0.015,0.019)	(0.222,0.261)
	3	-0.395	-0.481	-0.040	0.164	0.231	0.005	(0.074,0.137)	(0.017,0.021)	(0.237,0.283)
	1	-0.372	-0.487	-0.107	0.151	0.238	0.013	(0.097,0.159)	(0.002,0.024)	(0.181,0.204)
50	2	-0.377	-0.482	-0.106	0.154	0.232	0.013	(0.093,0.152)	(0.016,0.020)	(0.181,0.207)
	3	-0.440	-0.480	-0.086	0.238	0.231	0.009	(0.020,0.171)	(0.018,0.021)	(0.200,0.228)
	1	-0.404	-0.479	-0.129	0.177	0.230	0.018	(0.069,0.123)	(0.019,0.023)	(0.164,0.178)
70	2	-0.392	-0.482	-0.129	0.166	0.232	0.018	(0.082,0.134)	(0.016,0.021)	(0.163,0.180)
	3	-0.440	-0.479	-0.096	0.210	0.229	0.011	(0.030,0.089)	(0.020,0.023)	(0.193,0.215)
	1	-0.460	-0.477	-0.156	0.231	0.228	0.025	(0.013,0.068)	(0.021,0.024)	(0.140,0.149)
100	2	-0.472	-0.475	-0.151	0.239	0.226	0.023	(0.002,0.053)	(0.024,0.026)	(0.145,0.153)
	3	-0.431	-0.478	-0.123	0.200	0.228	0.017	(0.045,0.092)	(0.021,0.024)	(0.169,0.184)

Table 3.The biases, the MSEs and asymptotic confidence intervals of the MLEs of a, b and α under various censoring schemes of the k-level SSALT with APHC scheme for $a = 0.5, b = 0.5, \alpha = 0.4$ and $\Upsilon = 1.5$ when $k = (2, 3, 4)$

N	Scheme	Bias			MSE			Confidence Interval		
		$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$
					$k = 2$					
	1	-0.385	-0.476	-0.199	0.220	0.235	0.041	(0.019,0.210)	(-.009,0.058)	(0.186,0.217)
30	2	-0.422	-0.485	-0.184	0.219	0.235	0.035	(0.005,0.151)	(0.013,0.016)	(0.202,0.230)
	3	-0.324	-0.485	-0.246	0.106	0.235	0.063	(0.167,0.185)	(0.015,0.016)	(0.136,0.171)
	1	-0.428	-0.472	-0.225	0.258	0.232	0.052	(-0.003,0.14)	(0.001,0.054)	(0.165,0.185)
50	2	-0.382	-0.482	-0.212	0.158	0.232	0.046	(0.087,0.148)	(0.017,0.020)	(0.178,0.198)
	3	-0.403	-0.483	-0.201	0.182	0.233	0.042	(0.058,0.136)	(0.015,0.018)	(0.189,0.209)
	1	-0.418	-0.473	-0.252	0.227	0.228	0.064	(0.028,0.135)	(0.011,0.042)	(0.142,0.153)
	2	-0.402	-0.480	-0.245	0.176	0.231	0.060	(0.070,0.150)	(0.018,0.021)	(0.150,0.160)
	3	-0.479	-0.483	-0.236	0.277	0.234	0.056	(-0.03,0.072)	(0.012,0.021)	(0.157,0.172)
	1	-0.540	-0.469	-0.285	0.347	0.224	0.082	(-0.08,0.005)	(0.019,0.042)	(0.109,0.120)
100	2	-0.554	-0.475	-0.289	0.337	0.225	0.084	(-0.088,-0.02)	(0.024,0.026)	(0.106,0.116)
	3	-0.540	-0.480	-0.250	0.329	0.230	0.063	(-0.07,-0.002)	(0.016,0.025)	(0.146,0.154)
					$k = 3$					
	1	-0.414	-0.486	-0.185	0.213	0.236	0.036	(0.013,0.159)	(0.014,0.015)	(0.201,0.228)
30	2	-0.341	-0.485	-0.213	0.123	0.235	0.213	(0.129,0.188)	(0.015,0.016)	(0.169,0.188)
	3	-0.552	-0.496	-0.171	0.371	0.254	0.031	(-0.144,0.04)	(-0.02,0.035)	(0.214,0.243)
	1	-0.331	-0.482	-0.213	0.261	0.232	0.047	(-0.02,0.092)	(0.017,0.020)	(0.176,0.198)
50	2	-0.452	-0.481	-0.202	0.227	0.232	0.042	(0.005,0.09)	(0.014,0.024)	(0.189,0.208)
	3	-0.347	-0.484	-0.209	0.318	0.234	0.045	(-0.08,0.038)	(0.013,0.019)	(0.180,0.203)
	1	-0.440	-0.480	-0.238	0.216	0.230	0.057	(0.025,0.094)	(0.018,0.022)	(0.157,0.168)
70	2	-0.465	-0.479	-0.235	0.247	0.230	0.056	(-0.005,0.07)	(0.019,0.023)	(0.158,0.171)
	3	-0.555	-0.483	-0.221	0.337	0.234	0.049	(-0.09,-0.015)	(0.009,0.025)	(0.174,0.185)
	1	-0.544	-0.475	-0.295	0.457	0.226	0.087	(-0.184,-0.10)	(0.023,0.026)	(0.100,0.111)
100	2	-0.551	-0.475	-0.290	0.469	0.225	0.085	(-0.19,-0.108)	(0.024,0.026)	(0.104,0.116)
	3	-0.489	-0.480	-0.246	0.270	0.231	0.061	(-0.02,0.045)	(0.015,0.026)	(0.150,0.159)

					$k=4$					
	1	-0.337	-0.484	-0.234	0.117	0.235	0.057	(0.142,0.183)	(0.014,0.017)	(0.149,0.183)
30	2	-0.425	-0.479	-0.119	0.187	0.230	0.017	(0.047,0.102)	(0.018,0.023)	(0.262,0.300)
	3	-0.329	-0.485	-0.144	0.108	0.235	0.021	(0.169,0.173)	(0.015,0.016)	(0.253,0.259)
	1	-0.453	-0.479	-0.182	0.221	0.229	0.035	(0.013,0.082)	(0.020,0.023)	(0.207,0.230)
50	2	-0.463	-0.478	-0.174	0.231	0.229	0.032	(0.001,0.071)	(0.020,0.024)	(0.214,0.239)
	3	-0.440	-0.478	-0.164	0.298	0.229	0.030	(-0.066,0.02)	(0.020,0.024)	(0.219,0.252)
	1	-0.490	-0.476	-0.218	0.253	0.227	0.048	(-0.01,0.037)	(0.022,0.026)	(0.176,0.189)
70	2	-0.505	-0.475	-0.212	0.266	0.226	0.046	(-0.029,0.02)	(0.023,0.026)	(0.181,0.194)
	3	-0.440	-0.476	-0.170	0.298	0.226	0.033	(-0.06,-0.006)	(0.023,0.026)	(0.216,0.244)
	1	-0.557	-0.473	-0.257	0.322	0.224	0.067	(-0.079,-0.03)	(0.026,0.028)	(0.137,0.148)
100	2	-0.566	-0.473	-0.245	0.333	0.224	0.061	(-0.088,-0.04)	(0.026,0.028)	(0.148,0.162)
	3	-0.560	-0.473	-0.197	0.322	0.224	0.042	(-0.07,-0.042)	(0.025,0.028)	(0.192,0.214)

Table 4.Optimum time of changing stress level under various censoring schemes of the k-level SSALT with APHC scheme for $a=0.5, b=0.5, \alpha=0.2$ and when $k=(2,3,4)$

N	Scheme	τ_1					
		$k=2$		$k=3$		$k=4$	
		A Optimality	D Optimality	A Optimality	D Optimality	A Optimality	D Optimality
	1	1.590	0.173	1.596	0.510	0.770	0.526
30	2	1.507	0.678	0.567	0.250	0.932	0.481
	3	1.176	0.125	1.169	0.117	0.515	0.275
	1	0.741	1.447	1.297	0.154	0.473	0.738
50	2	1.202	0.635	1.124	0.264	0.400	0.625
	3	1.579	0.865	1.263	0.147	0.210	0.106
	1	0.573	2.510	0.382	0.338	0.347	0.190
70	2	1.079	0.150	1.120	0.350	0.884	0.117
	3	0.924	1.255	0.931	0.534	0.813	0.240
	1	0.502	0.766	0.315	0.310	0.323	0.357
100	2	0.472	0.646	0.522	0.306	0.464	0.310
	3	1.359	0.140	1.257	0.661	0.402	0.207

Table 5. Optimum time of changing stress level under various censoring schemes of the k-level SSALT with APHC scheme for $a = 0.5, b = 0.5, \alpha = 0.3$ and when $k = (2, 3, 4)$

N	Scheme	τ_1					
		$k = 2$		$k = 3$		$k = 4$	
		A Optimality	D Optimality	A Optimality	D Optimality	A Optimality	D Optimality
	1	1.402	0.237	1.281	0.346	0.637	0.526
30	2	1.334	0.447	0.856	0.638	0.899	0.402
	3	1.013	0.826	1.298	0.422	0.584	0.113
	1	0.533	0.461	1.176	0.680	0.353	0.509
50	2	1.005	0.596	0.951	0.293	0.312	0.250
	3	1.244	0.279	1.335	0.740	0.230	0.278
	1	0.492	1.102	0.321	0.122	0.315	0.562
70	2	1.084	1.763	0.408	0.194	0.565	0.250
	3	0.562	1.217	1.227	2.035	0.626	0.770
	1	0.370	1.157	0.367	0.731	0.366	0.173
100	2	0.362	0.250	0.432	0.250	0.340	0.153
	3	0.368	1.624	0.557	0.172	0.165	0.364

Table 6. Optimum time of changing stress level under various censoring schemes of the k-level SSALT testing with APHC scheme for a=0.5, b=0.5, $\alpha = 0.4$ and when $k = (2, 3, 4)$

N	Scheme	τ_1					
		$k = 2$		$k = 3$		$k = 4$	
		A optimality	D optimality	A optimality	D optimality	A optimality	D optimality
	1	1.549	0.284	0.435	0.304	0.568	0.526
30	2	1.384	0.451	0.842	0.214	0.216	0.463
	3	1.395	1.403	1.267	0.112	0.437	0.213
	1	0.485	0.389	0.951	0.181	0.570	0.447
50	2	0.971	1.161	0.790	0.872	0.299	0.476
	3	1.404	0.437	1.471	0.878	0.221	0.520
	1	0.418	2.719	0.338	0.270	0.392	0.122
70	2	1.049	1.575	0.744	0.770	0.184	0.500
	3	0.616	0.254	0.598	0.676	0.157	0.367
	1	0.303	1.036	0.360	0.170	0.316	0.250
100	2	0.307	0.375	0.499	0.480	0.323	0.250
	3	0.529	2.007	0.429	0.340	0.387	0.595

7 Conclusion

This paper concerns with the estimation problem and optimal test plans for k-level SSALT based on APHC data. A Lomax failure time distribution with scale parameter which is a log-linear function of the stress and a cumulative exposure model are assumed. The performance of the MLEs is evaluated using the mean square error criterion through numerical data. Asymptotic CIs have been established for the model parameters. In addition, the optimum times of changing stress levels are computed using A optimality and D optimality schemes. The calculations have been worked out based on different sample sizes and three selected progressive censored schemes.

In the numerical study, it is observed that the MSEs have its smallest values as k increases for almost all values of the parameters. In general, if the experimental time is not a major concern, then considering $k = 4$ is recommended in order to obtain better estimates of model parameters. The decision problem of obtaining appropriate number of failures under adaptive type-II progressive hybrid censored life testing experiment can save the total test time and increase the efficiency of statistical analysis. In this study, it is noted that the optimum number of changing stress levels is not the same but has relatively close values for the two optimality criteria.

Competing Interests

Authors have declared that no competing interests exist.

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