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An Extension of Exponentiated Lomax Distribution with Application to Lifetime Data

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Abstract

We introduce a new lifetime distribution referred to alpha power transformed exponentiated Lomax (APTEL) distribution. It includes new models as; the alpha power transformed exponentiated Pareto, alpha power transformed Pareto and alpha power transformed Lomax. Also, it includes former distributions as exponentiated Lomax, exponentiated Pareto, Pareto and Lomax. Several properties of the proposed distribution are studied. Maximum likelihood and least squares estimation methods are utilized to estimate the population parameters. A simulation study is carried out to assess the behavior of the proposed estimates via absolute bias, standard error and mean square error. Two real data sets are analyzed to illustrate the importance of the proposed distribution when compared with some present lifetime distributions.

Keywords: Alpha power transformation, moments, stochastic ordering, maximum likelihood, least squares.

1. Introduction

The Lomax (L) or Pareto type II distribution can be applied in many fields like actuarial science, economics and so on. It has been considered to be useful in reliability and life testing problems (Hassan and Al-Ghamdi 2009 and Hassan et al. 2016) in engineering and in survival analysis as an alternative distribution. In the lifetime context, the L model belongs to the family of decreasing failure rate (see Chahkandi and Ganjali 2009). Balkema and de Hann (1974) demonstrated that the L model arises as a limiting distribution of residual lifetimes at great age. For more information about the L distribution is given in Johnson et al. (1995).

Modified and extended versions of the L distribution have been derived and studied; examples including, Marshall-Olkin extended L distribution (Ghitany et al. 2007), exponentiated Lomax (EL) distribution (Abdul-Moniem and Abdel-Hameed 2012), Kumaraswamy EL distribution (Elbatal and Kareem 2014), beta L (BL), Kumaraswamy L (KuL) and McDonald L distributions (Lemonte and Cordeiro 2013), gamma-L (GaL) distribution (Cordeiro et al. 2015), transmuted EL distribution (Ashour and Eltehiwy 2013), exponential L distribution (El-Bassiouny et al. 2015), Weibull L (WL) distribution (Tahir et al. 2015), extended Poisson-Lomax distribution (Al-Zahrani 2015), transmuted WL distribution (Afify et al. 2015), Gumbel-L distribution (Tahir et al. 2016), power L distribution (Rady et al. 2016), EL geometric distribution (Hassan and Abd-Allah 2017), Gompartz L distribution (Oguntunde et al. 2017), power L Poisson distribution (Hassan and Nasser 2018), exponentiated Weibull L distribution (Hassan and Abd-Alla 2018) inverse power L distribution (Hassan and Abd-Alla 2019) and inverse exponentiated Lomax (Hassan and Mohammed 2019).

The cumulative distribution function (cdf) and the probability density function (pdf) of the L distribution are given, respectively, by

$$G(x;\lambda,\theta)=(1+\lambda x)^{-\theta},\ \ x,\lambda,\theta>0, \tag{1}$$

and

$$g(x;\lambda,\theta)=\lambda\theta(1+\lambda x)^{-(\theta+1)},\ \ x,\lambda,\theta>0.$$

where λ is the scale parameter and θ is the shape parameter. Abdul-Moniem and Abdel-Hameed (2012) proposed the EL distribution with an extra shape parameter β to the baseline distribution (1). The cdf and pdf of the EL distribution are given by

$$G(x;\lambda,\theta,\beta)=[1-(1+\lambda x)^{-\theta}]^{\beta},\ \ x,\lambda,\theta,\beta>0, \tag{2}$$

and

$$g(x;\lambda,\theta,\beta)=\lambda\theta\beta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta)}]^{\beta-1},\ \ x,\lambda,\theta,\beta>0. \tag{3}$$

The alpha power transformation (APT) is a recent method with an extra parameter that make the distributions richer and flexible for modeling real life data (Mahdavi and Kundu 2017). Let $F(x)$ be the cdf of a continuous random variable X, then according to Mahdavi and Kundu (2017), the cdf and pdf of APT are defined as follows

$$F(x;\alpha)=\begin{cases}\dfrac{\alpha^{G(x)}-1}{\alpha-1}, & \text{if } \alpha>0,\alpha\neq 1,\\ G(x), & \text{if } \alpha=1,\end{cases}$$

and

$$f(x;\alpha)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}g(x)\alpha^{G(x)}, & \text{if } \alpha>0,\alpha\neq 1,\\ g(x), & \text{if } \alpha=1.\end{cases} \tag{4}$$

Based on the APT method, some new distributions are considered like; APT exponential distribution (Mahdavi and Kundu 2017), APT Weibull distribution (Nassr et al. 2017), APT generalized exponential distribution (Dey et al. 2017), APT extended exponential distribution (Hassan et al., 2018), APT inverse Lindley distribution (Dey et al. 2018) and APT power Lindley distribution (Hassan et al. 2019).

In this paper, we introduce a new four-parameter distribution, the so called alpha power transformed exponentiated Lomax (APTEL) distribution using a similar idea to Mahdavi and Kundu (2017). We are motivated to introduce the APTEL distribution because it is capable of modeling decreasing and upside-down bathtub shaped hazard rates and it can be viewed as a suitable model for two real data applications compared with other competing lifetime distributions. This paper can be sorted as follows. In Section 2, the pdf, reliability function and hazard rate function of the APTEL distribution are defined. Some statistical properties of the APTEL are derived in Section 3. Estimators based on maximum likelihood (ML) and least squares (LS) methods as well as simulation studies are provided in Section 4. The proposed model is shown to give a better fit for two real data sets as explained in Section 5. Some concluding remarks are given in Section 6.

2. The APTEL Model

In this section, we introduce the APTEL distribution via APT. Special sub-models are provided.

Definition *A random variable* X *is said to have a four-parameter APTEL distribution if its pdf is given by inserting (2) and (3) in (4) as follows*

$$f(x;\nu)=\begin{cases}\dfrac{\theta\lambda\beta\log\alpha}{(\alpha-1)}(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}, & \text{if } \alpha>0, \alpha\neq 1,\\ \lambda\theta\beta(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}, & \text{if } \alpha=1,\end{cases} \tag{5}$$

where $\nu=(\alpha,\theta,\lambda,\beta)$ *is a set of parameters. The cdf of the APTEL distribution is given by*

$$F(x;\nu)=\begin{cases}\dfrac{\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}-1}{\alpha-1}, & \text{if } \alpha>0, \alpha\neq 1,\\ [1-(1+\lambda x)^{-\theta}]^{\beta}, & \text{if } \alpha=1.\end{cases} \tag{6}$$

A random variable X with distribution (6) is denoted by $X\sim \text{APTEL}(\nu)$. Some plots of the density (5) are displayed in Figure1. They reveal that the pdf of X is quite flexible and can take asymmetric forms, among others.

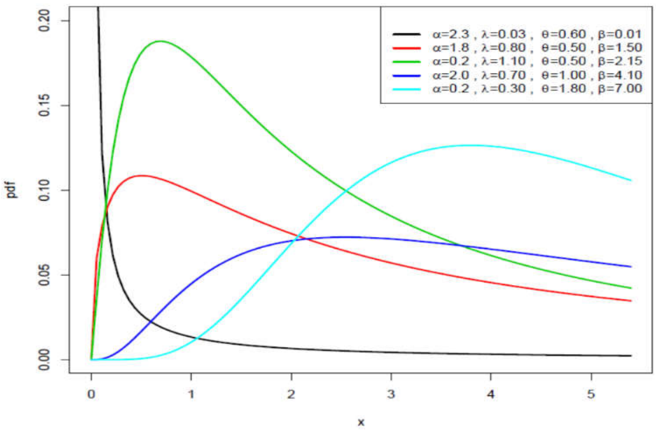


Figure 1 Plots of the APTEL density function for some parameter values

Sub-models of the APTEL distribution are considered as follows:

For $\lambda=1$, the pdf (5) provides the APT exponentiated Pareto distribution (new).

For $\lambda=1$ and $\beta=1$, the pdf (5) provides the APT Pareto distribution (new).

For $\beta=1$, the pdf (5) provides the EL distribution (Abdul-Moniem and Abdel-Hameed, 2012).

For $\alpha=1$, the pdf (5) provides the APT Lomax distribution (new).

For $\lambda=1$ and $\alpha=1$, the pdf (5) provides the exponentiated Pareto distribution (Gupta et al. 1998).

For $\lambda=1$, $\alpha=1$ and $\beta=1$, the pdf (5) provides the Pareto distribution.

For $\beta=1$ and $\alpha=1$, the pdf (5) provides the Lomax distribution (Lomax 1954).

Also, the reliability function, say $R(x;\nu)$, and hazard rate function (hrf), say $h(x;\nu)$, of X are given, respectively, as follows

$$R(x;\nu)=\begin{cases}(\alpha-1)^{-1}\left[\alpha-\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}\right], & \text{if } \alpha>0,\alpha\neq 1,\\ 1-[1-(1+\lambda x)^{-\theta}]^{\beta}, & \text{if } \alpha=1,\end{cases}$$

and

$$h(x;\nu)=\begin{cases}\dfrac{\theta\lambda\beta\log\alpha\,(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}}{\alpha-\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}}, & \text{if } \alpha>0,\alpha\neq 1,\\ \dfrac{\theta\lambda\beta\,(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}}{1-[1-(1+\lambda x)^{-\theta}]^{\beta}}, & \text{if } \alpha=1.\end{cases}$$

Some plots of the hrf are illustrated for some choices of parameters in Figure 2.

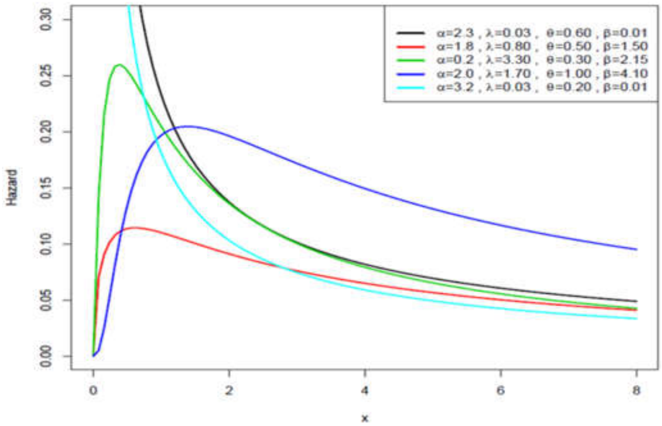


Figure 2 Plots of the APTEL hazard rate function for some parameter values

3. Statistical Properties

Here, some statistical properties of the APTEL distribution including; quantile function, moments, incomplete moments, probability weighted moments, Rényi entropy, and stochastic ordering are derived.

3.1. Quantile function

For $q\in(0,1)$, the quantile function of X is obtained by inverting (6) as follows

$$\frac{\alpha^{[1-(1+\lambda x_q)^{-\theta}]^{\beta}}-1}{\alpha-1}=q,$$

which yields

$$x_q=\frac{1}{\lambda}\left[\left\{1-\left((\log\alpha)^{-1}\log\left[q\left(\alpha-1\right)+1\right]\right)^{1/\beta}\right\}^{-1/\theta}-1\right],\quad 0<q<1. \tag{7}$$

The first quantile (25%), median (50%) and third quantile (75%) are obtained, respectively, by setting $q=0.25$, 0.5 and 0.75 in (7). Simulating the APTEL random variable is straightforward. If U is a uniform variate on the unit interval (0, 1), then the random variable $X=x_q$ at $q=U$ follows (5).

3.2. Moments

The s^{th} moment of a random variable X has the APTEL distribution is obtained as follows

$$\mu_s' = \frac{\theta\lambda\beta\log\alpha}{(\alpha-1)}\int_0^\infty x^s(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}\alpha^{[1-(1+\lambda x)^{-\theta}]^\beta}dx. \tag{8}$$

The power series can be written as

$$\alpha^k = \sum_{j=0}^\infty \frac{(\log\alpha)^j}{j!}k^j. \tag{9}$$

Hence, (8) can be expressed as follows

$$\mu_s' = \sum_{j=0}^\infty \frac{\theta\lambda\beta(\log\alpha)^{j+1}}{(\alpha-1)j!}\int_0^\infty x^s(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta j+\beta-1}dx.$$

Let $y=(1+\lambda x)^{-\theta}$ and using the binomial expansion, then

$$\mu_s' = \sum_{j=0}^\infty\sum_{m=0}^s \frac{\beta(\log\alpha)^{j+1}(-1)^m}{(\alpha-1)j!\lambda^s}\binom{s}{m}B\left(\beta j+\beta, 1-\frac{1}{\theta}(s-m)\right),$$

where $B(\cdot,\cdot)$ is the beta function. Further, the central moments (μ_s) and cumulants (κ_s), $s=1,2,\ldots$ of the APTEL distribution can be obtained from

$$\mu_s = E(X-\mu_1')^s = \sum_{i=0}^s(-1)^i\binom{s}{i}(\mu_1')^i\mu_{s-i}' \quad \text{and} \quad \kappa_s = \sum_{r=0}^{s-1}\binom{s-1}{r-1}\kappa_r\mu_{s-r}',$$

where $\kappa_1=\mu_1'$, $\kappa_2=\mu_2'-\mu_1'^2$, $\kappa_3=\mu_3'-3\mu_1'\mu_2'+\mu_1'^3$ and so on. The skewness and kurtosis measures can be evaluated from the ordinary moments using the well-known relationships. Numerical values of the μ_1', μ_2', μ_3', μ_4', variance ((σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the APTEL distribution for certain values of parameters are obtained and recorded in Table 1.

Table 1 Summary statistics of moments of APTEL distribution

μ_s'	$\alpha=0.5,\theta=7,$ $\beta=5,\lambda=0.5$	$\alpha=2.5,\theta=8,$ $\beta=2.5,\lambda=0.5$	$\alpha=5,\theta=5,$ $\beta=2,\lambda=0.5$	$\alpha=10,\theta=6,$ $\beta=1.5,\lambda=0.5$	$\alpha=20,\theta=6,$ $\beta=2.5,\lambda=1$
μ_1'	0.573	0.256	0.470	0.360	0.503
μ_2'	0.315	0.239	0.907	0.512	0.868
μ_3'	0.341	0.338	3.231	1.212	2.298
μ_4'	1.587	0.727	27.561	5.198	10.481
σ^2	0.242	1.627	0.686	0.383	0.616
CV	1.562	0.043	1.764	1.717	1.562
CS	2.636	2.602	3.801	3.177	2.572
CK	17.739	15.400	47.888	25.957	18.254

3.3. Incomplete moments

The s^{th} lower incomplete moment, say $\varpi_s(t)$, of the APTEL distribution is given by

$$\varpi_s(t) = \frac{\theta\lambda\beta\log\alpha}{(\alpha-1)}\int_0^t x^s(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}\alpha^{[1-(1+\lambda x)^{-\theta}]^\beta}dx. \tag{10}$$

Using the power expansion (9) and binomial expansion, then the s^{th} lower incomplete moment of the APTEL distribution is

$$\varpi_s(t)=\sum_{j=0}^{\infty}\sum_{m=0}^{s}\frac{\beta(\log\alpha)^{j+1}(-1)^m}{(\alpha-1)j!\lambda^s}\binom{s}{m}B\left(\beta j+\beta,1-\frac{1}{\theta}(s-m),(1+\lambda t)^{-\theta}\right),$$

where $B(\cdot,\cdot,z)$ is the incomplete beta function. The important application of the first incomplete moment is related to the Lorenz and Bonferroni curves defined by $L(q)=\varpi_1(q)/E(X)$ and $B(q)=L(q)/F(x)$, respectively. These curves are very useful in economics, demography, insurance, engineering and medicine.

The mean residual life (MRL) and the mean waiting time (MWT) are another application of the first incomplete moment. The MRL function is defined by $m_1(t)=[1-\varpi_1(t)]/R(t)-t$, which represents the expected additional life length for a unit which is alive at age t. The MWT is defined by $M_1(t)=t-\varpi_1(t)/F(t)$, which represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$.

3.4. Probability weighted moments

The class of probability weighted moments (PWM), denoted by $\Xi_{h,s}$, for a random variable X, is defined as follows

$$\Xi_{h,s}=E[X^hF(x)^s]=\int_{-\infty}^{\infty}x^hf(x)(F(x))^s\,dx. \tag{11}$$

Substituting (5) and (6) in (11), then the PWM of the APTEL distribution is

$$\Xi_{h,s}=\int_0^{\infty}x^h\frac{\theta\lambda\beta\log\alpha}{(\alpha-1)}(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}\left[\frac{\alpha^{[1-(1+\lambda x)^{-\theta}]^{\beta}}-1}{\alpha-1}\right]^s dx.$$

Using the binomial expansion, then

$$\Xi_{h,s}=\sum_{i=0}^{s}(-1)^{s-i}\binom{s}{i}\frac{\theta\lambda\beta\log\alpha}{(\alpha-1)^{s+1}}\int_0^{\infty}x^h(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta-1}\alpha^{(i+1)[1-(1+\lambda x)^{-\theta}]^{\beta}}dx. \tag{12}$$

Using (9) then (12) will be

$$\Xi_{h,s}=\sum_{i=0}^{s}\sum_{j=0}^{\infty}(-1)^{s-i}\binom{s}{i}\frac{\theta\lambda\beta(\log\alpha)^{j+1}(i+1)^j}{(\alpha-1)^{s+1}j!}\int_0^{\infty}x^h(1+\lambda x)^{-(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\beta+\beta j-1}dx.$$

Let $y=(1+\lambda x)^{-\theta}$ and using the binomial expansion, then

$$\Xi_{h,s}=\sum_{i=0}^{s}\sum_{j=0}^{\infty}\sum_{m=0}^{h}(-1)^{s-i+m}\binom{s}{i}\binom{h}{m}\frac{\lambda^{-h}\beta(\log\alpha)^{j+1}(i+1)^j}{(\alpha-1)^{s+1}j!}B(\beta+\beta j,-\frac{1}{\theta}(h-m)).$$

3.5. Rényi entropy

The Rényi entropy of a random variable represents a measure of variation of an uncertainty. The Rényi entropy is defined by

$$\delta_R(\omega)=(1-\omega)^{-1}\log\left(\int_0^{\infty}\left(f(x)\right)^{\omega}dx\right). \tag{13}$$

Substituting (5) in (13), then we have

$$\delta_R(\omega)=(1-\omega)^{-1}\log\left(\left(\frac{\theta\lambda\beta\log\alpha}{(\alpha-1)}\right)^{\omega}\int_0^{\infty}(1+\lambda x)^{-\omega(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\omega(\beta-1)}\alpha^{\omega[1-(1+\lambda x)^{-\theta}]^{\beta}}dx\right).$$

Using (9), then $\delta_R(\omega)$ is converted to

$$\delta_R(\omega)=(1-\omega)^{-1}\log\left(\left(\frac{\theta\lambda\beta\log\alpha}{(\alpha-1)}\right)^{\omega}\sum_{j=0}^{\infty}\frac{(\omega\log\alpha)^j}{j!}\int_0^{\infty}(1+\lambda x)^{-\omega(\theta+1)}[1-(1+\lambda x)^{-\theta}]^{\omega(\beta-1)+\beta j}dx\right).$$

Let, $y=(1+\lambda x)^{-\theta}$, then $\delta_R(\omega)$ will be

$$\delta_R(\omega)=(1-\omega)^{-1}\log\left(\left(\frac{\beta\log\alpha}{(\alpha-1)}\right)^{\omega}\sum_{j=0}^{\infty}\frac{(\omega\log\alpha)^j(\theta\lambda)^{\omega-1}}{j!}\mathrm{B}\left(\frac{\omega(\theta+1)-1}{\theta},\omega(\beta-1)+\beta j+1\right)\right).$$

3.6. Stochastic ordering

A random variable X is said to be stochastically smaller than Y, that is; $(X\le_{Sr}Y)$ If $F_X(x)\ge F_Y(x)$ for all x. In a similar way, X is said to be stochastically smaller than Y; if the following holds:

- Likelihood ratio order $(X\le_{lr}Y)$ if $\dfrac{f_X(x;\nu_1)}{f_Y(x,\nu_2)}$ is decreasing in x.
- Hazard rate order $(X\le_{hr}Y)$ if $h_X(x;\nu_1)\ge h_Y(x;\nu_2)$ for all x.
- Mean residual life order $(X\le_{mrl}Y)$ if $m_X(x;\nu_1)\ge m_Y(x;\nu_2)$ for all x.

The following results due to (Shaked and Shanthikumar 2007) are well known for establishing stochastic ordering of distributions.

$$(X\le_{lr}Y)\Rightarrow(X\le_{hr}Y)\Rightarrow(X\le_{mrl}Y)\Rightarrow(X\le_{Sr}Y).$$

Theorem *Let $X\sim APTEL_1(\nu_1)$ and $Y\sim APTEL_2(\nu_2)$. If $\alpha_1=\alpha_2=\alpha$, $\lambda_1\ge\lambda_2$, $\theta_1\ge\theta_2$ and $\beta_1\ge\beta_2$, then $(X\le_{lr}Y)$, $(X\le_{hr}Y)$, $(X\le_{mrl}Y)$ and $(X\le_{Sr}Y)$.*

Proof:

It is sufficient to show $\dfrac{f_X(x;\nu_1)}{f_Y(x,\nu_2)}$ is decreasing function of x; the likelihood ratio is

$$\frac{f_X(x;\nu_1)}{f_Y(x;\nu_2)}=\frac{(\alpha_2-1)\theta_1\lambda_1\beta_1\log\alpha_1(1+\lambda_1x)^{-(\theta_1+1)}[1-(1+\lambda_1x)^{-\theta_1}]^{\beta_1-1}\alpha_1^{[1-(1+\lambda_1x)^{-\theta_1}]^{\beta_1}}}{(\alpha_1-1)\theta_2\lambda_2\beta_2\log\alpha_2(1+\lambda_2x)^{-(\theta_2+1)}[1-(1+\lambda_2x)^{-\theta_2}]^{\beta_2-1}\alpha_2^{[1-(1+\lambda_2x)^{-\theta_2}]^{\beta_2}}},$$

thus

$$\begin{aligned}\frac{d}{dx}\log\left\{\frac{f_X(x;\nu_1)}{f_Y(x;\nu_2)}\right\}=&\frac{\lambda_2(\theta_2+1)}{(1+\lambda_2x)}-\frac{\lambda_1(\theta_1+1)}{(1+\lambda_1x)}+\frac{\lambda_1\theta_1(\beta_1-1)(1+\lambda_1x)^{-\theta_1-1}}{[1-(1+\lambda_1x)^{-\theta_1}]}-\frac{\lambda_2\theta_2(\beta_2-1)(1+\lambda_2x)^{-\theta_2-1}}{[1-(1+\lambda_2x)^{-\theta_2}]}\\&+\theta_1\lambda_1\beta_1(1+\lambda_1x)^{-\theta_1-1}[1-(1+\lambda_1x)^{-\theta_1}]^{\beta_1-1}\log\alpha_1\\&-\theta_2\lambda_2\beta_2(1+\lambda_2x)^{-\theta_2-1}(1+\lambda_2x)^{-\theta_2-1}[1-(1+\lambda_2x)^{-\theta_2}]^{\beta_2-1}\log\alpha_2.\end{aligned}$$

Hence, if $\alpha_1=\alpha_2=\alpha$, $\theta_1\ge\theta_2$, $\lambda_1\ge\lambda_2$ and $\beta_1\ge\beta_2$, then $\frac{d}{dx}\log\{f_X(x;\nu_1)/f_Y(x;\nu_2)\}\le 0$, which implies that $(X\le_{lr}Y)$, $(X\le_{hr}Y)$, $(X\le_{mrl}Y)$ and $(X\le_{Sr}Y)$.

4. Estimation and Simulation

In this section, the ML and LS methods of estimation are considered for estimating the parameters α, β, θ and λ of the APTEL distribution. The performance of estimates is studied via Monte Carlo simulation.

4.1. Maximum likelihood estimators

We derive the ML estimators of the parameters; α, β, θ and λ of the APTEL distribution based on a complete sample. Let $X_1, ..., X_n$ be a random sample from the APTEL distribution with observed values $x_1, ..., x_n$, then the log-likelihood function is given by

$$\ell = n\log\left[\log\alpha/(\alpha-1)\right] + n\log\theta + n\log\beta + n\log\lambda - (\theta+1)\sum_{i=1}^{n}\log(1+\lambda x_i) + (\beta-1)\sum_{i=1}^{n}\log[1-(1+\lambda x_i)^{-\theta}]$$

$$+\log\alpha\sum_{i=1}^{n}\left\{[1-(1+\lambda x_i)^{-\theta}]^{\beta}\ ]\right\}.$$

The ML equations of the APTEL distribution are obtained as follows

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\log(1+\lambda x_i) + (\beta-1)\sum_{i=1}^{n}\frac{\log(1+\lambda x_i)}{[(1+\lambda x_i)^{\theta}-1]} + \beta\log\alpha\sum_{i=1}^{n}[1-(1+\lambda x_i)^{-\theta}]^{\beta-1}(1+\lambda x_i)^{-\theta}\log(1+\lambda x_i),$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log[1-(1+\lambda x_i)^{-\theta}] + \log\alpha\sum_{i=1}^{n}[1-(1+\lambda x_i)^{-\theta}]^{\beta}\log[1-(1+\lambda x_i)^{-\theta}],$$

and

$$\frac{\partial\ell}{\partial\alpha} = \frac{n[\alpha-1-\alpha\log\alpha]}{\alpha(\alpha-1)\log\alpha} + \frac{1}{\alpha}\sum_{i=1}^{n}\left\{1-(1+\lambda x_i)^{-\theta}]^{\beta}\ ]\right\}.$$

Equating $\partial\ell/\partial\alpha, \partial\ell/\partial\beta$ and $\partial\ell/\partial\theta$ with zero and solving simultaneously, we obtain the ML estimators of α, β, θ and λ. To check that the global maximum has been attained, a number of starting values has been used.

4.2. Least squares estimator

The LS estimators were proposed by Swain et al. (1988) to estimate the parameters of the beta distributions. Consider a random sample of size n from the cdf (6) and let $x_{(1)}, ..., x_{(n)}$ be the corresponding ordred observation. So, the LS estimators of the population parameters of the APTEL distribution are obtained by minimizing the following quantity

$$\sum_{i=1}^{n}\left[\frac{\alpha^{[1-(1+\lambda x_{(i)})^{-\theta}]^{\beta}}-1}{\alpha-1} - \frac{i}{n+1}\right]^2,$$

with respect to α, β, θ and λ.

4.3. Numerical illustration

A simulation study is conducted to illustrate the behavior of the ML and LS estimates of the unknown parameters for the APTEL distribution. The performance of the estimates is evaluated in terms of their mean square errors (MSEs), absolute biases (ABs) and standard errors (SEs). The computations here are done via Mathcad 14 program. We generate 1000 samples of the APTEL distribution, where n = 10, 30, 50, 100 and by choosing λ = (0.3, 0.5, 0.8, 1.5), β = (0.8, 1.5, 1.8, 2.3), α = (0.5, 0.8, 1.3, 2) and θ = (0.5, 0.8, 1.3, 2). The MSEs, ABs and SEs of ML and LS estimates are computed and displayed in Table 2.

Table 2 MSEs, ABs and SEs of the APTEL distribution for ML and LS estimates

n	Methods	Properties	$\alpha=0.5,\theta=0.5,\lambda=0.3,\beta=0.8$				$\alpha=0.8,\theta=0.8,\lambda=0.5,\beta=1.5$			
			α	θ	λ	β	α	θ	λ	β
10	ML	MSE	0.248	0.859	0.088	0.241	0.638	0.513	0.192	0.426
		AB	0.497	0.557	0.293	0.484	0.798	0.041	0.361	0.499
		SE	0.001	0.074	0.004	0.008	0.000	0.072	0.025	0.042
	LS	MSE	0.397	0.864	0.092	0.499	1.411	0.724	1.046	1.911
		AB	0.282	0.822	0.277	0.693	0.620	0.734	0.616	1.333
		SE	0.056	0.043	0.012	0.014	0.101	0.043	0.082	0.037
30	ML	MSE	0.245	0.425	0.250	0.480	0.637	0.294	0.129	0.263
		AB	0.495	0.546	0.500	0.690	0.798	0.404	0.319	0.461
		SE	0.000	0.023	0.001	0.002	0.000	0.012	0.005	0.008
	LS	MSE	0.321	0.710	0.089	0.534	1.059	0.627	0.943	1.906
		AB	0.273	0.808	0.298	0.730	0.643	0.792	0.600	1.353
		SE	0.017	0.008	0.000	0.001	0.027	0.001	0.025	0.002
50	ML	MSE	0.244	0.217	0.063	0.051	0.637	0.289	0.128	0.198
		AB	0.492	0.148	0.244	0.210	0.798	0.513	0.314	0.417
		SE	0.000	0.009	0.001	0.002	0.000	0.003	0.002	0.003
	LS	MSE	0.255	0.537	0.089	0.533	0.536	0.624	0.924	1.904
		AB	0.201	0.682	0.299	0.730	0.380	0.791	0.609	1.347
		SE	0.009	0.005	0.000	0.000	0.013	0.000	0.015	0.001
100	ML	MSE	0.241	0.189	0.051	0.033	0.637	0.279	0.123	0.185
		AB	0.490	0.112	0.216	0.164	0.798	0.512	0.348	0.421
		SE	0.000	0.006	0.000	0.000	0.000	0.001	0.000	0.000
	LS	MSE	0.247	0.527	0.084	0.532	0.517	0.622	0.920	1.960
		AB	0.150	0.513	0.299	0.729	0.364	0.788	0.604	1.203
		SE	0.005	0.002	0.000	0.000	0.006	0.000	0.007	0.000

From Table 2, we observe that, as the sample sizes increase; the MSEs and ABs of ML and LS estimates of parameters are decreasing. The MSEs of the ML estimates are smaller than the MSEs of the corresponding LS estimates. Based on ML method; as the values of parameters increase, the MSEs for λ and β estimates increase in all values of parameters. Based on LS method, as the values of parameters increase, the MSEs for θ and λ estimates increase.

Table 2 (Continued)

n	Methods	Properties	$\alpha=1.3,\theta=1.3,\lambda=0.8,\beta=1.8$				$\alpha=2,\theta=2,\lambda=1.5,\beta=2.3$			
			α	θ	λ	β	α	θ	λ	β
10	ML	MSE	1.681	0.268	0.465	0.512	0.500	2.487	2.245	1.973
		AB	1.297	0.246	0.671	0.382	0.386	1.489	1.498	1.377
		SE	0.000	0.046	0.012	0.060	0.059	0.052	0.000	0.028
	LS	MSE	1.826	0.893	1.157	2.818	1.424	3.357	2.252	1.908
		AB	1.260	0.270	0.633	1.674	1.079	1.812	1.330	1.136
		SE	0.049	0.091	0.087	0.012	0.051	0.027	0.069	0.079
30	ML	MSE	1.679	0.191	0.453	0.413	0.413	2.393	2.237	1.540
		AB	1.296	0.341	0.681	0.589	0.547	1.479	1.496	1.212
		SE	0.000	0.009	0.002	0.009	0.011	0.010	0.000	0.009
	LS	MSE	1.728	0.432	0.460	2.807	1.319	3.344	2.210	1.636
		AB	1.307	0.213	0.495	1.605	1.065	1.801	1.467	1.195
		SE	0.005	0.021	0.016	0.001	0.014	0.003	0.008	0.015
50	ML	MSE	1.677	0.176	0.449	0.398	0.384	2.249	2.235	1.141
		AB	1.295	0.333	0.671	0.600	0.599	1.192	1.495	1.032
		SE	0.000	0.005	0.001	0.004	0.006	0.006	0.000	0.005
	LS	MSE	1.709	0.291	0.436	2.721	1.225	3.200	2.224	1.534
		AB	1.304	0.196	0.477	1.509	1.040	1.797	1.490	1.172
		SE	0.002	0.010	0.009	0.001	0.008	0.001	0.001	0.008
100	ML	MSE	1.673	0.128	0.433	0.370	0.254	2.158	2.232	0.882
		AB	1.294	0.285	0.667	0.593	0.460	1.114	1.487	0.862
		SE	0.000	0.002	0.000	0.001	0.002	0.001	0.000	0.004
	LS	MSE	1.690	0.211	0.371	2.636	0.894	3.659	2.237	0.851
		AB	1.300	0.154	0.414	1.413	0.728	1.912	1.496	0.711
		SE	0.000	0.004	0.004	0.000	0.006	0.000	0.000	0.006

5. Real Data Analysis

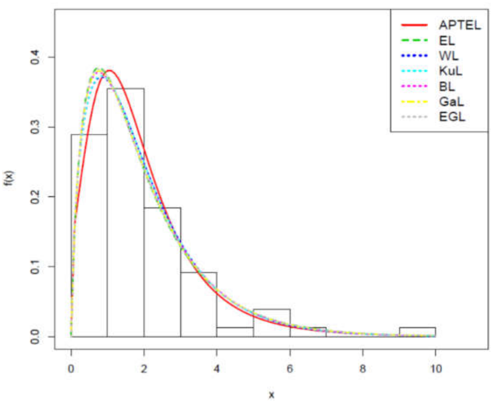
Here, we use two real data sets to compare the fits of the APTEL distribution with other corresponding distributions; that is, the EL, WL, KuL, BL, GaL and exponentiated generalized L (EGL) (Cordeiro et al. 2013) distributions. Result comparisons of the fitted models are made. The following criterions are used to select the distribution with the best fit: negative log-likelihood (−L) value, Akaike information criteria (AIC), Bayesian information criteria (BIC), consistent AIC (CAIC), Hannan and Quinn information criteria (HQIC). The value for the Kolmogorov-Smirnov (KS) statistic and the p-value are also provided.

We consider a first data set of the life of fatigue fracture of Kevlar 373/epoxy that are subjected to constant pressure at the 90% stress level until all had failed (Owoloko et al. 2015). The data are 0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0961. Comparisons criterion for the data are given in Table 3.

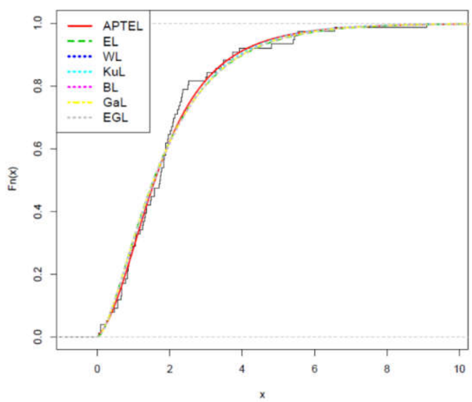
Table 3 The performances of the APTEL distribution with the other competing distributions

Model	−L	AIC	BIC	CAIC	HQIC	KS	p-value
APTEL	120.4329	248.8658	249.4292	255.1888	252.5917	0.08118390	0.6678827
EL	122.3134	250.6268	250.9601	257.6190	253.4212	0.09014059	0.5374227
WL	121.8335	251.6671	252.2305	260.9900	255.3930	0.09212608	0.5095990
KuL	121.8608	251.7217	252.2850	261.0446	255.4475	0.09138008	0.5199833
BL	122.2238	252.4477	253.0111	261.7706	256.1736	0.09241574	0.5055908
GaL	122.2385	250.4769	250.8102	257.4691	253.2713	0.09143058	0.5192776
EGL	122.2889	252.5777	253.1411	261.9007	256.3036	0.09054425	0.5317189

Plots of the estimated pdf and cdf of the APTEL distribution together with other competitor distributions for the second data set are represented in Figure 3. Also, P-P plots of the APTEL and other fitted distributions are displayed in Figure 4.



(a) Estimated pdf for the first data

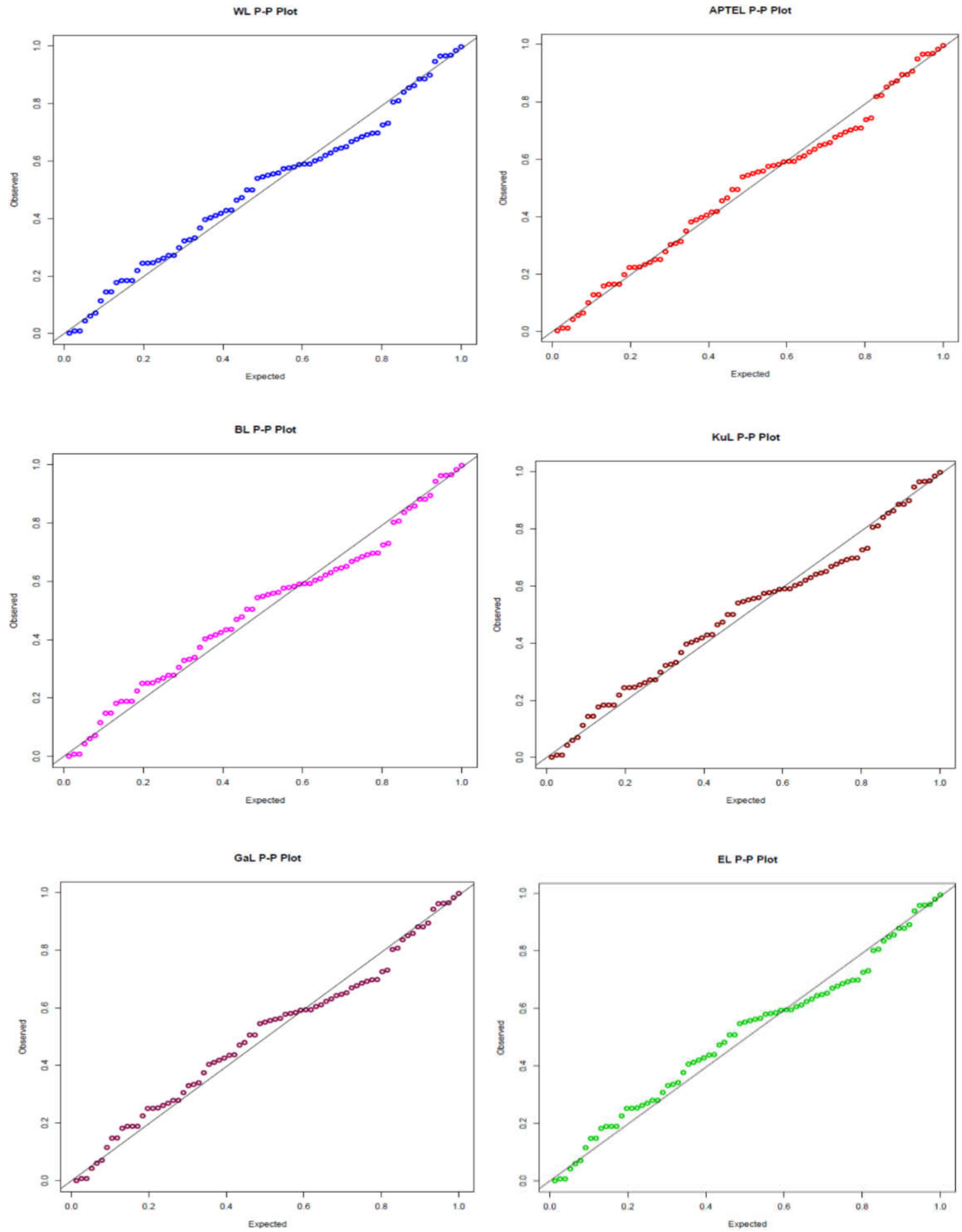


(b) Estimated cdf for the first data

Figure 3 Estimated pdf, empirical and estimated cdf of all models for the first data

This data represent the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee 1992). Comparisons criterion for the data are given in Table 4. The data are

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0.


WL P-P Plot
APTEL P-P Plot
BL P-P Plot
KuL P-P Plot
GaL P-P Plot
EL P-P Plot
Observed
Expected
0.0
0.2
0.4
0.6
0.8
1.0

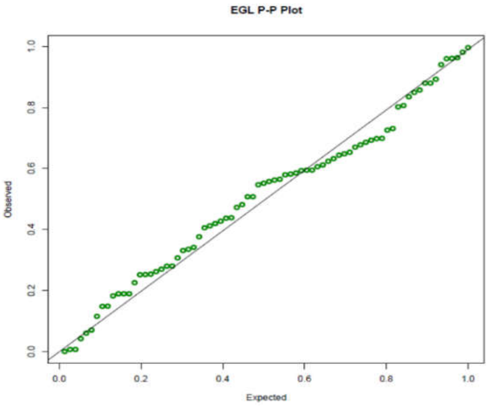


Figure 4 Fitted probability-probability plot for the first data set

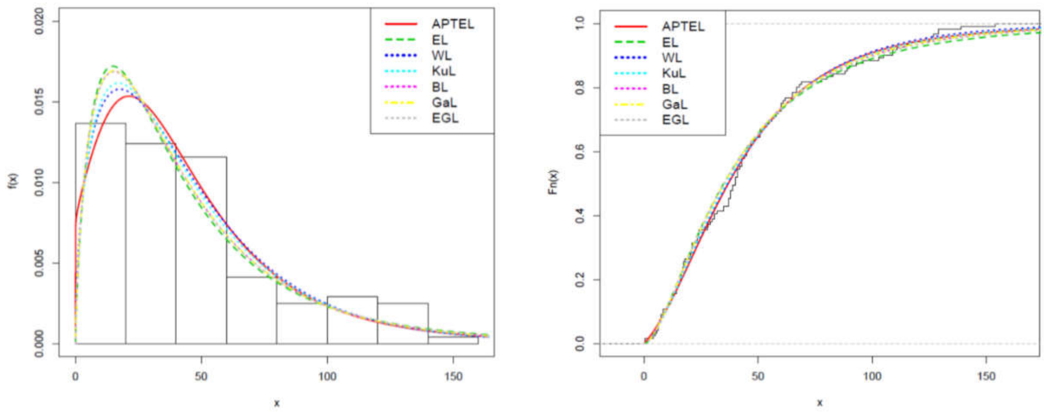
Table 4 The performances of the APTEL distribution with the other competing distributions

Model	−L	AIC	BIC	CAIC	HQIC	KS	p-value
APTEL	580.134	1168.26	1168.61	1177.45	1172.81	0.0706327	0.5820418
EL	583.3939	1172.78	1172.99	1181.17	1176.19	0.0953301	0.2214691
WL	580.2185	1168.43	1168.78	1179.62	1172.97	0.0748956	0.5058672
KuL	580.9609	1169.92	1170.26	1181.10	1174.46	0.0804302	0.4141534
BL	582.018	1172.03	1172.38	1183.21	1176.57	0.0937904	0.2375614
GaL	582.0117	1170.02	1170.22	1178.41	1173.43	0.0943289	0.2318364
EGL	582.2553	1172.51	1172.85	1183.69	1177.05	0.0915755	0.2622238

Plots of the estimated pdf and cdf of the APTEL distribution together with other competitor distributions for the second data set are demonstrated in Figure 5. Also, P-P plots of the APTEL and other fitted distributions are displayed in Figure 6.

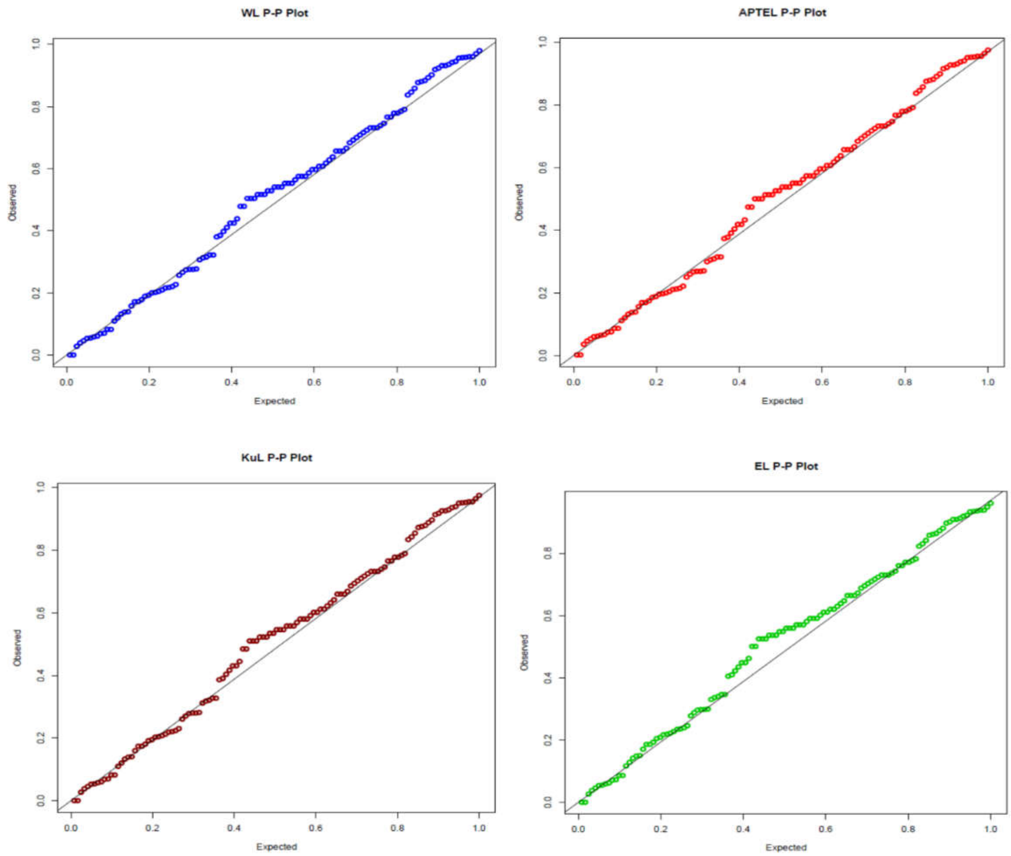
Tables 3 and 4 indicate that APTEL distribution gives the best fit and thus demonstrates superiority over the examined lifetime distributions in modeling the lifetime data sets under study. This conclusion was further supported by inspecting the PP plots, the density and cumulative distribution fit of the distributions for the real lifetime data sets.

The total time test (TTT) used for verifying the validity of the model. It allows identifying the shape of hrf graphically. Aarset (1987) showed that the hrf is constant if the TTT plot is graphically presented as a straight diagonal, the hrf is increasing (or decreasing) if the TTT plot is concave (or convex). The hrf is U-shaped (bathtub) if the TTT plot is firstly convex and then concave, if not, the hrf is unimodal. Figure 7 shows that the TTT plot is a unimodal; therefore it verifies our model validity.



(a) Estimated pdf for the second data (b) Estimated cdf for the second data

Figure 5 Estimated pdf, empirical and estimated cdf of all models for the second data



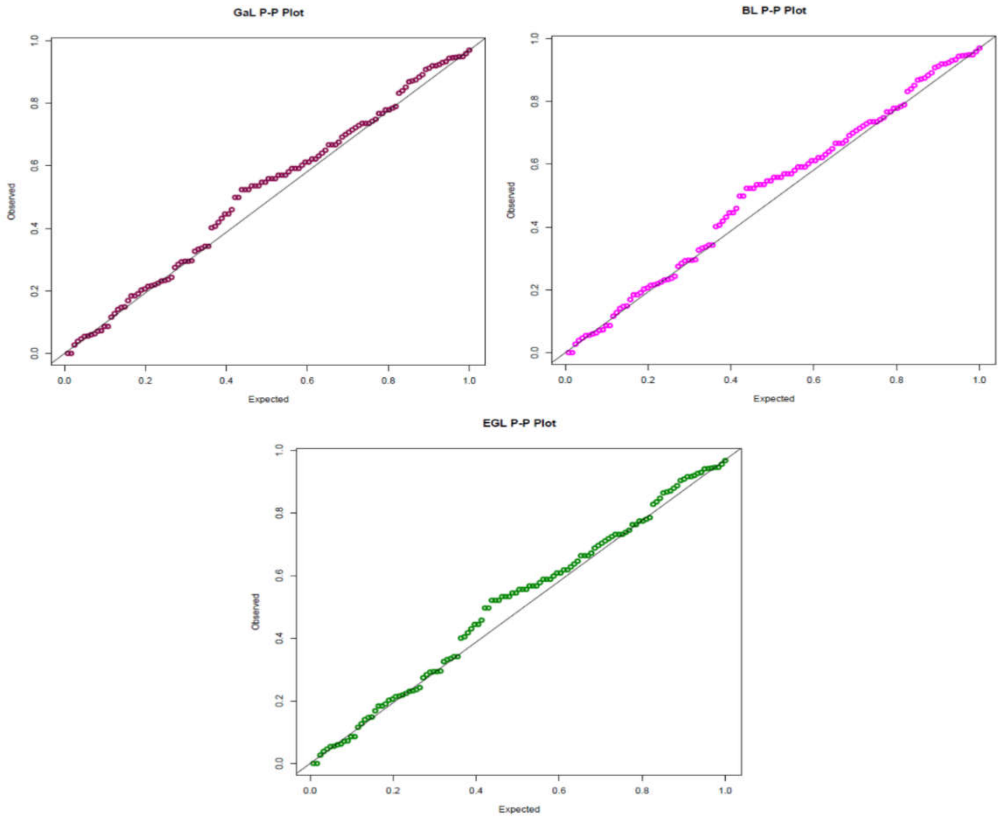
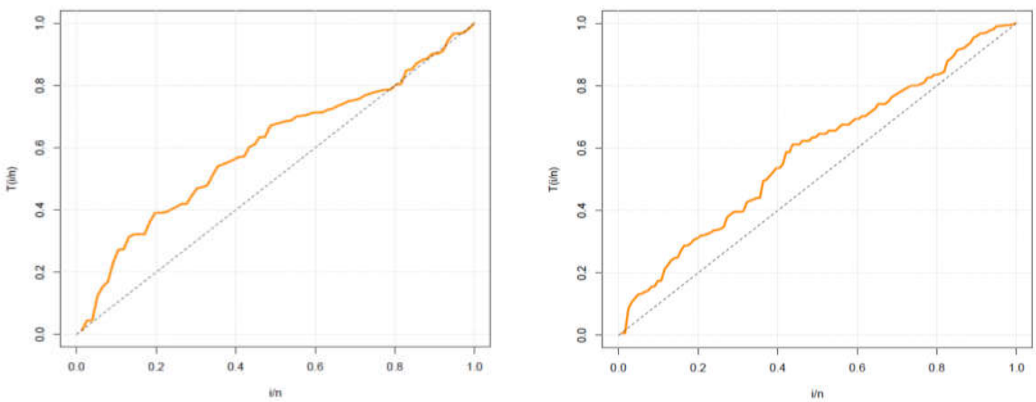


Figure 6 Fitted probability-probability plot for the second data set



(a) TTT-plot for the first data

(b) TTT-plot for the second data

Figure 7 The total time test plot (TTT-plot)

6. Concluding Remarks

In this paper, a new four-parameter lifetime distribution is proposed and the statistical properties such as, moments, mean residual life function, quantile function, Rényi entropy and stochastic ordering are discussed. The maximum likelihood and least squares methods for estimating the population parameters of the APTEL model are discussed. A simulation study indicates a good performance and accuracy of the maximum likelihood estimates than the least squares estimates. The application of the proposed distribution to two real lifetime data sets reveals its superiority over than some corresponding distributions.

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