Intelligent Automation & Soft Computing
DOI:10.32604/iasc.2021.017652
Article

Tech Science Press

A New Four-Parameter Moment Exponential Model with Applications to Lifetime Data

Abdullah Ali H. Ahmadini[1], Amal S. Hassan[2], Rokaya E. Mohamed[3,*], Shokrya S. Alshqaq[4] and Heba F. Nagy[5]

[1]Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia
[2]Faculty of Graduate Studies for Statistical Research, Department of Mathematical Statistics, Cairo University, Egypt
[3]Sadat Academy for Management Sciences, Department of Mathematics, Statistics and Insurance, Cairo, Egypt
[4]Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia
[5]Faculty of Graduate Studies for Statistical Research, Department of Mathematical Statistics, Cairo University, Egypt
*Corresponding Author: Rokaya E. Mohamed. Email: rokayaelmorsy@gmail.com
Received: 01 February 2021; Accepted: 06 March 2021

Abstract: In this research article, we propose and study a new model the so-called Marshal-Olkin Kumaraswamy moment exponential distribution. The new distribution contains the moment exponential distribution, exponentiated moment exponential distribution, Marshal Olkin moment exponential distribution and generalized exponentiated moment exponential distribution as special sub-models. Some significant properties are acquired such as expansion for the density function and explicit expressions for the moments, generating function, Bonferroni and Lorenz curves. The probabilistic definition of entropy as a measure of uncertainty called Shannon entropy is computed. Some of the numerical values of entropy for different parameters are given. The method of maximum likelihood is adopted for estimating the model parameters. We study the behavior of the maximum likelihood estimates for the model parameters using simulation study. A numerical study is performed to evaluate the behavior of the estimates with respect to their absolute biases, standard errors and mean square errors for different sample sizes and for different parameter values. Further, we conclude that the maximum likelihood estimates of the Marshal-Olkin Kumaraswamy moment exponential distribution perform well as the sample size increases. We take advantage of applied studies and offer two applications to real data sets that prove empirically the power of adjustment of the new model when compared to other lifetime distributions.

Keywords: Marshal-Olkin Kumaraswamy family; moment exponential distribution; quantile function; maximum likelihood estimation

1 Introduction

The modeling and analysis of lifetimes are important aspects of statistical work in a wide variety of technological fields. The procedure of adding one or two shape parameters to a class of distributions to obtain more flexibility, especially for studying tail behavior, is a well-known technique in the statistical literatures. Marshall et al. [1] proposed a method of adding a shape parameter to a family of distributions and many authors used their method to extend several well-known distributions.

The *cumulative distribution function* (cdf) and the *probability density function* (pdf) of the *Marshall-Olkin* (MO) family are defined as follows:

$$F_{MO}(x;\alpha,\zeta)=G(x;\zeta)\big/[1-\bar{\alpha}(\bar{G}(x;\zeta))], \tag{1}$$

and,

$$f_{MO}(x;\alpha,\zeta)=\alpha g(x;\zeta)\Big/[1-\bar{\alpha}(\bar{G}(x;\zeta))]^2, \tag{2}$$

where, $\alpha>0,\bar{\alpha}=1-\alpha$, and $\bar{G}(x;\zeta)=1-G(x;\zeta)$ is the survival function. The parameter $\bar{\alpha}$ is known as a tilt parameter and interpreted $\bar{\alpha}$ in terms of the behavior of the *hazard rate function* (hrf) of $\bar{F}(x)$. This ratio is increasing in x for $\bar{\alpha}\geq 1$ and decreasing in x for $\bar{\alpha}\in(0,1)$(see [2]). It is obvious that many new families can be derived from MO set up by considering different baseline distributions-G in Eq. (1). These new families are usually termed as MO extended-G distribution. The generalized MO proposed in [3] through exponentiating the MO survival function is defined as follows:

$$\bar{F}_{GMO}(x;\alpha,b,\zeta)=\left[\frac{\alpha(\bar{G}(x;\zeta))}{1-\bar{\alpha}(\bar{G}(x;\zeta))}\right]^b. \tag{3}$$

Tahir et al. [4] presented another generalization by exponentiating the cdf of the MO family; as follows:

$$F_{G2MO}(x;\alpha,b,\zeta)=\left[1-\frac{\alpha(\bar{G}(x;\zeta))}{1-\bar{\alpha}(\bar{G}(x;\zeta))}\right]^b. \tag{4}$$

[For more on MO distributions see [5–13]]. Cordeiro et al. [14] defined the *Kumaraswamy-G* (Kw-G) class with the cdf and pdf given by

$$F_{Kw}(x;a,b,\zeta)=1-[1-(G(x;\zeta))^a]^b, \tag{5}$$

and,

$$f_{Kw}(x;a,b,\zeta)=abg(x;\zeta)(G(x;\zeta))^{a-1}[1-(G(x;\zeta))^a]^{b-1}, \tag{6}$$

where $a>0$ and $b>0$ are shape parameters, in addition to those in the baseline distribution which partly govern skewness and variation in tail weights. Handique et al. [15] proposed a new extension of the MO family by considering the cdf and pdf of Kw-G distribution in (5) and (6) and call it *MO Kumaraswamy-G* (MOKw-G) distribution with cdf and pdf given by:

$$F_{MOKw}(x;a,b,\alpha,\zeta)=\frac{1-[1-(G(x;\zeta))^a]^b}{1-\bar{\alpha}[1-(G(x;\zeta))^a]^b}, \tag{7}$$

and,

$$f_{MOKw}(x;a,b,\alpha,\zeta)=\frac{ab\alpha g(x;\zeta)(G(x;\zeta))^a[1-(G(x;\zeta))^a]^{b-1}}{\left(1-\bar{\alpha}[1-(G(x;\zeta))^a]^b\right)^2}, \tag{8}$$

where $G(x;\zeta)$ is the baseline cdf depending on a parameter vector ζ; and $a, b > 0$, are additional shape parameters. The MOKw-G family generalizes the Kw-G family as well as the MO family.

The exponential distribution is a very popular statistical model and, probably, is one of the parametric models that most extensively applied in several fields [16]. Due to its importance, several studies introducing and/or studying extensions of the exponential distribution are available in the literatures. Some forms of exponential distribution are; the exponentiated exponential [17,18], beta exponential [19], beta generalized exponential [20], moment exponential [21], exponentiated moment exponential [22], generalized exponentiated moment exponential [23], extended exponentiated exponential [24], MO exponential Weibull [25], *MO generalized exponential* (MOGE) [26], *MO length-biased exponential* (MOLBE) [27], alpha power transformed extended exponential [28] and *MO Kumaraswamy exponential* (MOKwE) [29] distributions.

Moment distributions have a vital role in mathematics and statistics, in particular in probability theory, in the perspective research related to ecology, reliability, biomedical field, econometrics, survey sampling and in life-testing. Dara et al. [21] proposed the *moment exponential* (ME) distribution through assigning weight to the exponential distribution. They showed that their proposed model is more flexible model than the exponential distribution. The pdf of the ME distribution is specified by:

$$g(x;\beta) = \frac{x}{\beta^2}\, e^{\overline{\frac{-x}{\beta}}}, \qquad \beta, x > 0, \tag{9}$$

where, β is the scale parameter. The cdf corresponding to (9) is

$$G(x;\beta) = 1 - (1 + \frac{x}{\beta})\, e^{\overline{\frac{-x}{\beta}}}, \qquad \beta, x > 0. \tag{10}$$

In this paper, we introduce and study the *MO Kumaraswamy ME* (MOKwME) distribution. The MOKwME model includes as special cases the *generalized exponentiated ME* (GEME), *exponentiated ME* (EME), MOLBE, *Kumaraswamy ME* (KwME) and ME distributions, which are very important statistical models, especially for applied works. It is interesting to observe that its hazard rate function can be, increasing, decreasing, and upside-down bathtub. Accordingly, it can be used effectively to analyze lifetime data sets. Some statistical properties of the proposed model are provided. *Maximum likelihood* (ML) estimators of the model parameters are presented. A simulation study and an application of the suggested model on real life data set are given.

2 Model Formulation

The cdf and pdf of the MOKwME distribution are obtained by substituting (9) and (10) in (7) and (8) as follows:

$$F(x;\psi) = \frac{1 - [1 - (1 - \Lambda(x,\beta))^a]^b}{1 - \bar{\alpha}[1 - (1 - \Lambda(x,\beta))^a]^b}, \qquad a, b, \alpha, \beta > 0, \quad x > 0, \tag{11}$$

and,

$$f(x;\psi) = \frac{ab\alpha x}{\beta^2}\, e^{\overline{\frac{-x}{\beta}}} (1 - \Lambda(x,\beta))^{a-1} (1 - (1 - \Lambda(x,\beta))^a)^{b-1} \left(1 - \bar{\alpha}[1 - (1 - \Lambda(x,\beta))^a]^b\right)^{-2}. \tag{12}$$

where, $\Lambda(x,\beta) = \left(1 + {}^{x}/_{\beta}\right) e^{\frac{-x}{\beta}}, \psi = (a, b, \alpha, \beta)$ is a set of parameters. A random variable X has MOKwME distribution will be denoted by $X\sim$ MOKwME (a, b, α, β). The MOKwME distribution is a very flexible model that approaches to some distributions as follows:

- For $\alpha = 0$ and $b = 1$, we obtain EME distribution presented in [22]
- For $a = 1$ and $b = 1$, we obtain MOLBE distribution presented in [27].
- For $\alpha = 0, b = 1$, and $y = x^{1/\delta}$ we obtain, GEME distribution presented in [23].
- For $a = 1$, $b = 1$, and $\alpha = 0$, we obtain, ME distribution presented in [21].
- For $\alpha = 1$, we obtain, KwME distribution as a new model.

Next, we provide a simple motivation for the MOKwME distribution in the medical context as follows (see [15]): Consider a random sample $X_1, X_2, \ldots, X_N$, where the X_i's, $i = 1,2,\ldots,N$, be a sequence of identically independent distributed random variables with survival function $[1 - (1 - \Lambda(x,\beta))^a]^b$ then

- If N has a geometric distribution with parameter α; $(0 < \alpha < 1)$ independent of X_i's then the density of the random variable $W_1 = \min(X_1, X_2, \ldots, X_N)$ is that MOKwME (a, b, α, β).
- If N has a geometric distribution with parameter $1/\alpha$; $(\alpha > 1)$ independent of, X_i's then density of the random variable $W_2 = \max(X_1, X_2, \ldots, X_N)$ distributed as MOKwME $(\alpha^{-1}, a, b, \beta)$.

This setup is usually common in oncology, where N represents the amount of cells with metastasis potential and X_i denotes the time for the ith cell to metastasis. So, X represents the recurrence time of the cancer.

The survival and hazard rate functions of X are given, respectively, as follows:

$$\bar{F}(x;\psi) = \alpha[1 - (1 - \Lambda(x,\beta))^a]^b \left\{1 - \bar{\alpha}[1 - (1 - \Lambda(x,\beta))^a]^b\right\}^{-1}, \tag{13}$$

and

$$h(x;\psi) = \frac{a\,b}{\beta^2} x\, e^{\frac{-x}{\beta}} (1 - \Lambda(x,\beta))^{a-1} (1 - (1 - \Lambda(x,\beta))^a)^{-1} \left\{1 - \bar{\alpha}[1 - (1 - \Lambda(x,\beta))^a]^b\right\}^{-1}. \tag{14}$$

Plots of the pdf and hrf of the MOKwME distribution are displayed in Figs. 1 and 2, respectively, for different values of parameters. As seen from Fig. 1, the shapes of the pdf take different forms. Also, it is clear from Fig. 2 that the shapes of the hrf are reversed J-shaped, decreasing, increasing and upside-down bathtub at some selected values of parameters.

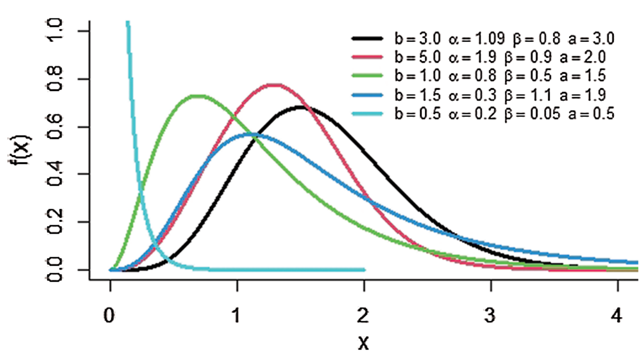

Figure 1: The pdf of the MOKwME distribution for some values of parameters

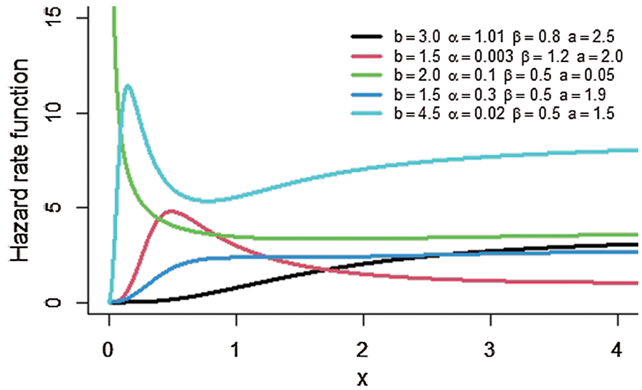

Figure 2: The hrf of the MOKwME distribution for some values of parameters

3 Statistical Properties

3.1 Expansion

Here, explicit expression for the MOKwME density function is provided. Since, the binomial expansion, for real non-integer value of k, is given by:

$$(1-y)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)y^j}{\Gamma(k)j!}, \quad |y| < 1,\ k > 0. \tag{15}$$

Using (15) in pdf (12), we obtain

$$f(x;\psi) = \sum_{j=0}^{\infty} (j+1)\, a\, b\, \alpha(\bar{\alpha})^j \frac{x}{\beta^2}\, e^{\frac{-x}{\beta}} \left(1-(1+\frac{x}{\beta})\, e^{\frac{-x}{\beta}}\right)^{a-1} \left(1-\left(1-(1+\frac{x}{\beta})\, e^{\frac{-x}{\beta}}\right)^{a}\right)^{b(j+1)-1}. \tag{16}$$

Again applying the binomial expansion in "previous equation", we obtain

$$f(x;\psi) = \sum_{i,j=0}^{\infty} (-1)^i (j+1)\, ab\alpha(\bar{\alpha})^j \binom{b(j+1)-1}{i} \frac{x}{\beta^2}\, e^{\frac{-x}{\beta}} \left(1-(1+\frac{x}{\beta})\, e^{\frac{-x}{\beta}}\right)^{a(i+1)-1}. \tag{17}$$

Hence the pdf of MOKwME distribution can be written as follows:

$$f(x;\psi) = \sum_{i,j=0}^{\infty} w_{i,j}\, a(i+1) g(x;\beta)(G(x;\beta))^{a(i+1)-1}, \tag{18}$$

where, $w_{i,j} = \frac{b\alpha(\bar{\alpha})^j}{(i+1)}(-1)^i (j+1)\binom{b(j+1)-1}{i}$.

Eq. (18) reveals that the MOKwME density function is a linear mixture of EME density functions with power parameter $a(i+1)$.

3.2 Moments

Here, we discuss the *sth* moment for the MOKwME distribution. The *sth* moment for the MOKwME distribution about zero is derived by using pdf (18) as follows:

$$\mu_s' = \sum_{i,j=0}^{\infty} w_{i,j} \, \frac{a(i+1)}{\beta^2} \int_0^{\infty} x^{s+1} \, e^{\frac{-x}{\beta}} \left(1 - (1 + \frac{x}{\beta}) \, e^{\frac{-x}{\beta}} \right)^{a(i+1)-1} dx. \tag{19}$$

Suppose $v = x/\beta$, then μ_s' can be written as follows:

$$\mu_s' = \sum_{i,j=0}^{\infty} w_{i,j} \; a(i+1)\beta^s \int_0^{\infty} v^{s+1} \, e^{-v} \, (1 - (1+v) \, e^{-v})^{a(i+1)-1} dv. \tag{20}$$

Using binomial expansion, then

$$\mu_s' = \sum_{i,j,\ell=0}^{\infty} \sum_{m=0}^{\ell} b_{i,j,m,\ell} \, \beta^s \frac{\Gamma(s+m+2)}{(\ell+1)^{s+m+2}}, s = 1, 2, \ldots \quad , \tag{21}$$

where $b_{i,j,m,\ell} = a(i+1)(-1)^{\ell} \binom{a(i+1)-1}{\ell} \binom{\ell}{m}$.

The mean of the MOKwME distribution is obtained by putting s =1 in (21). The sth central moment (μ_s) of X is given by

$$\mu_s = E(X - \mu_1')^s = \sum_{i=0}^{s} (-1)^i \binom{s}{i} (\mu_1')^i \mu_{s-i}'. \tag{22}$$

3.3 Incomplete Moments

The *sth* incomplete moment, say $\tau_s(t)$ is defined by:

$$\tau_s(t) = \int_{-\infty}^{t} x^s \, f(x) \, dx. \tag{23}$$

Hence, the *sth* moment of MOKwME is derived by substituting (18) in (23) as follows:

$$\tau_s(t) = \sum_{i,j,\ell=0}^{\infty} \sum_{m=0}^{\ell} b_{i,j,\ell,m} \, \beta^s \gamma \left(\frac{s+m+2}{(\ell+1)^{s+m+2}}, \frac{t}{\beta} \right), \tag{24}$$

where $\gamma(k,t) = \int_0^t x^{k-1} e^{-k} dx$ is the lower incomplete gamma function. The Bonferroni and Lorenz curves and the Gini indices have applications in economics, reliability, demography, insurance and medicine (see [30]). The Lorenz curve of MOKwME distribution is given as follows:

$$L[X] = \frac{1}{\mu'_1} \int_0^{x} tf(t)dt = \frac{\sum\limits_{i,j,\ell=0}^{\infty} \sum\limits_{m=0}^{\ell} b_{i,j,\ell,m} \, \gamma \left(\frac{m+3}{(\ell+1)^{m+3}}, \frac{x}{\beta} \right)}{\sum\limits_{i,j,\ell=0}^{\infty} \sum\limits_{m=0}^{\ell} b_{i,j,\ell,m} \, \frac{\Gamma(m+3)}{(\ell+1)^{m+3}}}. \tag{25}$$

The Bonferroni curve of MOKwME distribution is obtained as

$$B_F[X] = \frac{1}{\mu'_1 F(x)} \int_0^x tf(t)dt = \frac{1}{F(x;\psi)} \frac{\sum_{i,j,\ell=0}^{\infty} \sum_{m=0}^{\ell} b_{i,j,\ell,m}\, \gamma\left(\frac{m+3}{(\ell+1)^{m+3}}, \frac{x}{\beta}\right)}{\sum_{i,j,\ell=0}^{\infty} \sum_{m=0}^{\ell} b_{i,j,\ell,m} \frac{\Gamma(m+3)}{(\ell+1)^{m+3}}}. \tag{26}$$

3.4 Moments of Residual Life Function

The residual life plays an important role in life testing situations and reliability theory. The *nth* moment of the residual life is defined by:

$$\varpi_n(t) = E[(X-t)^n | X > t] = \frac{1}{\bar{F}(t)} \int_t^{\infty} (x-t)^n f(x)dx. \tag{27}$$

Using the binomial expansion and pdf (18), then $\varpi_n(t)$ can be written as follows:

$$\varpi_n(t) = \frac{1}{\bar{F}(t;\psi)} \sum_{r=0}^{n} \sum_{i,j=0}^{\infty} \binom{n}{r} w_{i,j} a(i+1)(-t)^{n-r} \int_t^{\infty} \frac{x^{r+1}}{\beta^2} e^{\frac{-x}{\beta}} \left(1 - (1+\frac{x}{\beta}) e^{\frac{-x}{\beta}}\right)^{a(i+1)-1} dx. \tag{28}$$

So, after simplification the *nth* moment of the residual life of MOKwME distribution is obtained as follows:

$$\varpi_n(t) = \frac{1}{\bar{F}(t;\psi)} \sum_{r=0}^{n} \sum_{i,j,\ell=0}^{\infty} \sum_{m=0}^{\ell} \binom{n}{r} b_{i,j,\ell,m} (-t)^{n-r} \beta^r \Gamma\left(\frac{r+m+2}{(\ell+1)^{r+m+2}}, \frac{t}{\beta}\right), \tag{29}$$

where $\Gamma(k,t) = \int_t^{\infty} x^{k-1} e^{-k} dx$ is the upper incomplete gamma and $\bar{F}(t;\psi)$ is the survival function of MOKwME distribution In particular, *the mean residual life* (MRL) which represents the expected additional life length for a unit which is alive at age *t* is obtained by substituting *n* =1 in (29).

3.5 Quantile

The *P-th* quantile function (also called the percentile of order *p*) of the MOKwME distribution is of the form:

$$\frac{1 - [1 - (1 - \Lambda(x,\beta))^a]^b}{1 - \bar{\alpha}[1 - (1 - \Lambda(x,\beta))^a]^b} = P. \tag{30}$$

In particular, the median, denoted by M, can be obtained from (30) by substituting $P = 0.5$ and solving the following:

$$\frac{1 - [1 - (1 - \Lambda(x,\beta))^a]^b}{1 - \bar{\alpha}[1 - (1 - \Lambda(x,\beta))^a]^b} = 0.5. \tag{31}$$

Solving the Eq. (30) numerically, the percentage points are computed for some selected values of the parameters. These values are provided in Tab. 1.

Table 1: Percentage points for a, b, α and β

a	β	$\alpha = 0.1,\ b = 0.5$				$\alpha = 1,\ b = 1.5$			
		50%	75%	85%	95%	50%	75%	85%	95%
1	2	1.501	2.801	3.990	7.341	2.582	4.068	5.049	7.003
	3	2.252	4.201	5.984	11.012	3.873	6.102	7.573	10.505
	4	3.002	5.601	7.979	14.683	5.165	8.136	10.098	14.006
	5	3.753	7.002	9.974	18.354	6.456	10.171	12.622	17.508
2	2	2.849	4.347	5.610	8.991	4.107	5.692	6.696	8.655
	3	4.274	6.521	8.415	13.486	6.123	8.538	10.044	12.982
	4	5.698	8.695	11.220	17.981	8.214	11.384	13.393	17.309
	5	7.123	10.868	14.026	22.477	10.239	14.201	16.741	21.637

4 Shannon Entropy

Shannon [31] introduced the probabilistic definition of entropy as a measure of uncertainty. It is also a useful instrument for comparing two or more distributions. The Shannon entropy of a random variable X is defined by:

$$SH(f) = -\left(\int_{-\infty}^{\infty} f(x)\ \log(f(x))dx \right). \tag{32}$$

The Shannon entropy for the MOKwME distribution with pdf (12) is as follows:

$$\begin{aligned} SH(f) = -\int_{0}^{\infty} ab\alpha \frac{x}{\beta^2}\, e^{\frac{-x}{\beta}} (1-\Lambda(x,\beta))^{a-1}(1-(1-\Lambda(x,\beta))^{a})^{b-1}\left(1-\bar{\alpha}[1-(1-\Lambda(x,\beta))^{a}]^{b}\right)^{-2} \\ \log\left[ab\alpha \frac{x}{\beta^2}\, e^{\frac{-x}{\beta}} (1-\Lambda(x,\beta))^{a-1}(1-(1-\Lambda(x,\beta))^{a})^{b-1}\left(1-\bar{\alpha}[1-(1-\Lambda(x,\beta))^{a}]^{b}\right)^{-2} \right] dx. \end{aligned} \tag{33}$$

Since the theoretical result of entropy is not in a closed form, some of the numerical values of entropy for different parameters are given in Tab. 2.

Table 2: Shannon Entropy for some values of a, b, α and β

a	β	$\alpha = 0.1,\ b = 0.5$	$\alpha = 1,\ b = 1.5$
1	2	0.641	0.857
	3	0.984	1.033
	4	1.227	1.158
	5	1.415	1.255
2	2	0.470	0.462
	3	0.642	0.550
	4	0.763	0.613
	5	0.857	0.661

5 Maximum Likelihood Estimators

We consider the estimation of the unknown parameters of the MOKwME distribution using the ML method. Let $X_1,\ X_2,\ \ldots,\ X_n$ be the observed values from the MOKwME distribution with set of parameters $\psi = (a, b, \alpha, \beta)^T$. The log-likelihood function, denoted by LnL, based on complete sample for the vector of parameters ψ can be expressed as

$$\begin{aligned} LnL = {} & n\ln a + n\ln b + n\ln\alpha - 2n\ln\beta + \sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\frac{x_i}{\beta} + (a-1)\sum_{i=1}^{n}\ln\left(1-\Lambda(x_i,\beta)\right) \\ & + (b-1)\sum_{i=1}^{n}\ln(1-(1-\Lambda(x_i,\beta))^a) - 2\sum_{i=1}^{n}\ln\Big(1-\bar{\alpha}[1-(1-\Lambda(x_i,\beta))^a]^b\Big), \end{aligned} \tag{34}$$

The partial derivatives of the log-likelihood function with respect to a, b, α and β components of the score vector $U(\psi) = \partial LnL/\partial\psi = (U(a), U(b), U(\alpha), U(\beta))^T$ can be obtained as follows:

$$\begin{aligned} U(a) = {} & \frac{n}{a} + \sum_{i=1}^{n}\ln(1-\Lambda(x_i,\beta)) - (b-1)\sum_{i=1}^{n}\frac{\ln(1-\Lambda(x_i,\beta))}{((1-\Lambda(x_i,\beta))^{-a}-1)} \\ & - 2\sum_{i=1}^{n}\frac{\bar{\alpha}b\ln(1-\Lambda(x_i,\beta))(1-\Lambda(x_i,\beta))^a[1-(1-\Lambda(x_i,\beta))^a]^{b-1}}{\Big(1-\bar{\alpha}[1-(1-\Lambda(x_i,\beta))^a]^b\Big)}, \end{aligned} \tag{35}$$

$$U(b) = \frac{n}{b} + \sum_{i=1}^{n}\ln(1-(1-\Lambda(x_i,\beta))^a) + 2\sum_{i=1}^{n}\frac{\ln[1-(1-\Lambda(x_i,\beta))^a]}{\Big((1/\bar{\alpha})[1-(1-\Lambda(x_i,\beta))^a]^{-b}-1\Big)}, \tag{36}$$

$$U(\alpha) = \frac{n}{\alpha} + 2\sum_{i=1}^{n}\frac{[1-(1-\Lambda(x_i,\beta))^a]^b}{\Big(1-\bar{\alpha}[1-(1-\Lambda(x_i,\beta))^a]^b\Big)}, \tag{37}$$

$$\begin{aligned} U(\beta) = {} & \frac{-2n}{\beta} + \sum_{i=1}^{n}\frac{x_i}{\beta^2} - \sum_{i=1}^{n}\frac{(a-1)}{(1-\Lambda(x_i,\beta)}\frac{\partial\Lambda(x_i,\beta)}{\partial\beta} + \sum_{i=1}^{n}\frac{a(b-1)(1-\Lambda(x_i,\beta))^{a-1}}{(1-(1-\Lambda(x_i,\beta))^a)}\frac{\partial\Lambda(x_i,\beta)}{\partial\beta} \\ & + 2\sum_{i=1}^{n}\frac{ab\bar{\alpha}[1-(1-\Lambda(x_i,\beta))^a]^{b-1}(1-\Lambda(x_i,\beta))^{a-1}}{\Big(1-\bar{\alpha}[1-(1-\Lambda(x_i,\beta))^a]^b\Big)}\frac{\partial\Lambda(x_i,\beta)}{\partial\beta}, \end{aligned} \tag{38}$$

and,

$$\frac{\partial\Lambda(x_i,\beta)}{\partial\beta} = \frac{x_i^2}{\beta^3}e^{\dfrac{-x_i}{\beta}}. \tag{39}$$

The ML estimators of the model parameters are determined by solving the non-linear equations $U(a) = 0,\ U(b) = 0,\ U(\alpha) = 0,$ and $U(\beta) = 0$. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative technique.

6 Simulation Study

A numerical study is performed to evaluate the performance of the estimates with respect to their *absolute biases* (ABs), *standard errors* (SEs) and *mean square errors* (MSEs) for different sample sizes and for different parameter values. The numerical procedures are described through the following steps:

Step 1: A random sample $X_1,\ldots,X_n$ of sizes n =10, 20, 30, 50 and 100 are selected, these random samples are generated from the MOKwME distribution.

Step 2: Four different set values of the parameters are selected as, $Set1 \equiv (\alpha = 0.3, \beta = 0.8, a = 0.8, b = 0.5)$, $Set2 \equiv (\alpha = 0.6, \beta = 0.8, a = 0.5, b = 0.7)$, $Set3 \equiv (\alpha = 0.1, \beta = 1.3, a = 1.1, b = 0.9)$ and $Set4 \equiv (\alpha = 0.9, \beta = 1.3, a = 0.3, b = 1.2)$.

Step 3: For each sample size, the ML estimates (MLEs) of α, β, a and b are computed.

Step 4: Steps from 1 to 3 are repeated 1000 times, then, the ABs, SEs and MSEs of the estimates are computed.

Numerical results are reported in Tabs. 3 and 4, from these tables, the following observations can be detected on the behavior of estimated parameters from the MOKwME distribution.

- The ABs, SEs and MSEs decrease as sample sizes increase (see Tabs. 3 and 4).
- The ABs of a decrease as the value of b increases (see Tab. 3). Also, the ABs of β increase as the value of β increases, in approximately, most of the situations.
- For fixed values of β and as the values of b and α increase, the ABs and MSEs are decreasing, in approximately most of situations (see Tab. 3). As the values of α increase and for fixed values of β, the ABs and MSEs for all estimates decrease (see Tab. 4).

Table 3: ABs, SEs and MSEs of MOKwME parameter estimates for Set 1and Set 2

		Set1≡(α = 0.3, β = 0.8, a = 0.8, b=0.5)				Set2≡(α = 0.6, β = 0.8, a = 0.5, b=0.7)			
n	Measure	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
10	MSE	0.090	0.169	0.633	0.250	1.360	0.173	0.194	0.616
	AB	0.299	0.124	0.795	0.500	0.329	0.408	0.353	0.482
	SE	0.000	0.101	0.000	0.000	0.354	0.025	0.084	0.196
20	MSE	0.088	0.060	0.566	0.248	0.857	0.142	0.179	0.592
	AB	0.296	0.244	0.749	0.498	0.301	0.367	0.314	0.406
	SE	0.000	0.051	0.000	0.000	0.196	0.019	0.064	0.146
30	MSE	0.085	0.059	0.561	0.246	0.694	0.112	0.161	0.494
	AB	0.291	0.244	0.746	0.496	0.356	0.325	0.269	0.424
	SE	0.000	0.000	0.000	0.000	0.138	0.015	0.054	0.102
50	MSE	0.068	0.054	0.410	0.215	0.519	0.106	0.157	0.493
	AB	0.260	0.068	0.545	0.432	0.390	0.318	0.291	0.466
	SE	0.000	0.000	0.000	0.000	0.086	0.010	0.038	0.074
100	MSE	0.066	0.024	0.342	0.187	0.271	0.089	0.131	0.299
	AB	0.174	0.024	0.231	0.354	0.486	0.286	0.226	0.502
	SE	0.000	0.000	0.000	0.000	0.019	0.009	0.028	0.022

7 Real Data Applications

In this section, we fit the MOKwME distribution into two distinct real data sets and we compare the performance with those of MOLBE, GEME, MOGE, EME, ME and MOKwE distributions. In each real

Table 4: ABs, SEs and MSEs of MOKwME parameter estimates for Set 3and Set 4

		Set3≡(α = 0.1, β = 1.3, a = 1.1, b = 0.9)				Set4≡(α = 0.9, β = 1.3, a = 0.3, b = 1.2)			
n	Measure	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
10	MSE	0.010	0.950	0.870	0.806	0.810	1.163	1.147	1.430
	AB	0.100	0.967	0.908	0.897	0.900	1.078	1.043	1.196
	SE	0.000	0.038	0.067	0.011	0.000	0.000	0.077	0.004
20	MSE	0.001	0.772	0.678	0.802	0.810	0.950	0.996	1.334
	AB	0.099	0.843	0.657	0.893	0.900	0.975	0.976	1.145
	SE	0.000	0.034	0.044	0.010	0.000	0.000	0.046	0.003
30	MSE	0.001	0.594	0.456	0.794	0.810	0.950	0.994	1.322
	AB	0.098	0.533	0.635	0.891	0.900	0.975	0.976	1.121
	SE	0.000	0.023	0.042	0.000	0.000	0.000	0.037	0.002
50	MSE	0.001	0.409	0.410	0.718	0.808	0.759	0.927	1.067
	AB	0.098	0.851	0.581	0.844	0.899	0.871	0.942	1.017
	SE	0.000	0.017	0.038	0.000	0.000	0.000	0.028	0.001
100	MSE	0.001	0.395	0.387	0.712	0.793	0.759	0.861	0.543
	AB	0.098	0.547	0.505	0.821	0.890	0.871	0.893	0.067
	SE	0.000	0.014	0.015	0.000	0.000	0.000	0.025	0.000

data set, the MLEs and their corresponding SEs (in parentheses) of the model parameters are obtained. *-2 log-likelihood* (-2LnL), *Akaike information criterion* (AIC), the *correct Akaike information criterion* (CAIC), *Bayesian information criterion* (BIC), *Hannan-Quinn information criterion* (HQIC), *Anderson-Darling* (A^*) statistic, *Cramér- von Mises* (W^*) statistic and *Kolmogorov-Smirnov* (K-S) statistic are used to assess the effectiveness of the models. The model with the smallest value of these measures gives a better representation of the data set than the others.

First Real Data Set: The first data refer to Smith et al. [32] which represent the strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data are:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2.0, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.50, 1.54, 1.60, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.70, 1.77, 1.84, 0.84, 1.24, 1.30, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.70, 1.78, 1.89.

The MLEs and their corresponding SEs (in parentheses) of the model parameters are listed in Tab. 5. Also, the above suggested statistical measures of all models are listed in Tab. 6. It is observed, from Tab. 6, that the MOKwME distribution gives a better fit than other fitted models.

The empirical pdf and estimated cdf for the first data set are provided in Figs. 3 and 4.

Second Real Data: The second data were discussed in Ristić et al. [26], which represent the strength data measured in GPA, the single carbon fibers, and impregnated 1000-carbon fiber tows. Single fibers were tested under tension at gauge length 1 mm. The data are:

Table 5: MLEs of all models and the corresponding SEs (in parentheses) for the first data

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
MOKwME	–	–	18.426	–	1.959	2.041	87.374
	–	–	(21.488)	–	(1.839)	(0.861)	(179.452)
MOLBE	–	–	–	0.001	31.696	–	–
	–	–	–	(0.0017)	(26.351)	–	–
GEME	–	–	0.393	6.399	20.544	–	–
	–	–	(0.142)	(1.44)	(21.726)	–	–
MOGE	5.484	94.469	47.044	–	–	–	–
	(0.567)	(72.603)	(36.566)	–	–	–	–
EME		–	12.925	–	0.313	–	–
	–	–	(3.641)	–	(0.026)	–	–
ME	–	–	–	–	0.753	–	–
	–	–	–	–	(0.067)	–	–
MOKwE	0.551	–	20.755	–	–	4.603	37.667
	(0.399)	–	(24)	–	–	(2.096)	(53.661)

Table 6: Statistics measures for the first data

Distribution	–2LnL	AIC	BIC	CAIC	HQIC	W*	A*	K-S
MOKwME	24.568	32.568	41.141	33.258	35.94	0.225	2.108	0.468
MOLBE	144.923	148.923	153.209	149.123	150.609	1.228	22.8	1.000
GEME	29.735	35.735	42.164	36.142	38.264	1.283	24.89	0.884
MOGE	32.983	38.983	45.413	39.39	41.512	0.285	2.268	0.845
EME	60.161	64.161	68.448	64.361	65.847	0.259	2.253	0.997
ME	132.635	140.635	136.778	141.324	135.477	1.154	21.373	1.000
MOKwE	25.002	33.002	41.574	33.691	36.373	0.233	2.322	0.484

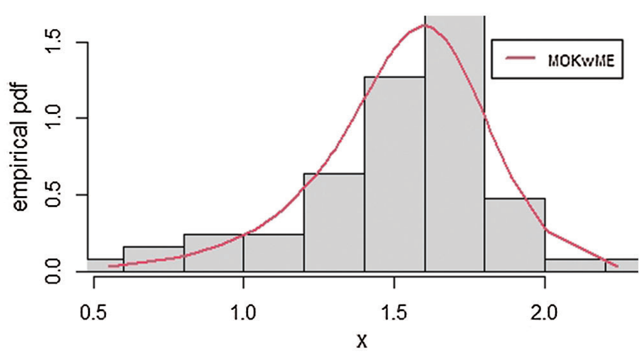

Figure 3: The empirical pdf of the MOKwME distribution for the first real data

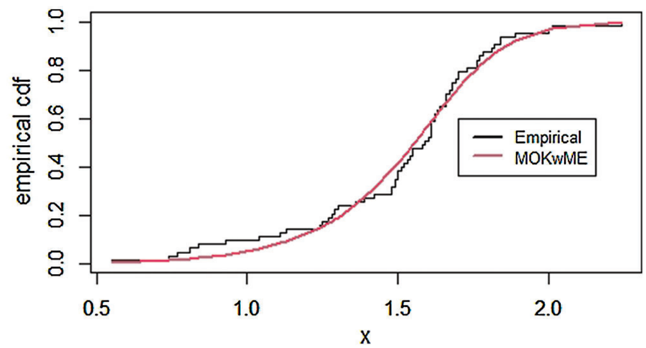

Figure 4: The empirical cdf of the MOKwME distribution for the first real data

2.247, 2.64, 2.908, 3.099, 3.126, 3.245, 3.328, 3.355, 3.383, 3.572, 3.581, 3.681, 3.726, 3.727, 3.728, 3.783, 3.785, 3.786, 3.896, 3.912, 3.964, 4.05, 4.063, 4.082, 4.111, 4.118, 4.141, 4.246, 4.251, 4.262, 4.326, 4.402, 4.457, 4.466, 4.519, 4.542, 4.555, 4.614, 4.632, 4.634, 4.636, 4.678, 4.698, 4.738, 4.832, 4.924, 5.043, 5.099, 5.134, 5.359, 5.473, 5.571, 5.684, 5.721, 5.998, 6.06.

The MLEs and the SEs of the model parameters are listed in Tab. 7 whereas Tab. 8 gives the statistics measures of all models. It is observed, from Tab. 8, that the MOKwME distribution gives a better fit than other fitted models.

Table 7: MLEs of all models and the corresponding SEs (in parentheses) for the second data

Distribution	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{a}$	$\hat{b}$
MOKwME	–	–	12.504	–	0.619	23.655	1.509
	–	–	(76.031)	–	(1.491)	(107.352)	(4.701)
MOLBE	–	–	–	0.006	36.284	–	–
	–	–	–	(0.008)	(23.94)	–	–
GEME	–	–	81.268	0.905	0.529	–	–
	–	–	(31.778)	(0.089)	(0.087)	–	–
MOGE	1.9	62.842	48.377	–	–	–	–
	(0.191)	(59.043)	(38.121)	–	–	–	–
EME		–	43.247	–	0.68	–	–
	–	–	(17.653)	–	(0.06)	–	–
ME	–	–	–	–	2.13	–	–
	–	–	–	–	(0.201)	–	–
MOKwE	0.502	–	0.0069	–	–	30.207	0.362
	(0.012)	–	(0.011)	–	–	(7.306)	(1.302)

Table 8: Statistics measures for the second data

Distribution	-2LnL	AIC	BIC	CAIC	HQIC	W*	A*	K-S
MOKwME	136.039	144.039	143.031	144.823	136.524	0.395	3.592	0.001
MOLBE	245.592	249.592	249.088	249.818	246.562	1.808	71.707	1.000
GEME	143.934	149.934	149.178	150.395	145.389	0.427	3.627	0.612
MOGE	141.107	147.107	146.352	147.569	142.563	0.555	5.892	0.895
EME	142.479	146.479	145.975	146.705	143.449	0.458	4.066	0.254
ME	233.212	241.212	234.96	241.996	233.697	1.758	64.685	1.000
MOKwE	144.949	152.949	151.942	153.734	146.89	0.482	4.459	0.039

The empirical pdf and cdf for the second data set are provided in Figs. 5 and 6.

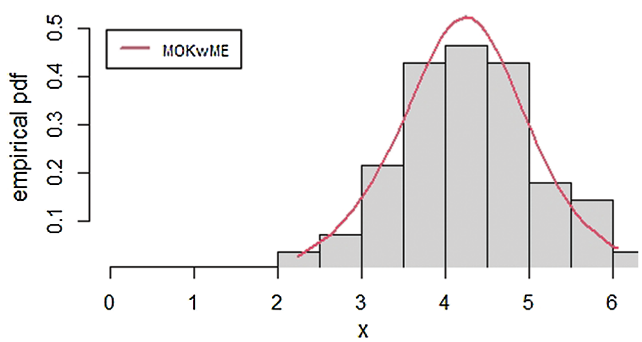

Figure 5: The empirical pdf of the MOKwME distribution for the second real data

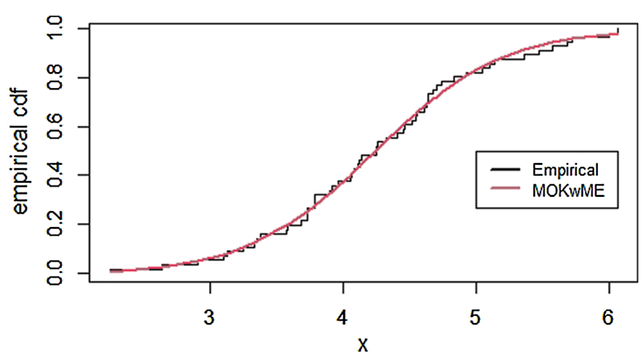

Figure 6: The empirical cdf of the MOKwME distribution for the second real data

8 Concluding Remarks

In this paper, we introduce a new probability model called the Marshal-Olkin Kumaraswamy moment exponential. The new model includes; exponentiated moment exponential, generalized exponentiated moment exponential, Marshall-Olkin length-biased exponential and moment exponential distributions. At the same time, it contains the Kumaraswamy moment exponential distribution as a new model. We study some of its structural properties including an expansion for the density function and explicit expressions for the moments, generating function, Bonferroni and Lorenz curves. The maximum likelihood method is employed for estimating the model parameters. The usefulness of the new model is illustrated by means of an application to real data set.

Acknowledgement: The authors would like to thank the editor and the anonymous referees for their valuable and very constructive comments, which have greatly improved the contents of the paper

Funding Statement: The authors received no specific funding for this study.

Conflicts of Interest: The authors declare no conflict of interest.

References

[1] A. W. Marshall and I. Olkin, "A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families," *Biometrika*, vol. 84, no. 3, pp. 641–652, 1997.

[2] A. K. Nanda and S. Das, "Stochastic orders of the Marshall-Olkin extended distribution," *Statistics & Probability Letters*, vol. 82, no. 2, pp. 295–302, 2012.

[3] K. Jayakumar and T. Mathew, "On a generalization to Marshall-Olkin scheme and its application to Burr type XII distribution," *Statistical Papers*, vol. 49, no. 3, pp. 421–439, 2008.

[4] M. H. Tahir and S. Nadarajah, "Parameter induction in continuous univariate distributions: Well-established G families," *Anais da Academia Brasileira de Ciências*, vol. 87, no. 2, pp. 539–568, 2015.

[5] H. Barakat, M. Ghitany and E. Al-Hussaini, "Asymptotic distributions of order statistics and record values under the Marshall-Olkin parametrization operation," *Communications in Statistics-Theory and Methods*, vol. 38, no. 13, pp. 2267–2273, 2009.

[6] M. Alizadeh, M. Tahir, G. M. Cordeiro, M. Mansoor, M. Zubair *et al.,* "The Kumaraswamy Marshal-Olkin family of distributions," *Journal of the Egyptian Mathematical Society*, vol. 23, no. 3, pp. 546–557, 2015.

[7] R. Bantan, A. S. Hassan and M. Elsehetry, "Generalized Marshall Olkin inverse Lindley distribution with applications," *CMC-Computers, Materials & Continua*, vol. 64, no. 3, pp. 1505–1526, 2020.

[8] W. Barreto-Souza, A. J. Lemonte and G. M. Cordeiro, "General results for the Marshall and Olkin's family of distributions," *Anais da Academia Brasileira de Ciências*, vol. 85, no. 1, pp. 3–21, 2013.

[9] G. M. Cordeiro, A. J. Lemonte and E. M. Ortega, "The Marshall-Olkin family of distributions: Mathematical properties and new models," *Journal of Statistical Theory and Practice*, vol. 8, no. 2, pp. 343–366, 2014.

[10] C. R. Dias, G. M. Cordeiro, M. Alizadeh, P. R. D. Marinho and H. F. C. Coêlho, "Exponentiated Marshall-Olkin family of distributions," *Journal of Statistical Distributions and Applications*, vol. 3, no. 1, pp. 106, 2016.

[11] K. K. Jose and E. Krishna, "Marshall-Olkin extended uniform distribution," *ProbStat Forum*, vol. 4, no. October, pp. 78–88, 2011.

[12] I. Elbatal and M. Elgarhy, "Extended Marshall-Olkin length-Biased exponential distribution: Properties and applications," *Advances and Applications in Statistics*, vol. 64, no. 1, pp. 113–125, 2020.

[13] M. Haq, G. G. Hamedani, M. Elgarhy and P. L. Ramos, "Marshall-Olkin power Lomax distribution: Properties and estimation based on complete and censored samples," *International Journal of Statistics and Probability*, vol. 9, no. 1, pp. 48–62, 2020.

[14] G. M. Cordeiro and M. de Castro , "A new family of generalized distributions," *Journal of Statistical Computation and Simulation*, vol. 81, no. 7, pp. 883–898, 2011.

[15] L. Handique, S. Chakraborty and G. G. Hamedani, "The Marshall-Olkin-Kumaraswamy-G family of distributions," *Journal of Statistical Theory and Applications*, vol. 16, no. 4, pp. 427–447, 2017.

[16] A. J. Lemonte, "A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function," *Computational Statistics & Data Analysis*, vol. 62, pp. 149–170, 2013.

[17] R. D. Gupta and D. Kundu, "Generalized exponential distributions: Theory & methods," *Australian & New Zealand Journal of Statistics*, vol. 41, no. 2, pp. 173–188, 1999.

[18] R. D. Gupta and D. Kundu, “Exponentiated exponential family: An alternative to gamma and Weibull distributions,” *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, vol. 43, no. 1, pp. 117–130, 2001.

[19] S. Nadarajah and S. Kotz, “The beta exponential distribution,” *Reliability Engineering & System Safety*, vol. 91, no. 6, pp. 689–697, 2006.

[20] W. Barreto-Souza, A. H. Santos and G. M. Cordeiro, “The beta generalized exponential distribution,” *Journal of Statistical Computation and Simulation*, vol. 80, no. 2, pp. 159–172, 2010.

[21] S. T. Dara and M. Ahmad, *in Recent Advances in Moment Distribution and Their Hazard Rates*. Germany: LAP LAMBERT Academic Publishing, 2012.

[22] S. A. Hasnain, “Exponentiated moment exponential distributions,” Ph.D. dissertation. National College of Business Administration & Economics, Lahore, Pakistan, 2013.

[23] Z. Iqbal, S. A. Hasnain, M. Salman, M. Ahmad and G. Hamedani, “Generalized exponentiated moment exponential distribution,” *Pakistan Journal of Statistics*, vol. 30, no. 4, pp. 537–554, 2014.

[24] S. Abu-Youssef, B. Mohammed and M. Sief, “An extended exponentiated exponential distribution and its properties,” *International Journal of Computer Applications*, vol. 121, no. 5, pp. 1–6, 2015.

[25] T. K. Pogány, A. Saboor and S. Provost, “The Marshall-Olkin exponential Weibull distribution,” *Hacettepe Journal of Mathematics and Statistics*, vol. 44, no. 6, pp. 1579, 2015.

[26] M. M. Ristić and D. Kundu, “Marshall-Olkin generalized exponential distribution,” *METRON*, vol. 73, no. 3, pp. 317–333, 2015.

[27] M. A. Haq, R. M. Usman, S. Hashmi and A. I. Al-Omeri, “The Marshall-Olkin length-biased exponential distribution and its applications,” *Journal of King Saud University - Science*, vol. 31, no. 2, pp. 246–251, 2019.

[28] A. S. Hassan, R. E. Mohamed, M. Elgarhy and A. Fayomi, “Alpha power transformed extended exponential distribution: properties and applications,” *Journal of Nonlinear Sciences & Applications*, vol. 12, no. 04, pp. 239– 251, 2018.

[29] S. Chakraborty and L. Handique, “The generalized Marshall-Olkin-Kumaraswamy-G family of distributions,” *Journal of Data Science*, vol. 15, no. 3, pp. 391–422, 2017.

[30] C. Kleiber and S. Kotz, in *Statistical Size Distributions in Economics and Actuarial Sciences*, Vol. 470, Hoboken, New Jersey: John Wiley & Sons, Inc., 2003. [Online] https://onlinelibrary.wiley.com/doi/pdf/10.1002/0471457175.

[31] C. E. Shannon, “A mathematical theory of communication,” *Bell System Technical Journal*, vol. 27, no. 3, pp. 379–423, 1948.

[32] R. L. Smith and J. C. Naylor, “A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution,” *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, vol. 36, no. 3, pp. 358–369, 1987.