Double Censoring Partial Probability Weighted Moments Estimation of the Generalized Exponential Distribution

By

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Abstract

The method of partial probability weighted moments (PPWM) was used to estimate the parameters of the generalized exponential distribution from censored samples. The performance of the PPWM method was studied under doubly censored samples. In particular, the results of PPWM estimation for right and left censored samples were obtained as special cases. To study the properties of the new estimators a comprehensive numerical study was carried out using Mathcad 13 software. The performance of the obtained estimators against the different censoring levels was investigated in terms of bias and mean square error (MSE). The simulation study showed that the censoring level has a significant effect on the performance of the PPWM estimation. The biases and MSEs increase as the censoring level increase. In addition, the Pearson system technique was used to fit a suitable distribution for the PPWM estimators. Most of the PPWM estimators follow Pearson type I, IV, and VI distributions.

Keywords: Partial Probability Weighted Moments, Generalized Exponential Distribution, Parameter estimation, Pearson system, Censored Samples.

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1. Introduction

In many practical applications such as life testing experiments, it is quite common not to observe complete data but only observe some forms of censored data. This may be based on cost or time consideration. In censored samples there exists the situation of incomplete lifetimes either from one side or from two sides. If the first few observations or the last few ones of a given sample are unknown, then the incomplete lifetimes from one side are referred to as single censored observations. On the other hand, if the first few observations and the last few ones of a given sample are unknown, then the incomplete lifetimes from two sides are referred to as doubly censored observations.

The maximum likelihood method may be employed for fitting a distribution to a censored sample. However, the method often breaks down in the process of maximizing the likelihood function, especially when the distribution is lower or upper bounded (Wang, 1996). To overcome the shortcomings of the maximum likelihood method, Wang (1990a) introduced the method of partial probability weighted moments (PPWM), which is an extension of the method of probability weighted moments (PWM), to fit distribution functions to censored samples. This method possesses the same merit as the original method of PWM.

Wang (1990a, b) applied the method of PPWM for estimating the parameters and quantiles of the generalized extreme value (GEV) distribution using two different techniques. The results of the two studies showed that moderately high left censoring threshold can be used for high quantile estimation with only slight increments in standard error and mean square error (MSE) compared with estimation from uncensored samples. Hosking (1995) reported that a PPWM estimator of the parameters of a reverse Gumbel distribution is only slightly less efficient than maximum likelihood estimator under type I right censoring. Also, he mentioned that these estimators are almost equivalent for censoring levels above 0.5 and for larger samples. Also, Wang (1996) derived a unified expression of PPWM with left censoring for the three types of the GEV distribution. In addition, Hassan (2005) used the method of PPWM to estimate the parameters of the generalized Pareto distribution from censored samples.

The aim of this article is to estimate the unknown parameters of the generalized exponential distribution from doubly censored samples by the method of PPWM. Then, the PPWM estimators of the parameters from left and right censored samples will be obtained as special cases. To illustrate the properties of the new estimators, an extensive numerical study will be performed. Furthermore, the sampling distribution of the PPWM parameter estimators will be obtained by using the Pearson system technique.

This article is organized as follows. In section 2, the definition of PPWM and their sample estimators from doubly, left and right censored samples is reviewed. Section 3 discusses the PPWM estimation of the generalized exponential distribution from doubly, left and right censored samples. A simulation study is performed using the Mathcad 13 software to investigate the properties of the PPWM estimators from censored samples in section 4. Finally, the sampling distributions of the PPWM estimators are obtained in section 5. The conclusions are included in section 6. Tables and some figures are included in the appendix.

2. PPWM and Their Sample Estimators

The concept of probability weighted moments was introduced by Greenwood *et al* (1979) and indicated to be of potential interest for distributions that may be written in inverse form. For a random variable X with cumulative distribution function $F(x) = P(X \le x)$ and inverse distribution function $x = x(F)$; the probability weighted moments are the quantities:

$$M_{p,r,s} = E[X^p F^r (1-F)^s] = \int_0^1 [x(F)]^p F^r [1-F]^s \, dF \, , \qquad (2.1)$$

where p, r, and s are real numbers.

Wang (1990a) extended the concept of PWM and introduced PPWM. The general form of PPWM with double bound censoring for the random variable X is defined as follows:

$$M'''_{p,r,s} = E[X^p [F(X)]^r [1-F(X)]^s] = \int_c^d [x(F)]^p F^r (1-F)^s \, dF \, , \qquad (2.2)$$

where $c = F(x_{01})$ and $d = F(x_{02})$, x_{01} and x_{02} are lower and upper bound censoring thresholds, respectively; and p, r, s are real numbers. When $p = 1$ and $s = 0$, $M'''_{p,r,s}$ become:

$$\beta'''_r = M'''_{1,r,0} = E[X[F(X)]^r] = \int_c^d x(F) F^r \, dF \, . \qquad (2.3)$$

It should be noted that when the upper bound $d = 1$ and the lower bound $c = 0$ in equation (2.2), the PPWM with double bound censoring, $M'''_{p,r,s}$, is reduced to the original PWM in complete samples, $M_{p,r,s}$, defined in equation (2.1).

Wang (1990a) introduced an unbiased estimator b_r''' for β_r''' that is based on the ordered complete sample $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$ of size n from the distribution F. It is defined as follows:

$$b_r''' = \frac{1}{n}\sum_{i=1}^{n} \frac{(i-1)(i-2)...(i-r)}{(n-1)(n-2)...(n-r)} x_{(i)}^{***}, \tag{2.4}$$

where $x_{(i)}^{***} = \begin{cases} 0 & x_{(i)} \le x_{01} \\ x_{(i)} & x_{01} < x_{(i)} \le x_{02} \\ 0 & x_{(i)} > x_{02} \end{cases}$.

The PPWM with lower bound (left) censoring, when $p = 1$ and $s = 0$, can be defined as a special case from PPWM with double bound censoring, β_r''', by putting $d = 1$ in equation (2.3). Therefore, the PPWM with left censoring when $p = 1$ and $s = 0$ is defined as follows:

$$\beta_r' = M_{1,r,0}' = E[X[F(X)]^r] = \int_c^1 x(F)F^r dF. \tag{2.5}$$

The unbiased estimator b_r' of β_r' which was derived by Wang (1990a) is defined as follows:

$$b_r' = \frac{1}{n}\sum_{i=1}^{n} \frac{(i-1)(i-2)...(i-r)}{(n-1)(n-2)...(n-r)} x_{(i)}^{*} \tag{2.6}$$

where $x_{(i)}^{*} = \begin{cases} 0 & x_{(i)} \le x_{01} \\ x_{(i)} & x_{(i)} > x_{01} \end{cases}$.

It is also based on the ordered complete sample $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$ of size n from the distribution F.

The PPWM with upper bound (right) censoring, when $p = 1$ and $s = 0$, can be defined as a special case from β_r''' by putting $c = 0$ in equation (2.3). Thus, the PPWM with right censoring when $p = 1$ and $s = 0$ is given by:

$$\beta_r'' = M_{1,r,0}'' = E[X[F(X)]^r] = \int_0^d x(F)F^r dF. \tag{2.7}$$

Wang (1990b) introduced the following statistic as an unbiased estimator for β_r'':

$$b_r'' = \frac{1}{n}\sum_{i=1}^{n} \frac{(i-1)(i-2)...(i-r)}{(n-1)(n-2)...(n-r)} x_{(i)}^{**}, \tag{2.8}$$

where $x_{(i)}^{**} = \begin{cases} 0 & x_{(i)} > x_{02} \\ x_{(i)} & x_{(i)} \le x_{02} \end{cases}$,

which is also based on the ordered complete sample $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$ of size n from a distribution function *F*.

3. PPWM Estimation of the Generalized Exponential Distribution

In this section, the PPWM estimation of the parameters of the generalized exponential distribution from doubly, left and right censored samples is introduced.

The generalized exponential distribution was introduced and studied quite extensively by Gupta and Kundu (1999, 2001, 2002, 2003). It is observed that this distribution can be considered for situations where a skewed distribution for a non-negative random variable is needed. Also, it is observed that it can be used quite effectively to analyze lifetime data in place of gamma, Weibull and log-normal distributions. The generalized exponential distribution has the following distribution function:

$$F(x;\alpha,\lambda) = (1-e^{-\lambda x})^{\alpha}; \qquad \alpha,\lambda > 0 \tag{3.1}$$

for $x > 0$ and 0 otherwise. The corresponding density function is:

$$f(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}e^{-\lambda x}; \qquad \alpha,\lambda > 0 \tag{3.2}$$

for $x > 0$ and 0 otherwise.

Here α is the shape parameter and λ is the scale parameter. If the shape parameter $\alpha = 1$, then the generalized exponential distribution coincides with the exponential distribution with a scale parameter λ.

The PPWM with double bound censoring, β_r'''', for the generalized exponential distribution can be found as follows:

$$\beta_r'''' = \frac{1}{\lambda}\int_c^d \sum_{j=1}^{\infty} \frac{F^{\frac{j}{\alpha}}}{j} F^r \, dF$$

$$= \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+r+1} - c^{\frac{j}{\alpha}+r+1}}{j(\frac{j}{\alpha}+r+1)}\right]. \tag{3.3}$$

Writing equation (3.3) for $r = 0$ and 1 gives

$$\beta_0''' = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+1} - c^{\frac{j}{\alpha}+1}}{j(\frac{j}{\alpha}+1)}\right], \tag{3.4}$$

and

$$\beta_1''' = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+2} - c^{\frac{j}{\alpha}+2}}{j(\frac{j}{\alpha}+2)}\right]. \tag{3.5}$$

The PPWM estimators of α and λ from doubly censored samples, $\hat{\alpha}_1$ and $\hat{\lambda}_1$, can be obtained by solving equations (3.4) and (3.5) in terms of α and λ; where β_0''' and β_1''' are replaced by their sample estimators, b_0''' and b_1''', given by equation (2.4). Therefore, equations (3.4) and (3.5) yield:

$$\hat{\lambda}_1 = \frac{\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_1}+1} - c^{\frac{j}{\hat{\alpha}_1}+1}}{j(\frac{j}{\hat{\alpha}_1}+1)}\right]}{b_0'''}, \tag{3.6}$$

and

$$b_1'''\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_1}+1} - c^{\frac{j}{\hat{\alpha}_1}+1}}{j(\frac{j}{\hat{\alpha}_1}+1)}\right] - b_0'''\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_1}+2} - c^{\frac{j}{\hat{\alpha}_1}+2}}{j(\frac{j}{\hat{\alpha}_1}+2)}\right] = 0. \tag{3.7}$$

Equation (3.7) has to be solved by iteration to obtain $\hat{\alpha}_1$. Then the corresponding scale parameter estimator, $\hat{\lambda}_1$, is determined by substituting the value of $\hat{\alpha}_1$ into equation (3.6).

The procedure of PPWM estimation from left censored samples is similar to that one from doubly censored samples. PPWM with left (lower bound) censoring can be obtained from PPWM with double bound censoring as a special case. In other words, all the equations obtained in section 2 will be obtained here but the right censoring level, *d*, will be set to equal one. Therefore, the general expression of PPWM with left censoring, β_r', for the generalized exponential distribution can be obtained by putting $d = 1$ in equation (3.3).

$$\beta_r' = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{1 - c^{\frac{j}{\alpha}+r+1}}{j(\frac{j}{\alpha}+r+1)}\right]. \tag{3.8}$$

Writing equation (3.8) for $r =$ 0 and 1 gives:

$$\beta_0' = \frac{1}{\lambda}\sum_{j=1}^{\infty}[\frac{1-c^{\frac{j}{\alpha}+1}}{j(\frac{j}{\alpha}+1)}], \tag{3.9}$$

and

$$\beta_1' = \frac{1}{\lambda}\sum_{j=1}^{\infty}[\frac{1-c^{\frac{j}{\alpha}+2}}{j(\frac{j}{\alpha}+2)}]. \tag{3.10}$$

The PPWM estimators of α and λ from left censored samples, $\hat{\alpha}_2$ and $\hat{\lambda}_2$, can be obtained by solving equations (3.9) and (3.10) in terms of α and λ; where β_0' and β_1' are replaced by their sample estimators, b_0' and b_1', given by equation (2.6). Therefore, equations (3.9) and (3.10) yield:

$$\hat{\lambda}_2 = \frac{\sum_{j=1}^{\infty}[\frac{1-c^{\frac{j}{\hat{\alpha}_2}+1}}{j(\frac{j}{\hat{\alpha}_2}+1)}]}{b_0'}, \tag{3.11}$$

and

$$b_1'\sum_{j=1}^{\infty}[\frac{1-c^{\frac{j}{\hat{\alpha}_2}+1}}{j(\frac{j}{\hat{\alpha}_2}+1)}] - b_0'\sum_{j=1}^{\infty}[\frac{1-c^{\frac{j}{\hat{\alpha}_2}+2}}{j(\frac{j}{\hat{\alpha}_2}+2)}] = 0. \tag{3.12}$$

The solution of equation (3.12) requires some iterative technique obtain $\hat{\alpha}_2$. Thus the value of $\hat{\lambda}_2$ can be obtained by substituting the value of $\hat{\alpha}_2$ into equation (3.11).

Similarly, the PPWM with right (upper bound) censoring can be obtained as a special case from the PPWM with double bound censoring. The form of PPWM with right censoring, β_r'', for the generalized exponential distribution can be obtained by putting $c = 0$ in equation (3.3), that is:

$$\beta_r'' = \frac{1}{\lambda}\sum_{j=1}^{\infty}[\frac{d^{\frac{j}{\alpha}+r+1}}{j(\frac{j}{\alpha}+r+1)}]. \tag{3.13}$$

Writing equation (3.13) for $r = 0$ and 1 gives:

$$\beta_0'' = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+1}}{j(\frac{j}{\alpha}+1)}\right], \tag{3.14}$$

and

$$\beta_1'' = \frac{1}{\lambda}\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\alpha}+2}}{j(\frac{j}{\alpha}+2)}\right]. \tag{3.15}$$

The PPWM estimators of α and λ from right censored samples, $\hat{\alpha}_3$ and $\hat{\lambda}_3$, can be obtained by solving equations (3.14) and (3.15) in terms of α and λ; where β_0'' and β_1'' are replaced by their sample estimators, b_0'' and b_1'', given by equation (2.8). Therefore, equations (3.14) and (3.15) yield:

$$\hat{\lambda}_3 = \frac{\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_3}+1}}{j(\frac{j}{\hat{\alpha}_3}+1)}\right]}{b_0''}, \tag{3.16}$$

and

$$b_1''\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_3}+1}}{j(\frac{j}{\hat{\alpha}_3}+1)}\right] - b_0''\sum_{j=1}^{\infty}\left[\frac{d^{\frac{j}{\hat{\alpha}_3}+2}}{j(\frac{j}{\hat{\alpha}_3}+2)}\right] = 0. \tag{3.17}$$

Equation (3.17) has to be solved by iteration to obtain $\hat{\alpha}_3$. Therefore, $\hat{\lambda}_3$ can be obtained by substituting $\hat{\alpha}_3$ into equation (3.16).

4. Numerical Experiments and Discussions

Monte Carlo simulations have been performed, using Mathcad (version 13) software, to investigate the properties of the PPWM estimation for the generalized exponential distribution from censored samples. The investigated properties are biases and MSEs of the PPWM estimators of the two parameters α and λ. Different sample sizes were used in the experiments; which are; $n = 15$, 20, 30, 50 and 100. In addition, five shape parameter values; $\alpha = 0.5$, 1.0, 1.5, 2.0 and 2.5; were considered with scale parameter $\lambda = 1$ throughout. Different levels of censoring were considered, namely, $c =$ 0.1 (0.1) 0.4 and $d = 0.6$ (0.1) 0.9.

For each combination of the values of n and α; 10,000 random samples were generated from the generalized exponential distribution using the following transformation:

$$x_i = \left(-\frac{1}{\lambda}\right)\ln\left(1-(u_i)^{1/\alpha}\right) \qquad , \ i = 1,\dots,n$$

where $u_1,\dots,u_n$ are random sample from uniform(0,1).

For each sample the two parameters α and λ were estimated under three cases, which are doubly, left and right PPWM. In the three cases, the solutions of equations (3.7), (3.12) and (3.17) require some iterative technique.

Simulation results are summarized in tables (1) and (2) for double censoring; tables (3) and (4) for left censoring and tables (5) and (6) for right censoring. These tables give the biases and MSEs of the parameters estimators. They are included at the appendix. From these tables, the following observations can be made on the properties of PPWM estimation from doubly, left and right censored samples for the parameters of the generalized exponential distribution:

(1) For the different levels of censoring and for the same value of the shape parameter α, the biases and MSEs of all the PPWM parameter estimators decrease as the sample size increases. This indicates that the method of PPWM provides consistent estimators for α and λ from doubly, left and right censored samples. This can be seen clearly in figures (1)-(4) for double censoring, figures (5)-(8) for left censoring, and figures (9)-(12) for right censoring. All these figures are included in the appendix.

(2) It is noticed that the censoring level has a significant effect on the performance of PPWM estimation in terms of bias and MSE. In other words, the biases and MSEs increase as the censoring level increase whether it was double, left or right. In more details:

   (i) For doubly censored samples, it can be noted that for the same value of α within the same sample size, the double censoring level: 0.1-0.9 gives the smallest bias and MSE for the estimators $\hat{\alpha}_1$ and $\hat{\lambda}_1$, while the level: 0.3-0.8 gives the largest bias and MSE. This happens in most of the cases. However, there are some cases in which the double censoring level: 0.3-0.9 produces the largest MSE. For example, this happens when $\alpha = 0.5$ and $n = 15$ and 30 for the estimators of α [see figure (1)]. In addition, there are a few cases, in which the double censoring level: 0.3-0.9 or 0.2-0.9 gives the smallest bias while the level 0.2-0.8 gives the largest one. This happens, for example, when $\alpha = 2.5$ and $n = 20$ for both the estimators of α and λ. Also, it happens when $\alpha = 2$ and $n = 30$ for the estimators of α [see table (1)].

(ii) For left censored samples, the biases and MSEs increase as the left censoring level increase. See figures (5)-(8) which displays the MSE against the different left censoring levels. In other words, the smallest left censoring level, $c = 0.1$, gives the smallest bias and MSE for the PPWM estimators $\hat{\alpha}_2$ and $\hat{\lambda}_2$; while the largest left censoring level gives the largest bias and MSE. However, there are some cases for $\hat{\alpha}_2$ and $\hat{\lambda}_2$ in which the left censoring level $c = 0.3$ or $c = 0.4$ give a bias that is slightly smaller than the bias given by the censoring level $c = 0.1$.

(iii) For right censored samples, it can be noted that the biases and MSEs for both $\hat{\alpha}_3$ and $\hat{\lambda}_3$ increase as the right censoring level increase (considering $d = 0.9$ is the lowest right censoring level and $d = 0.6$ is the highest one) for the same value of α within the same sample size. In other words, the right censoring level $d = 0.9$ gives the smallest bias and MSE, while $d = 0.6$ gives the largest bias and MSE in all the cases. See figures (9)-(12) which displays the MSEs against the different right censoring levels

(3) The results of PPWM estimation from doubly censored samples can be considered good and reasonable except for large values of α ($\alpha = 2$ and 2.5) where the biases and MSEs of $\hat{\alpha}_1$ and $\hat{\lambda}_1$ are large and they do not improve very much as the sample size increases. In addition, the biases of $\hat{\alpha}_1$ are smaller than those of $\hat{\lambda}_1$ when $\alpha = 0.5$ and 1. The opposite is true when $\alpha = 1.5$, 2 and 2.5. On the other hand, the MSEs of $\hat{\lambda}_1$ are much less than those of $\hat{\alpha}_1$ except when $\alpha = 0.5$, where the opposite is true.

(4) In the case of left censoring, it is noticed that for all values of α (except when $\alpha = 0.5$) the MSEs of $\hat{\alpha}_2$ are very large especially for high censoring levels $c = 0.3$ and 0.4. However, the MSEs of $\hat{\lambda}_2$ are very good and there are no big differences between the MSEs under different censoring levels within the same sample size.

(5) In the case of right censoring, it can be noted that when $\alpha = 2$ and 2.5, the MSEs of $\hat{\alpha}_3$ are large especially when $d = 0.6$ and 0.7. In addition, the MSEs of $\hat{\lambda}_3$ are much less than those of the estimators of α when $\alpha = 1.5$, 2 and 2.5. The opposite is true when $\alpha = 0.5$ and 1.

5. Sampling Distributions of the PPWM Estimators

In this section, the sampling distributions of the PPWM parameters estimators from doubly, left and right censored samples are obtained. This system was originated by Karl Pearson (1895). The Pearson system embeds seven basic types of distribution together in a single parametric framework. The selection approach is based on computing a certain quantity, *K*, which is a function of the first four central moments, that is:

$$K = \frac{\beta_1(\beta_2 + 3)^2}{4(4\beta_2 - 3\beta_1)(2\beta_2 - 3\beta_1 - 6)}, \tag{5.1}$$

where the two moment ratios $\beta_1 = \mu_3^2 / \mu_2^3$ and $\beta_2 = \mu_4 / \mu_2^2$ denote the skewness and kurtosis measures, respectively. μ_r is the *r*th central moment.

The implementation of the Pearson system approach could be summarized in the following steps:

Step (1): Calculate the first four central moments from the resulting 10,000 PPWM estimates for each case in doubly, left and right censored samples.

Step (2): Use the moment estimates to compute β_1 and β_2.

Step (3): Use β_1 and β_2 to calculate *K*.

Step (4): Select an appropriate distribution from the Pearson family according to the values of β_1, β_2 and *K*.

As a result of computer simulation; four Pearson distributions were fitted to the PPWM estimators which are Pearson type I, III, IV and VI distributions. Pearson type I distribution has the probability density function:

$$p(x) = k(x - a_1)^{m_1}(a_2 - x)^{m_2}, \quad a_1 < x < a_2 \tag{5.2}$$

where a_1 and a_2 are the roots of the equation $c_0 + c_1 x + c_2 x^2 = 0$ with $m_1 = \frac{a + a_1}{c_2(a_2 - a_1)}$ and $m_2 = -\frac{a + a_2}{c_2(a_2 - a_1)}$. While Pearson type III has the density function:

$$p(x) = k(c_0 + c_1 x)^m \exp(-x/c_1), \tag{5.3}$$

where $m = \frac{1}{c_1}(\frac{c_0}{c_1} - a)$. If $c_1 > 0$ then the range of x will be $x > -c_0/c_1$ but if $c_1 < 0$, then the range is taken to be $x < -c_0/c_1$.

In addition, the density function of Pearson type IV is given by:

$$p(x) = k[C_0 + c_2(x + C_1)^2]^{-(2c_2)^{-1}} \exp(-\frac{a - C_1}{\sqrt{c_2 C_0}} \tan^{-1} \frac{x + C_1}{\sqrt{C_0/c_2}}), \quad -\infty < x < \infty \tag{5.4}$$

where $C_0 = c_0 - \frac{1}{4} c_1^2 c_2^{-1}$ and $C_1 = \frac{1}{2} c_1 c_2^{-1}$. Whereas, the density function of Pearson type VI is:

$$p(x) = k(x - a_1)^{m_1} (x - a_2)^{m_2}. \quad x > a_2 \tag{5.5}$$

The sampling distributions of the PPWM estimators for each value of α under each level of censoring and for different sample sizes are listed in tables (7), (8) and (9). In addition, the probability density functions of these distributions are provided for some cases in figures (13)-(16). From tables (7), (8) and (9) it is noticed that:

(1) In the case of doubly censored samples, most of the PPWM estimators of α and λ follow Pearson type I distribution. However, there are many cases in which the estimators follow Pearson type VI distribution, especially, for the estimators of λ. Aslo, there are a few cases in which the estimators of λ follow Pearson type IV or III [see table (7)].

(2) In the case of left censored samples, most of the estimators of α follow Pearson type I or type VI distribution, a few cases follow Pearson type IV or type III. On the other hand, most of the estimators of λ follow Pearson type IV or type VI, while a few cases follow Pearson type I distribution [see table (8)].

(3) In the case of right censored samples, most of the PPWM estimators of α and λ follow Pearson type I distribution, a few cases follow Pearson type IV, type VI or type III distribution [see table (9)].

6. Conclusions

The method of PPWM has provided consistent estimators for the unknown parameters of the generalized exponential distribution from doubly, left and right censored samples. Also, it is clear that the censoring level has a significant effect on the performance of the PPWM estimation. In other words, the biases and MSEs increase as the censoring increases whether it was double, left or right. Moreover, the effect of right censoring on the MSEs of the estimators of the shape parameter α is much less than the effect of left censoring in most of the cases, especially for large values of α (α = 1.5, 2, 2.5). On the other hand, the MSEs of the estimators of λ in the case of left censoring are less than those in the case of right censoring for all the cases.

Regarding the sampling distributions of the PPWM parameter estimators; it can be concluded that most of the PPWM parameter estimators follow Pearson type I, VI or IV distribution except for a few cases in which the PPWM parameter estimators follow Pearson type III distribution especially in the cases of double and left censoring when α =0.5 and 1.

Appendix

Table (1): Biases of PPWM Parameter Estimators From Doubly Censored Samples

n	Left censoring level	Right censoring level	$\hat{\alpha}_1$					$\hat{\lambda}_1$				
	c	d	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.1	0.9	-0.085	-0.224	-0.715	-1.223	-1.671	-0.141	-0.266	-0.476	-0.586	-0.616
	0.2	0.9	-0.093	-0.263	-0.795	-1.250	-1.769	-0.185	-0.359	-0.539	-0.602	-0.654
	0.3	0.9	-0.115	-0.321	-0.856	-1.276	-1.842	-0.269	-0.424	-0.606	-0.634	-0.690
	0.1	0.8	-0.198	-0.497	-0.982	-1.546	-2.017	-0.378	-0.548	-0.665	-0.764	-0.783
	0.2	0.8	-0.221	-0.547	-1.042	-1.567	-2.020	-0.446	-0.583	-0.712	-0.769	-0.780
	0.3	0.8	-0.263	-0.623	-1.129	-1.640	-2.176	-0.552	-0.670	-0.760	-0.812	-0.842
20	0.1	0.9	-0.044	-0.142	-0.617	-1.093	-1.666	-0.072	-0.219	-0.415	-0.515	-0.622
	0.2	0.9	-0.055	-0.181	-0.647	-1.183	-1.643	-0.127	-0.279	-0.453	-0.568	-0.618
	0.3	0.9	-0.080	-0.251	-0.781	-1.226	-1.631	-0.209	-0.374	-0.534	-0.609	-0.614
	0.1	0.8	-0.158	-0.423	-0.974	-1.448	-2.050	-0.311	-0.423	-0.648	-0.702	-0.802
	0.2	0.8	-0.186	-0.482	-0.997	-1.479	-2.078	-0.380	-0.453	-0.672	-0.715	-0.818
	0.3	0.8	-0.227	-0.573	-1.106	-1.559	-2.048	-0.482	-0.630	-0.749	-0.756	-0.791
30	0.1	0.9	0.002	-0.049	-0.528	-0.986	-1.519	0.004	-0.106	-0.363	-0.467	-0.548
	0.2	0.9	-0.003	-0.062	-0.576	-0.984	-1.494	-0.036	-0.165	-0.413	-0.481	-0.537
	0.3	0.9	-0.015	-0.142	-0.583	-1.094	-1.557	-0.093	-0.278	-0.450	-0.544	-0.562
	0.1	0.8	-0.104	-0.343	-0.868	-1.376	-1.930	-0.198	-0.375	-0.585	-0.677	-0.739
	0.2	0.8	-0.127	-0.394	-0.926	-1.477	-1.943	-0.265	-0.445	-0.627	-0.723	-0.737
	0.3	0.8	-0.178	-0.491	-1.011	-1.466	-2.009	-0.391	-0.552	-0.682	-0.719	-0.770
50	0.1	0.9	0.025	0.036	-0.349	-0.808	-1.379	0.041	-0.025	-0.244	-0.386	-0.502
	0.2	0.9	0.041	0.037	-0.380	-0.811	-1.298	0.054	-0.071	-0.288	-0.407	-0.474
	0.3	0.9	0.047	0.021	-0.339	-0.926	-1.359	0.016	-0.158	-0.327	-0.469	-0.515
	0.1	0.8	-0.045	-0.217	-0.756	-1.311	-1.882	-0.080	-0.253	-0.502	-0.634	-0.720
	0.2	0.8	-0.066	-0.273	-0.826	-1.336	-1.862	-0.140	-0.330	-0.552	-0.651	-0.705
	0.3	0.8	-0.109	-0.387	-0.914	-1.449	-1.907	-0.262	-0.464	-0.628	-0.709	-0.721
100	0.1	0.9	0.029	0.082	-0.096	-0.550	-1.190	0.055	0.035	-0.095	-0.275	-0.435
	0.2	0.9	-0.041	0.131	-0.095	-0.541	-1.126	0.060	0.029	-0.127	-0.283	-0.400
	0.3	0.9	0.064	0.169	-0.168	-0.458	-1.039	0.065	-0.022	-0.211	-0.284	-0.406
	0.1	0.8	0.006	-0.055	-0.584	-1.148	-1.790	0.028	-0.092	-0.390	-0.537	-0.680
	0.2	0.8	-0.002	-0.112	-0.654	-1.105	-1.725	-0.007	-0.174	-0.441	-0.558	-0.637
	0.3	0.8	-0.026	-0.201	-0.713	-1.231	-1.745	-0.086	-0.295	-0.504	-0.599	-0.658

Table (2): MSEs of PPWM Parameter Estimators From Doubly Censored Samples

n	Left censoring level	Right censoring level	$\hat{\alpha}_1$					$\hat{\lambda}_1$				
	c	d	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$	$\alpha = 2$	$\alpha = 2.5$
15	0.1	0.9	0.128	0.547	1.145	2.204	3.541	0.789	0.492	0.477	0.511	0.519
	0.2	0.9	0.154	0.759	1.380	2.459	3.949	0.880	0.580	0.536	0.554	0.563
	0.3	0.9	0.201	1.094	1.805	2.827	4.482	0.847	0.660	0.609	0.598	0.626
	0.1	0.8	0.131	0.562	1.346	2.722	4.472	0.917	0.616	0.619	0.683	0.704
	0.2	0.8	0.150	0.658	1.501	2.837	4.540	0.921	0.719	0.678	0.710	0.709
	0.3	0.8	0.170	0.772	1.671	3.076	5.043	1.012	0.794	0.758	0.774	0.812
20	0.1	0.9	0.116	0.516	1.019	1.955	3.608	0.664	0.415	0.415	0.450	0.537
	0.2	0.9	0.146	0.745	1.324	2.307	3.807	0.677	0.495	0.489	0.500	0.546
	0.3	0.9	0.199	1.080	1.561	2.745	4.075	0.760	0.578	0.550	0.565	0.559
	0.1	0.8	0.122	0.510	1.296	2.544	4.573	0.791	0.549	0.596	0.636	0.717
	0.2	0.8	0.140	0.624	1.492	2.649	4.724	0.848	0.628	0.646	0.653	0.743
	0.3	0.8	0.164	0.767	1.634	2.917	4.629	0.938	0.738	0.729	0.752	0.424
30	0.1	0.9	0.095	0.440	0.952	1.777	3.203	0.473	0.316	0.370	0.408	0.471
	0.2	0.9	0.130	0.700	1.219	2.070	3.337	0.527	0.394	0.419	0.437	0.464
	0.3	0.9	0.189	1.047	1.753	2.608	3.791	0.642	0.490	0.493	0.513	0.510
	0.1	0.8	0.106	0.458	1.196	2.408	4.223	0.683	0.483	0.544	0.601	0.652
	0.2	0.8	0.125	0.584	1.331	2.630	4.304	0.732	0.568	0.597	0.651	0.658
	0.3	0.8	0.155	0.730	1.541	2.744	4.500	0.802	0.668	0.686	0.670	0.687
50	0.1	0.9	0.068	0.350	0.697	1.569	2.891	0.301	0.221	0.248	0.359	0.428
	0.2	0.9	0.101	0.607	0.992	1.882	2.951	0.384	0.295	0.304	0.385	0.409
	0.3	0.9	0.170	1.112	1.638	2.450	3.608	0.481	0.397	0.371	0.449	0.454
	0.1	0.8	0.084	0.385	1.009	2.248	4.072	0.535	0.384	0.449	0.552	0.627
	0.2	0.8	0.107	0.527	1.165	2.358	4.066	0.615	0.476	0.507	0.570	0.616
	0.3	0.8	0.145	0.710	1.417	2.649	4.276	0.704	0.589	0.593	0.638	0.640
100	0.1	0.9	0.036	0.218	0.519	1.131	2.536	0.161	0.124	0.159	0.253	0.379
	0.2	0.9	0.056	0.467	0.867	1.381	2.524	0.200	0.189	0.212	0.271	0.343
	0.3	0.9	0.114	0.997	1.401	2.071	3.132	0.299	0.280	0.278	0.297	0.376
	0.1	0.8	0.054	0.282	0.816	1.959	3.879	0.354	0.257	0.352	0.469	0.598
	0.2	0.8	0.078	0.443	0.983	1.966	3.672	0.441	0.353	0.408	0.480	0.551
	0.3	0.8	0.122	0.669	1.233	2.314	3.872	0.567	0.464	0.471	0.534	0.574

Table (3): Biases of PPWM Parameter Estimators from Left Censored Samples

n	Left censoring level	$\hat{\alpha}_2$					$\hat{\lambda}_2$				
	c	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.1	0.042	0.128	0.116	-0.087	-0.512	0.143	0.074	0.004	-0.090	-0.199
	0.2	0.061	0.215	0.190	-0.111	-0.549	0.154	0.058	-0.044	-0.149	-0.243
	0.3	0.102	0.344	0.194	-0.111	-0.517	0.146	0.018	-0.119	-0.213	-0.283
	0.4	0.166	0.498	0.234	-0.085	-0.587	0.104	-0.059	-0.208	-0.286	-0.355
20	0.1	0.028	0.093	0.107	-0.029	-0.334	0.098	0.056	0.005	-0.059	-0.142
	0.2	0.043	0.177	0.195	0.009	-0.358	0.104	0.053	-0.020	-0.104	-0.185
	0.3	0.079	0.313	0.318	0.065	-0.346	0.118	0.029	-0.070	-0.162	-0.234
	0.4	0.135	0.494	0.388	0.098	-0.361	0.090	-0.026	-0.145	-0.230	-0.297
30	0.1	0.018	0.051	0.064	0.023	-0.183	0.062	0.026	-0.002	-0.035	-0.093
	0.2	0.024	0.109	0.157	0.099	-0.150	0.063	0.029	-0.013	-0.061	-0.125
	0.3	0.049	0.234	0.313	0.213	-0.076	0.071	0.023	-0.037	-0.102	-0.168
	0.4	0.099	0.439	0.445	0.251	-0.095	0.068	-0.003	-0.091	-0.168	-0.230
50	0.1	0.006	0.016	0.022	0.003	-0.104	0.028	0.001	-0.011	-0.027	-0.060
	0.2	0.010	0.045	0.110	0.084	0.256	0.028	0.003	-0.007	-0.037	-0.031
	0.3	0.016	0.119	0.232	0.237	0.094	0.021	0.006	-0.020	-0.056	-0.102
	0.4	0.049	0.276	0.458	0.407	0.202	0.032	0.003	-0.039	-0.096	-0.148
100	0.1	-0.003	-0.008	-0.020	-0.045	-0.093	-0.0004	-0.012	-0.023	-0.032	-0.043
	0.2	-0.0001	0.005	-0.005	0.011	-0.025	0.003	-0.013	-0.028	-0.034	-0.048
	0.3	0.004	0.031	0.075	0.117	0.099	0.003	-0.013	-0.025	-0.039	-0.058
	0.4	0.013	0.097	0.224	0.310	0.261	0.004	-0.013	-0.032	-0.053	-0.083

Table (4): MSEs of PPWM Parameter Estimators from Left Censored Samples

n	Left censoring level	$\hat{\alpha}_2$					$\hat{\lambda}_2$				
	c	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.1	0.078	0.461	0.924	1.350	2.032	0.411	0.216	0.157	0.158	0.208
	0.2	0.122	1.020	2.009	2.554	3.232	0.453	0.241	0.194	0.200	0.239
	0.3	0.259	2.438	3.640	4.529	5.330	0.475	0.275	0.233	0.242	0.267
	0.4	0.610	5.408	6.433	7.741	8.180	0.478	0.304	0.273	0.287	0.314
20	0.1	0.050	0.304	0.702	1.120	1.685	0.255	0.148	0.113	0.117	0.153
	0.2	0.077	0.747	1.623	2.253	2.923	0.275	0.173	0.139	0.153	0.185
	0.3	0.172	1.983	3.533	4.493	5.112	0.323	0.203	0.182	0.197	0.220
	0.4	0.433	4.820	6.568	7.920	8.325	0.336	0.242	0.228	0.243	0.262
30	0.1	0.029	0.162	0.448	0.823	1.272	0.148	0.085	0.069	0.071	0.097
	0.2	0.042	0.377	1.087	1.839	2.475	0.157	0.102	0.090	0.098	0.125
	0.3	0.084	1.219	2.776	4.057	4.900	0.182	0.130	0.124	0.136	0.161
	0.4	0.260	3.549	5.677	7.371	8.174	0.211	0.168	0.160	0.180	0.203
50	0.1	0.016	0.082	0.238	0.479	0.820	0.077	0.047	0.039	0.038	0.050
	0.2	0.021	0.164	0.611	1.158	3.399	0.080	0.056	0.050	0.054	0.047
	0.3	0.034	0.498	1.650	2.929	3.898	0.088	0.073	0.072	0.078	0.096
	0.4	0.094	1.684	4.433	6.441	7.868	0.108	0.100	0.104	0.114	0.132
100	0.1	0.007	0.034	0.101	0.223	0.402	0.035	0.022	0.019	0.018	0.017
	0.2	0.009	0.063	0.208	0.551	0.982	0.036	0.026	0.024	0.024	0.027
	0.3	0.014	0.138	0.588	1.399	2.346	0.040	0.033	0.033	0.036	0.041
	0.4	0.026	0.426	1.893	3.748	5.268	0.047	0.047	0.050	0.055	0.065

Table (5): Biases of PPWM Parameter Estimators from Right Censored Samples

n	Right censoring level	$\hat{\alpha}_3$					$\hat{\lambda}_3$				
	d	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.9	-0.082	-0.236	-0.348	-0.581	-0.985	-0.129	-0.251	-0.196	-0.323	-0.393
	0.8	-0.195	-0.474	-0.588	-1.073	-1.577	-0.372	-0.486	-0.429	-0.550	-0.622
	0.7	-0.270	-0.618	-0.849	-1.355	-1.868	-0.540	-0.649	-0.604	-0.697	-0.755
	0.6	-0.313	-0.712	-1.019	-1.533	-2.055	-0.654	-0.739	-0.724	-0.799	-0.842
20	0.9	-0.046	-0.170	-0.228	-0.435	-0.846	-0.077	-0.227	-0.135	-0.256	-0.344
	0.8	-0.159	-0.408	-0.473	-0.971	-1.482	-0.302	-0.466	-0.362	-0.500	-0.585
	0.7	-0.240	-0.565	-0.778	-1.271	-1.814	-0.488	-0.593	-0.556	-0.655	-0.731
	0.6	-0.290	-0.668	-0.968	-1.474	-1.995	-0.608	-0.732	-0.692	-0.769	-0.819
30	0.9	-0.010	-0.065	-0.076	-0.206	-0.670	-0.017	-0.144	-0.044	-0.163	-0.281
	0.8	-0.101	-0.315	-0.332	-0.789	-1.327	-0.185	-0.313	-0.270	-0.413	-0.523
	0.7	-0.189	-0.492	-0.649	-1.149	-1.693	-0.375	-0.493	-0.471	-0.593	-0.677
	0.6	-0.249	-0.613	-0.844	-1.370	-1.907	-0.527	-0.707	-0.608	-0.715	-0.781
50	0.9	0.024	0.010	0.044	-0.024	-0.335	0.049	0.024	0.023	-0.073	-0.165
	0.8	-0.045	-0.200	-0.148	-0.575	-1.092	-0.074	-0.135	-0.154	-0.310	-0.433
	0.7	-0.130	-0.397	-0.448	-0.968	-1.550	-0.256	-0.403	-0.337	-0.501	-0.618
	0.6	-0.197	-0.534	-0.709	-1.259	-1.811	-0.417	-0.591	-0.516	-0.658	-0.737
100	0.9	0.023	0.051	0.135	0.126	-0.055	0.045	0.057	0.035	0.006	-0.062
	0.8	0.007	-0.067	0.057	-0.288	-0.767	0.032	-0.024	-0.018	-0.173	-0.308
	0.7	-0.052	-0.255	-0.215	-0.738	-1.319	-0.081	-0.304	-0.181	-0.384	-0.523
	0.6	-0.124	-0.419	-0.493	-1.059	-1.653	-0.248	-0.492	-0.365	-0.551	-0.669

Table (6): MSEs of PPWM Parameter Estimators from Right Censored Samples

n	Right censoring level	$\hat{\alpha}_3$					$\hat{\lambda}_3$				
	d	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.9	0.118	0.442	1.098	1.988	2.892	0.783	0.432	0.405	0.378	0.386
	0.8	0.128	0.505	1.259	2.192	3.605	0.878	0.531	0.542	0.538	0.565
	0.7	0.141	0.585	1.345	2.512	4.184	0.934	0.671	0.648	0.664	0.689
	0.6	0.149	0.650	1.475	2.784	4.657	1.007	0.792	0.734	0.752	0.784
20	0.9	0.103	0.394	1.041	1.854	2.648	0.592	0.360	0.343	0.328	0.343
	0.8	0.117	0.453	1.191	2.050	3.390	0.786	0.554	0.486	0.494	0.525
	0.7	0.130	0.536	1.272	2.347	4.028	0.851	0.598	0.611	0.615	0.659
	0.6	0.140	0.604	1.392	2.659	4.470	0.948	0.715	0.698	0.719	0.753
30	0.9	0.084	0.324	0.884	1.613	2.317	0.434	0.273	0.271	0.249	0.288
	0.8	0.098	0.395	1.084	1.804	3.091	0.656	0.337	0.416	0.419	0.474
	0.7	0.112	0.476	1.126	2.139	3.704	0.763	0.555	0.528	0.554	0.601
	0.6	0.123	0.549	1.239	2.434	4.213	0.839	0.703	0.622	0.656	0.706
50	0.9	0.062	0.239	0.672	1.273	1.842	0.296	0.208	0.176	0.171	0.201
	0.8	0.076	0.316	0.896	1.497	2.608	0.499	0.326	0.312	0.332	0.392
	0.7	0.090	0.403	0.950	1.852	3.380	0.642	0.435	0.430	0.471	0.544
	0.6	0.103	0.481	1.081	2.224	3.943	0.748	0.560	0.540	0.595	0.656
100	0.9	0.030	0.134	0.440	0.801	1.227	0.142	0.089	0.084	0.088	0.113
	0.8	0.048	0.210	0.642	1.076	1.959	0.331	0.198	0.194	0.219	0.285
	0.7	0.061	0.292	0.726	1.470	2.844	0.487	0.353	0.311	0.366	0.453
	0.6	0.074	0.382	0.839	1.860	3.533	0.611	0.522	0.413	0.491	0.582

Table (7): Sampling Distribution of the PPWM Estimators from Doubly Censored Samples

n	Left censoring level	Right censoring level	$\hat{\alpha}_1$					$\hat{\lambda}_1$				
	c	d	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.1	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**I**	**I**
	0.2	0.9	**I**	**I**	**I**	**I**	**I**	**IV**	**VI**	**I**	**I**	**I**
	0.3	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**VI**	**I**	**III**	**I**
	0.1	0.8	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**VI**	**I**
	0.2	0.8	**I**	**I**	**I**	**I**	**I**	**III**	**I**	**VI**	**VI**	**VI**
	0.3	0.8	**I**	**I**	**I**	**I**	**I**	**VI**	**III**	**I**	**III**	**III**
20	0.1	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**I**	**I**
	0.2	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**I**	**I**
	0.3	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**VI**	**VI**	**VI**	**I**
	0.1	0.8	**I**	**I**	**I**	**I**	**I**	**III**	**I**	**I**	**I**	**I**
	0.2	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.3	0.8	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**I**	**VI**
30	0.1	0.9	**I**	**I**	**I**	**I**	**I**	**IV**	**I**	**I**	**I**	**I**
	0.2	0.9	**I**	**I**	**I**	**I**	**I**	**III**	**I**	**I**	**I**	**I**
	0.3	0.9	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.1	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.2	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.3	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
50	0.1	0.9	**I**	**VI**	**I**	**I**	**I**	**I**	**I**	**IV**	**IV**	**I**
	0.2	0.9	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.3	0.9	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.1	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.2	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.3	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
100	0.1	0.9	**VI**	**VI**	**I**	**I**	**I**	**VI**	**VI**	**IV**	**IV**	**IV**
	0.2	0.9	**VI**	**I**	**I**	**I**	**I**	**VI**	**VI**	**IV**	**IV**	**I**
	0.3	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**I**	**I**
	0.1	0.8	**III**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.2	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.3	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**

Table (8): Sampling Distribution of the PPWM Estimators from Left Censored Samples

n	Left censoring level	$\hat{\alpha}_2$					$\hat{\lambda}_2$				
	c	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.1	**VI**	**III**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**I**
	0.2	**III**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**I**
	0.3	**I**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**I**	**I**
	0.4	**I**	**I**	**I**	**I**	**I**	**VI**	**VI**	**VI**	**I**	**I**
20	0.1	**VI**	**VI**	**I**	**I**	**I**	**IV**	**IV**	**IV**	**IV**	**I**
	0.2	**VI**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**I**
	0.3	**III**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**I**	**I**
	0.4	**I**	**I**	**I**	**I**	**IV**	**VI**	**IV**	**IV**	**I**	**I**
30	0.1	**IV**	**VI**	**III**	**I**	**VI**	**IV**	**IV**	**IV**	**IV**	**IV**
	0.2	**VI**	**VI**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**VI**
	0.3	**VI**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**VI**
	0.4	**III**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**I**
50	0.1	**IV**	**IV**	**VI**	**VI**	**IV**	**VI**	**VI**	**VI**	**IV**	**IV**
	0.2	**IV**	**VI**	**VI**	**I**	**I**	**IV**	**VI**	**IV**	**IV**	**IV**
	0.3	**IV**	**VI**	**I**	**I**	**I**	**IV**	**VI**	**IV**	**IV**	**IV**
	0.4	**VI**	**I**	**I**	**I**	**I**	**VI**	**IV**	**IV**	**IV**	**IV**
100	0.1	**VI**	**VI**	**IV**	**VI**	**VI**	**VI**	**VI**	**IV**	**IV**	**VI**
	0.2	**VI**	**IV**	**VI**	**VI**	**VI**	**VI**	**IV**	**IV**	**IV**	**IV**
	0.3	**IV**	**VI**	**VI**	**VI**	**I**	**VI**	**IV**	**VI**	**IV**	**IV**
	0.4	**VI**	**VI**	**III**	**I**	**I**	**VI**	**VI**	**IV**	**IV**	**IV**

Table (9): Sampling Distribution of the PPWM Estimators from Right Censored Samples

n	Right censoring level	$\hat{\alpha}_3$					$\hat{\lambda}_3$				
	d	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5	α = 0.5	α = 1	α = 1.5	α = 2	α = 2.5
15	0.9	**I**	**I**	**I**	**I**	**I**	**IV**	**VI**	**I**	**I**	**I**
	0.8	**I**	**I**	**I**	**I**	**I**	**VI**	**III**	**I**	**I**	**I**
	0.7	**I**	**I**	**I**	**I**	**I**	**VI**	**VI**	**I**	**I**	**I**
	0.6	**I**	**I**	**I**	**I**	**I**	**VI**	**I**	**I**	**I**	**I**
20	0.9	**I**	**I**	**I**	**I**	**I**	**IV**	**III**	**I**	**I**	**I**
	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.7	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.6	**I**	**I**	**I**	**I**	**I**	**IV**	**I**	**I**	**I**	**I**
30	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**VI**	**I**	**I**	**I**
	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.7	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.6	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
50	0.9	**I**	**I**	**I**	**I**	**I**	**VI**	**VI**	**IV**	**IV**	**I**
	0.8	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.7	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.6	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
100	0.9	**VI**	**VI**	**VI**	**IV**	**IV**	**VI**	**VI**	**IV**	**IV**	**VI**
	0.8	**IV**	**I**	**I**	**I**	**I**	**I**	**I**	**IV**	**I**	**I**
	0.7	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**
	0.6	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**	**I**

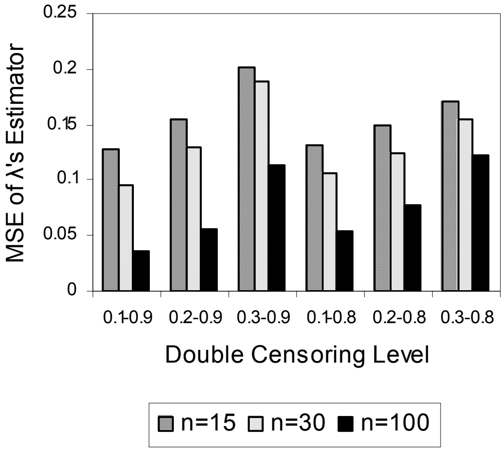

Figure (1): MSE of $\hat{\alpha}_1$ versus the different double censoring levels for $\alpha = 0.5$

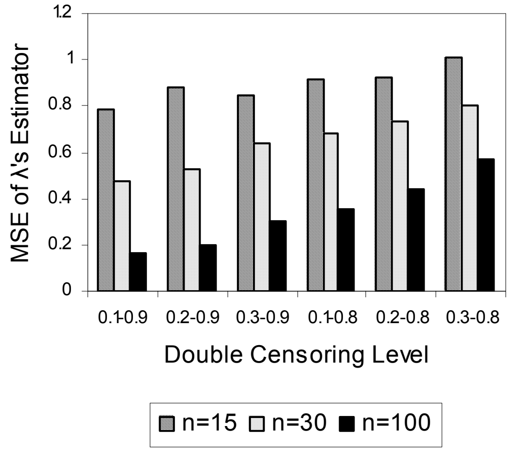

Figure (2): MSE of $\hat{\lambda}_1$ versus the different double censoring levels for $\alpha = 0.5$

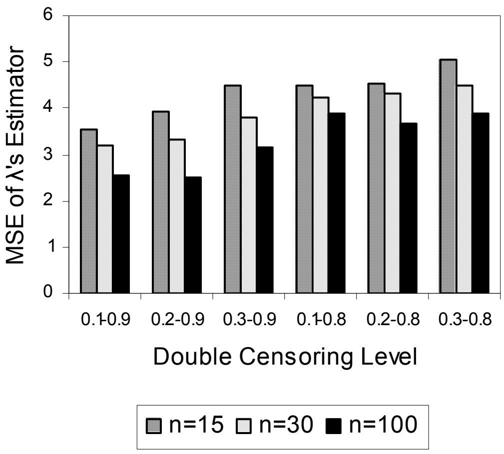

Figure (3): MSE of $\hat{\alpha}_1$ versus the different double censoring levels for $\alpha = 2.5$

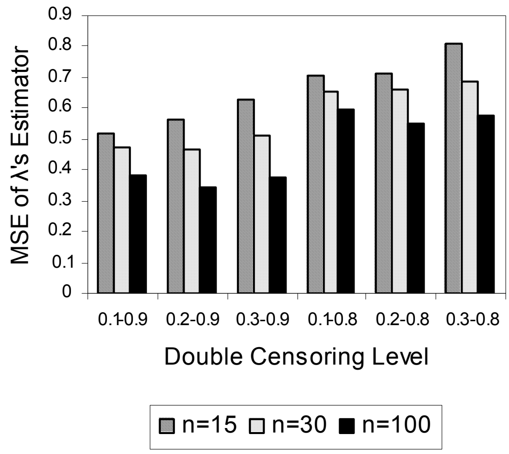

Figure (4): MSE of $\hat{\lambda}_1$ versus the different double censoring levels for $\alpha = 2.5$

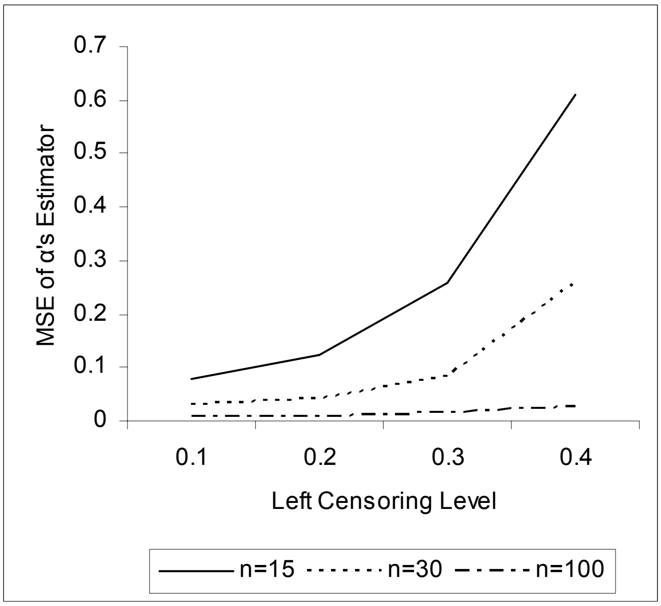

Figure (5): MSE of $\hat{\alpha}_2$ versus the different left censoring levels for $\alpha = 0.5$

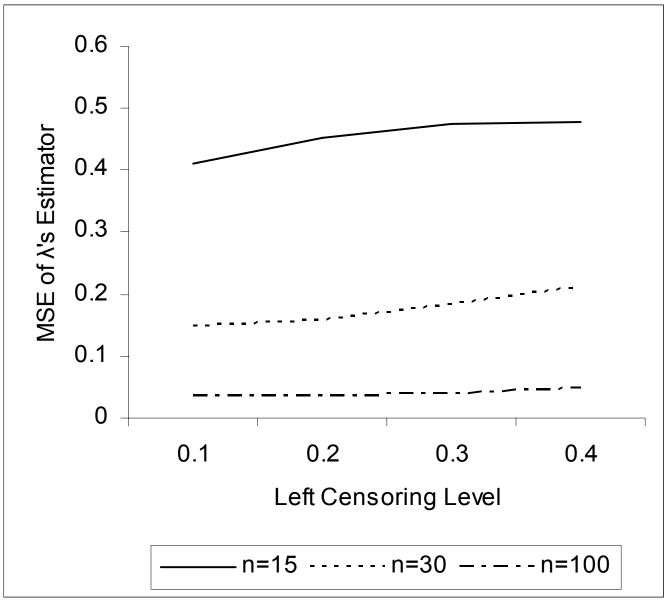

Figure (6): MSE of $\hat{\lambda}_2$ versus the different left censoring levels for $\alpha = 0.5$

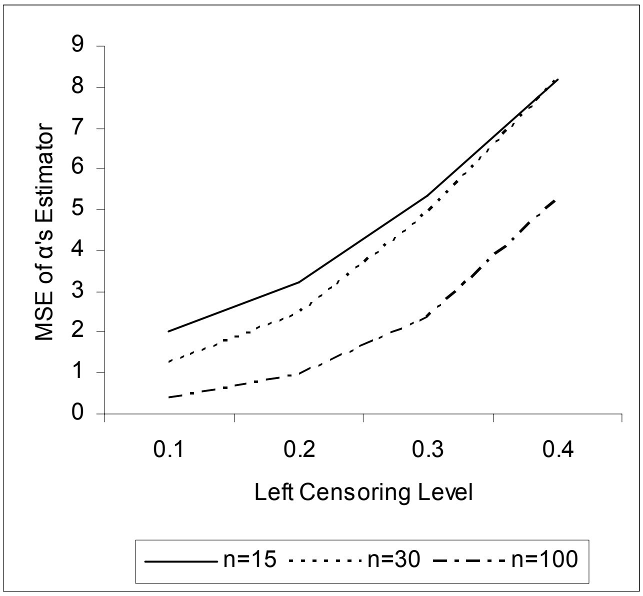

Figure (7): MSE of $\hat{\alpha}_2$ versus the different left censoring levels for $\alpha = 2.5$

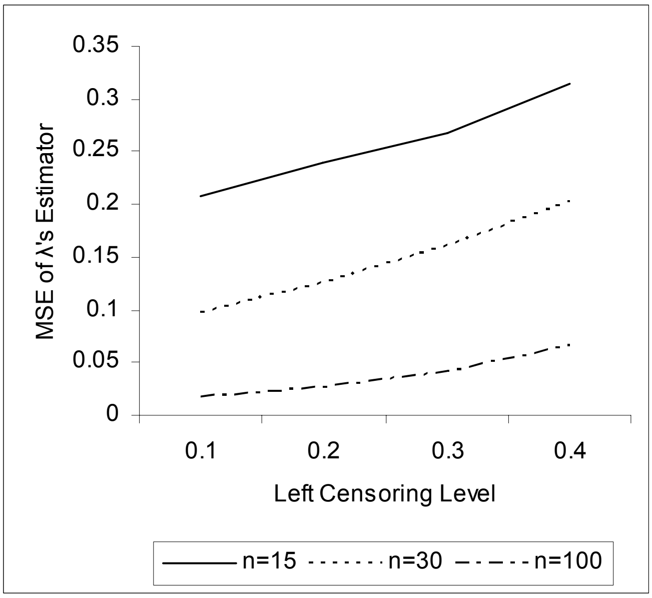

Figure (8): MSE of $\hat{\lambda}_2$ versus the different left censoring levels for $\alpha = 2.5$

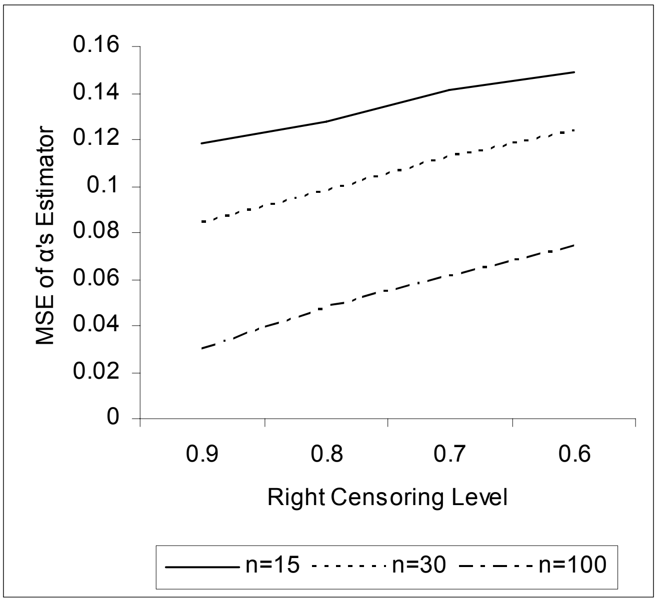

Figure (9): MSE of $\hat{\alpha}_3$ versus the different right censoring levels for $\alpha = 0.5$

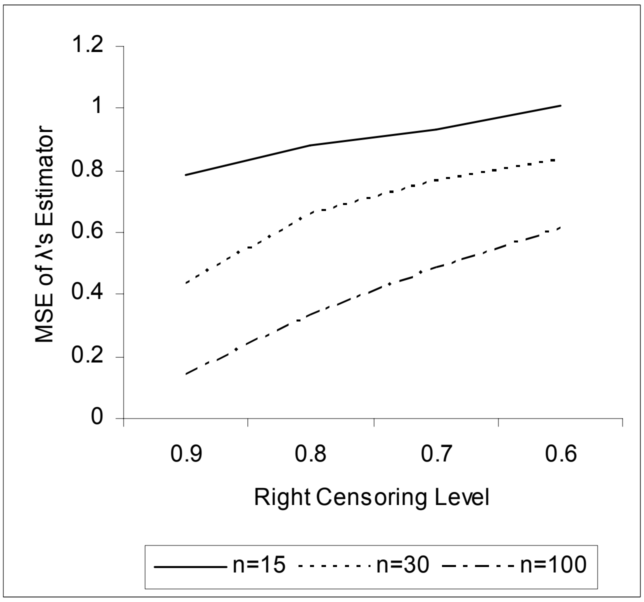

Figure (10): MSE of $\hat{\lambda}_3$ versus the different right censoring levels for $\alpha = 0.5$

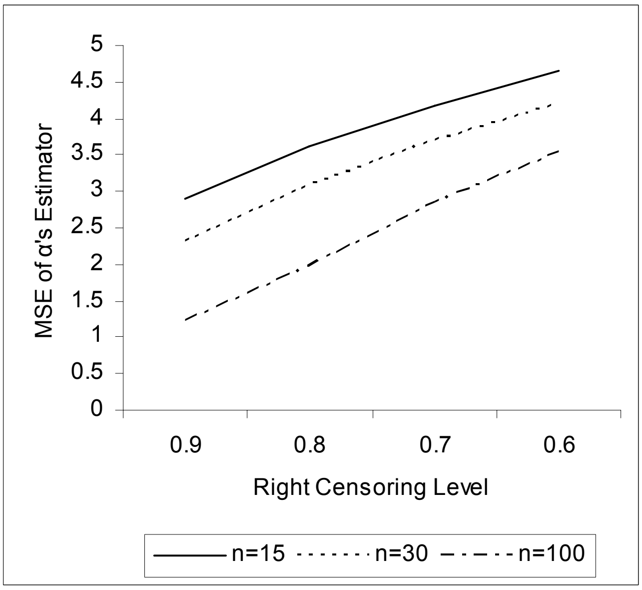

Figure (11): MSE of $\hat{\alpha}_3$ versus the different right censoring levels for $\alpha = 2.5$

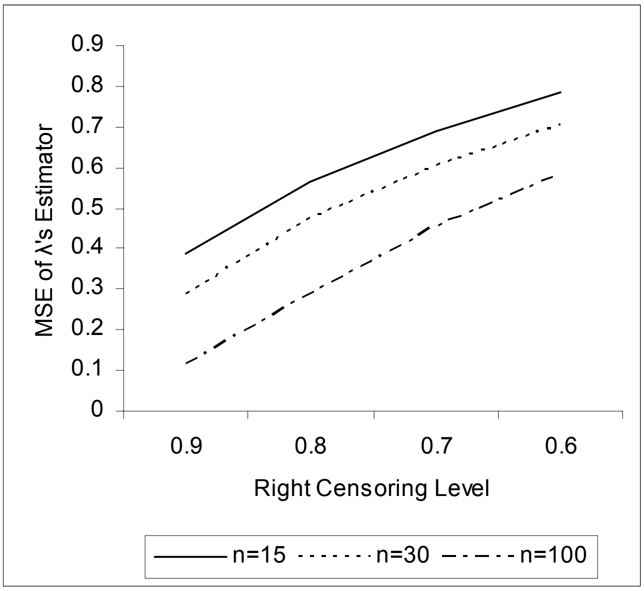

Figure (12): MSE of $\hat{\lambda}_3$ versus the different right censoring levels for $\alpha = 2.5$

Pearson Type VI distribution

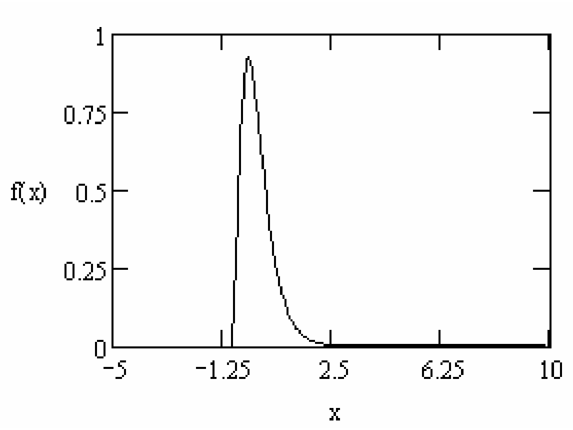

Figure (13): Sampling distribution of $\hat{\alpha}_1$ for $c = 0.1,\ d = 0.9,\ \alpha = 1$ and $n = 50$.

Pearson Type III distribution

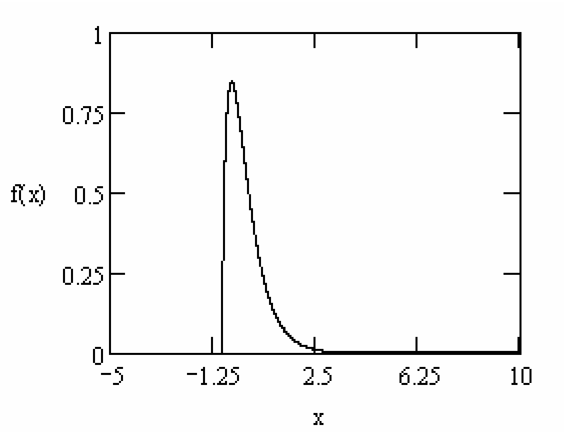

Figure (14): Sampling distribution of $\hat{\alpha}_2$ for $c = 0.1,\ \alpha = 1.5$ and $n = 30$.

Pearson Type I distribution

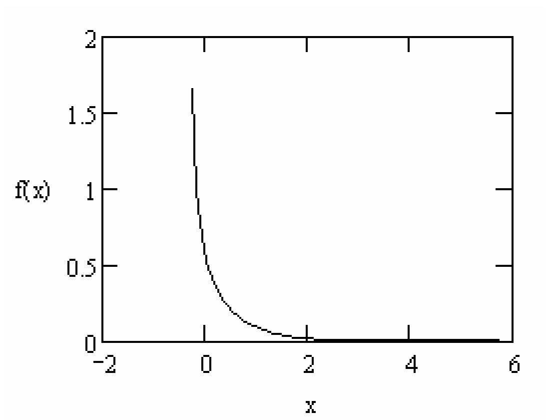

Figure (15): Sampling distribution of $\hat{\lambda}_1$ for $c = 0.3,\ d = 0.8,\ \alpha = 2$ and $n = 20$.

Pearson Type IV distribution

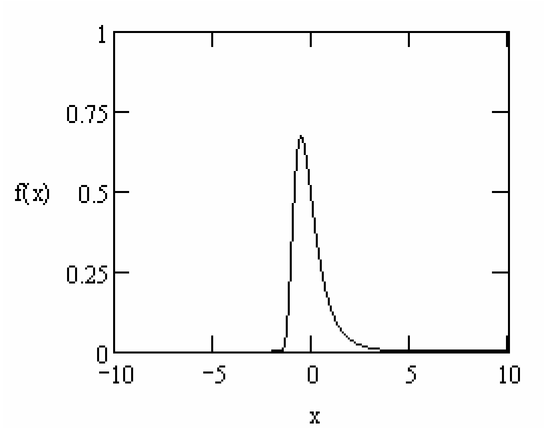

Figure (16): Sampling distribution of $\hat{\lambda}_3$ for $d = 0.9,\ \alpha = 0.5$ and $n = 15$.

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