Assiut University Journal of Multidisciplinary Scientific Research (AUNJMSR)
Faculty of Science, Assiut University, Assiut, Egypt.
Printed ISSN 2812-5029
Online ISSN 2812-5037
Vol. 53(1): 154- 189 (2024)
https://aunj.journals.ekb.eg

# Statistical Inference for the Unit Gompertz Power Series Distribution Using Ranked Set Sampling with Applications

**Manal M. Yousef [1,*], Amal S. Hassan [2] and Ehab M. Almetwally [3,4]**

[1]Department of Mathematics, Faculty of Science, New Valley University, EL-Khargah 72511, Egypt;
[2]Faculty of Graduate Studies for Statistical Research, Cairo University, Giza 12613, Egypt ;
[3]Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia
[4]Faculty of Business Administration, Delta University for Science and Technology, Gamasa 11152, Egypt

*Corresponding Author: manal.mansour@sci.nvu.edu.eg

## ARTICLE INFO

**Article History:**
Received: 2023-11-07
Accepted: 2023-12-05
Online: 2023-12-28

**Keywords**:
Unit Gompertz distribution, ranked set sampling, order statistics, LINEX loss function, Metropolis-Hastings; power series distribution.

## ABSTRACT

In this article, we present the unit Gompertz-power series distribution class, which is formed by combining the unit Gompertz and power series distributions. This new class of distributions includes some special cases, such as the unit Gompertz binomial, the unit Gompertz Poisson, the unit Gompertz geometric, and the unit Gompertz logarithmic distributions. The new class's mathematical properties, such as quantiles, moments, generating function, order statistics, and Renyi entropy are investigated. The unit Gompertz-power series distribution's sub-models are thoroughly examined. The maximum likelihood and Bayesian methodologies are used to estimate the model parameters for the complete and ranked set sampling. The Bayesian estimation approach under the squared error and linear exponential loss functions is examined. Finally, we demonstrate applications of two real data sets to highlight the adaptability and potential of the one sub-model in the new class of distributions.

## 1. INTRODUCTION

Statistical distributions are critical in fitting data to real-world phenomena. It is widely employed to simulate and analyze data in a variety of fields, including engineering, biology, economics, finance, and medical sciences. They occasionally work well with various types of data; however, they are frequently inflexible enough to analyze the complex behavior displayed by data. As a result, there has recently been a motivation to propose more flexible distributions that can fit any type of data with any degree of complexity. A highly effective strategy to construct more flexible distributions is to generate new families of distributions. Different kinds of generators can be employed to develop new families of distributions for any continuous distribution. Compounding lifetime distributions have been produced by combining the distribution when the lifetime can be expressed as the minimum of a series of independent and identically (iid) random variables with a discrete random variable, that is $Z_{(1)} = \min_{1\leq i \leq M} Z_i$, if the components are in a series. Another procedure can be produced by combining the distribution when the lifetime can be expressed as the maximum of iid random variables with a discrete random variable, that is $Z_{(M)} = \max_{1\leq i \leq M} Z_i$, if the components are in a parallel. It is assumed that in both scenarios, $Z_i$ and $M$ are independent continuous random variables.

Compounding the exponential random variable together with a geometric random variable was first proposed by Adamidis and Loukas [1]. This distribution is referred to as the exponential geometric distribution. Several studies have utilized this technique to generate many probability distributions with the power series (PS) model in the literature; for example, the extended Weibull-PS distributions [2], the Burr XII-PS distributions [3], the Lindley-PS [4], the Gompertz-PS [5], the Burr-Weibull PS [6], the exponentiated power Lindley-PS [7], the exponentiated power generalized Weibull-PS [8], the T–R{Y} PS distributions [9], power function-PS distributions [10], inverse gamma PS distributions [11], inverse exponentiated Lomax-PS distributions [12], beta exponential-PS distributions [13], inverse Lindley PS distributions [14], the unit exponentiated half logistic-PS distributions [15], power quasi Lindley-PS distributions [16] and the unit Burr XII-PS [17] distributions.

Benjamin Gompertz first presented the Gompertz distribution, and actuaries and demographers quickly took to it. This distribution, which is an extension of the exponential distribution, has numerous applications across a variety of fields, particularly in the fields of medicine and actuarial science. It has some relationships with some well-known distributions, including the Weibull, extreme value (Gumbel distribution), exponential, double exponential, and generalized logistic distributions (see Willekens [18]). This makes it useful for actuarial and medical studies. Using the transformation $Z = e^{Y}$, where random variable $Y$ has the Gompertz distribution, Mazucheli et al. [19] proposed a new transformed distribution. The authors emphasized that for some particular data sets, the newly transformed distribution provides a much better fit than the beta and Kumaraswamy distributions. The new transformed distribution is known as the unit Gompertz (UG) distribution. The probability density function (PDF) for the UG distribution with parameters $\eta$ and $\zeta$ is as follows

$$f(z;\eta,\zeta) = \eta\zeta z^{-\zeta-1} e^{-\eta(z^{-\zeta}-1)}, \tag{1}$$

For $z \in (0,1)$, $\eta > 0$, and $\zeta > 0$. Hereafter, we denote the UG distribution, with PDF(1), by $Z \sim UG(\eta,\zeta)$. The corresponding cumulative distribution function (CDF) is given by

$$F(z;\eta,\zeta) = e^{-\eta(z^{-\zeta}-1)}, \quad z \in (0,1) \tag{2}$$

The PDF of this distribution can take on a variety of shapes, whereas the hazard rate function can be constant, increasing, or upside-down bathtub-shaped. For more information on the properties of the UG distribution, see Mazucheli et al. [19]. Kumar et al. [20] investigated both classical and Bayesian estimation of model parameters based on lower record values and inter-record times. Jha et al. [21] discussed the problem of estimating the reliability of the multi-component stress strength of UG distribution under progressive type II censoring samples. Arshad et al. [22] examined this model in the context of dual generalized order statistics. For some recent references, see for example Ahmed and Aftab [23], Akata et al. [24] and Alsadat et al. [25].

The main goal of this paper is to propose a new class of compound distributions known as the UG power series (UGPS) by compounding the UG and PS distributions. The following are our motivations for presenting the UGPS distributions:

- To create specific models with various density functions, such as symmetric, left-skewed, right-skewed, U-shaped, J-shaped, and N-shaped functions. Aside from that, the hazard rate functions (HRFs), which include monotonic and nonmonotonic forms.
- To discuss some of its statistical properties such as quantiles, moments, generating function, order statistics, and Renyi entropy.
- To estimate one model of the UGPS parameters. Maximum likelihood (ML), and Bayesian procedures are suggested to estimate the unknown parameters of the UG geometric (UGG) based on simple random sampling (SRS) and ranked set sampling (RSS), and then recommend the best estimates using a simulation study.
- The usage of the UGPS distributions was supported by two real-world analyses that demonstrated their superior performance against competitor distributions.

The paper is structured as follows: In Section 2, we present the class of the UGPS distributions. In addition, we describe the density, survival, and hazard rate functions as well as some of their properties and some special distributions. In Section 3, we calculate the quantile and moments of UGPS distributions. Furthermore, we discuss order statistics via their moments and entropy. Section 4 examines parameter estimation using ML, and Bayesian techniques, as well as inference for the SRS and RSS. Section 5 includes a simulation study. Section 6 provides applications to two real-world data sets. The paper is finally concluded in Section 7.

## 2. THE CLASS OF UGPS DISTRIBUTION

Let $Z_1,\ldots,Z_M$ be UG random variables that are iid. Here, we consider $M$ to be a discrete random variable with a PS distribution (truncated at zero) and a probability mass function defined as

$$P(M=m;\gamma)=\frac{a_m\gamma^m}{C(\gamma)},\ m=1,2,3\ldots, \tag{3}$$

The coefficients $a_m$ 's depend only on $m$ and $C(\gamma)=\sum_{m=1}^{\infty} a_m \gamma^m$ (for $\gamma>0$) is assumed finite. The Poisson, logarithmic, geometric, and binomial distributions are examples of the PS distributions (truncated at zero) listed in Table 1 according to density (3). Detailed information on the PS class of distributions can be found in [26].

Table 1: Useful quantities for some PS distributions

| **Distributions** | $a_m$ | $C(\gamma)$ | $C'(\gamma)$ | $C''(\gamma)$ | $(C(\gamma))^{-1}$ | $\gamma$ |
|---|---|---|---|---|---|---|
| **Poisson** | $(m!)^{-1}$ | $e^{\gamma}-1$ | $e^{\gamma}$ | $e^{\gamma}$ | $\log(\gamma+1)$ | $\gamma\in(0,\infty)$ |
| **Logarithmic** | $(m)^{-1}$ | $-\log(1-\gamma)$ | $(1-\gamma)^{-1}$ | $(1-\gamma)^{-2}$ | $1-e^{-\gamma}$ | $\gamma\in(0,1)$ |
| **Geometric** | 1 | $\gamma(1-\gamma)^{-1}$ | $(1-\gamma)^{-2}$ | $2(1-\gamma)^{-3}$ | $\gamma(1+\gamma)^{-1}$ | $\gamma\in(0,1)$ |
| **Binomial** | $\binom{n}{m}$ | $(1+\gamma)^{n}-1$ | $n(1+\gamma)^{n-1}$ | $\dfrac{n(n-1)}{(\gamma+1)^{2-n}}$ | $(\gamma-1)^{1/n}-1$ | $\gamma\in(0,\infty)$ |

Let $Z_{(1)} = \min_{1\le i\le M} Z_i$, then the conditional CDF of $Z|M=m$ is given by

$$F_{Z_{(1)}|M=m}(z)=\left[1-e^{-\eta(z^{-\zeta}-1)}\right]^{m}. \tag{4}$$

The UGPS class of distributions, that we denoted by UGPS $(\eta,\zeta,\gamma)$, is defined by the marginal CDF of Z, i.e,

$$G(z;\eta,\zeta,\gamma)=1-\frac{C\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}{C(\gamma)},\quad 0<z<1. \tag{5}$$

The PDF of the UGPS $(\eta,\zeta,\gamma)$ distribution is given by

$$g(z;\eta,\zeta,\gamma)=\eta\zeta\gamma z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}\frac{C'\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}{C(\gamma)},\quad 0<z<1. \tag{6}$$

The survival function and the HRF of the UGPS class of distributions are given, respectively, by

$$S(z;\eta,\zeta,\gamma)=\frac{C\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}{C(\gamma)},\quad 0<z<1. \tag{7}$$

and

$$H(z;\eta,\zeta,\gamma)=\frac{\eta\zeta\gamma z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}C'\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}{C\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}. \tag{8}$$

**Remark 1.** Let $Z_{(M)}=max_{1\leq i\leq M}\ Z_i$, then the CDF of $Z_{(M)}$ is

$$G(z_{(M)})=\frac{C\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}{C(\gamma)},\qquad 0<z<1.$$

**Proposition 1.** The unit Gompertz distribution is the limiting case of the UGPS class of distributions when $\gamma\to 0^{+}$.

**Proof**

$$\lim_{\gamma\to 0^+}G(z)=1-\lim_{\gamma\to 0^+}\frac{\sum_{m=1}^{\infty}a_m\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)^m}{\sum_{m=1}^{\infty}a_m\gamma^m}$$

$$=1-\lim_{\gamma\to 0^+}\frac{\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\left[a_1+\sum_{m=2}^{\infty}ma_m\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)^{m-1}\right]}{a_1+\sum_{m=2}^{\infty}ma_m\gamma^{m-1}}.$$

Applying the limit on the right side, we have

$$\lim_{\gamma\to 0^+}G(z)=e^{-\eta(z^{-\zeta}-1)}, \tag{9}$$

which is the distribution function of the UG distribution as defined in (2).

**Proposition 2.** The infinite mixture representation of the UGPS can be expressed in terms of UG distribution with parameters $m\eta$ and $\zeta$.

**Proof**

Since $C'(\gamma)=\sum_{m=1}^{\infty}ma_m\gamma^{m-1}$, the PDF (6) can be rewritten as follows:

$$g(z;\eta,\zeta,\gamma)=\sum_{m=1}^{\infty}\frac{a_m\gamma^m}{C(\gamma)}\, m\eta\zeta\gamma z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}\left(1-e^{-\eta(z^{-\zeta}-1)}\right)^{m-1}$$
$$=\sum_{m=1}^{\infty}P(M=m)\, mf(z;\eta,\zeta)\left(1-F(z;\eta,\zeta)\right)^{m-1}$$
$$=\sum_{m=1}^{\infty}P(M=m)\, f_{(1)}(z;m\eta,\zeta), \qquad (10)$$

where $f_{(1)}(z;m\eta,\zeta)$ is the PDF of the UG distribution with parameters $m\eta$ and $\zeta$.

### 2.1. Unit Gompertz logarithmic distribution

The unit Gompertz logarithmic (UGL) distribution is a special case of UGPS and is defined by the CDF (5) with $-\log(1-\gamma)$ resulting in

$$G(z;\eta,\zeta,\gamma)=1-\frac{\log\left(1-\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}{\log(1-\gamma)},\quad 0<z<1. \qquad (11)$$

where $\gamma\in(0,1)$. The associated PDF and HRF are

$$g(z;\eta,\zeta,\gamma)=\frac{\eta\gamma\zeta z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}}{\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)-1\right)\log(1-\gamma)},\quad 0<z<1. \qquad (12)$$

and

$$H(z;\eta,\zeta,\gamma)=\frac{\eta\gamma\zeta z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}}{\left(\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)-1\right)\log\left(1-\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)\right)}. \qquad (13)$$

The plots of PDF and HRF of the UGL distribution for different values of $\eta,\zeta$ and $\gamma$ are given in Figure 1.

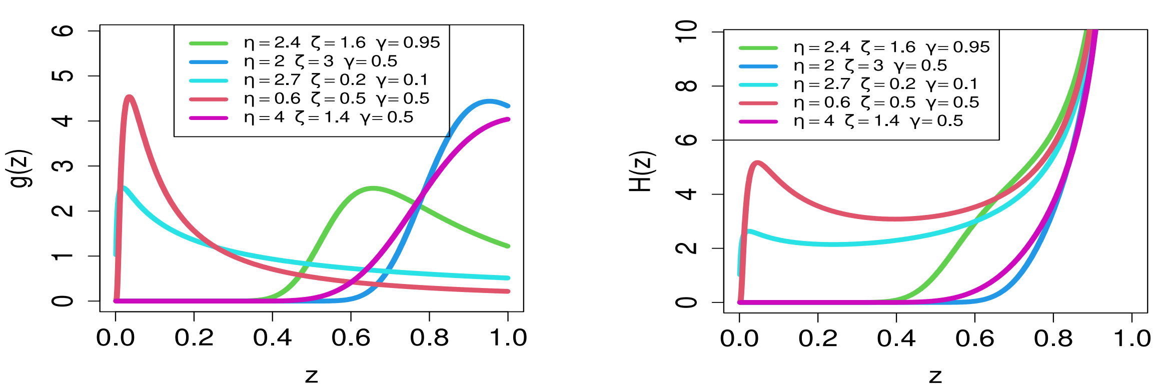


Figure 1: Plots of the UGL density function and hazard rate function

### 2.2. Unit Gompertz Poisson distribution

The unit Gompertz Poisson (UGP) distribution is a special case of UGPS and is defined the CDF (5) with $e^{\gamma}-1$ resulting in

$$G(z;\eta,\zeta,\gamma)=\frac{e^{\gamma}-e^{\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)}}{e^{\gamma}-1}, \quad 0<z<1, \tag{14}$$

where $\gamma\in(0,\infty)$. The associated PDF and HRF are

$$g(z;\eta,\zeta,\gamma)=\eta\gamma\zeta z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}\frac{e^{\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)}}{e^{\gamma}-1}, \quad 0<z<1, \tag{15}$$

and

$$H(z;\eta,\zeta,\gamma)=\frac{\eta\gamma\zeta z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}e^{\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)}}{e^{\gamma\left(1-e^{-\eta(z^{-\zeta}-1)}\right)}-1}. \tag{16}$$

The plots of PDF and HRF of the UGP distribution for different values of $\eta,\zeta$ and $\gamma$ are given in Figure 2.

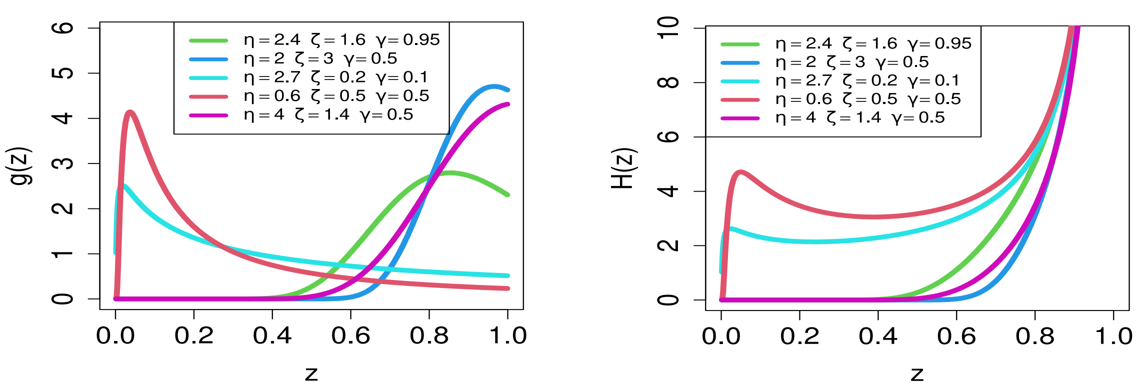


Figure 2: Plots of the UGP density function and hazard rate function

### 2.3 Unit Gompertz Geometric distribution

The unit Gompertz geometric (UGG) distribution is a special case of UGPS and is defined by the CDF (5) with $\gamma(1-\gamma)^{-1}$ resulting in

$$G(z;\eta,\zeta,\gamma)=\frac{e^{-\eta(z^{-\zeta}-1)}}{1-\gamma(1-e^{-\eta(z^{-\zeta}-1)})},\quad 0<z<1, \tag{17}$$

where $\gamma\in(0,1)$. The associated PDF and HRF are

$$g(z;\eta,\zeta,\gamma)=\frac{\eta\zeta(1-\gamma)z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}}{(1-\gamma(1-e^{-\eta(z^{-\zeta}-1)}))^{2}},\quad 0<z<1, \tag{18}$$

and

$$H(z;\eta,\zeta,\gamma)=\frac{\eta\zeta z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}}{(1-e^{-\eta(z^{-\zeta}-1)})(1-\gamma(1-e^{-\eta(z^{-\zeta}-1)}))},\quad 0<z<1. \tag{19}$$

The plots of PDF and HRF of the UGG distribution for different values of $\eta,\zeta$ and $\gamma$ are given in Figure 3.

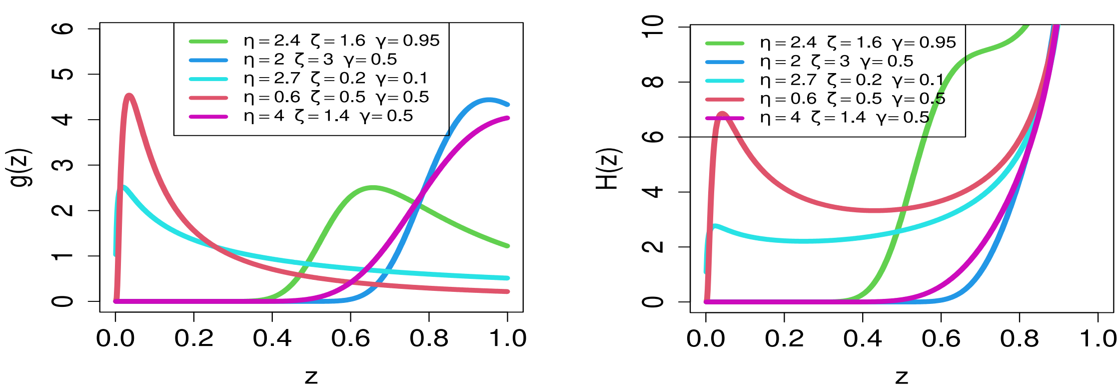


Figure 3: Plots of the UGG density function and hazard rate function

## 2.4. Unit Gompertz Binomial distribution

The unit Gompertz binomial (UGB) distribution is a special case of UGPS and is defined by the CDF (5) with $(1+\gamma)^n-1$ resulting in

$$G(z;\eta,\zeta,\gamma)=1-\frac{(1+\gamma(1-e^{-\eta(z^{-\zeta}-1)}))^{n}-1}{(1+\gamma)^{n}-1},\quad 0<z<1, \tag{20}$$

where $\gamma\in(0,1)$. The associated PDF and HRF are

$$g(z;\eta,\zeta,\gamma)=\eta\zeta\gamma z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}\,\frac{n(1+\gamma(1-e^{-\eta(z^{-\zeta}-1)}))^{n-1}}{(1+\gamma)^{n}-1},\quad 0<z<1, \tag{21}$$

and

$$H(z;\eta,\zeta,\gamma)=\eta\zeta\gamma z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}\frac{n(1+\gamma(1-e^{-\eta(z^{-\zeta}-1)}))^{n-1}}{(1+\gamma(1-e^{-\eta(z^{-\zeta}-1)}))^{n}-1}. \quad (22)$$

The plots of PDF and HRF of the UGB distribution for $n$=100, and different values of $\eta,\zeta$ and $\gamma$ are given in Figure 4.

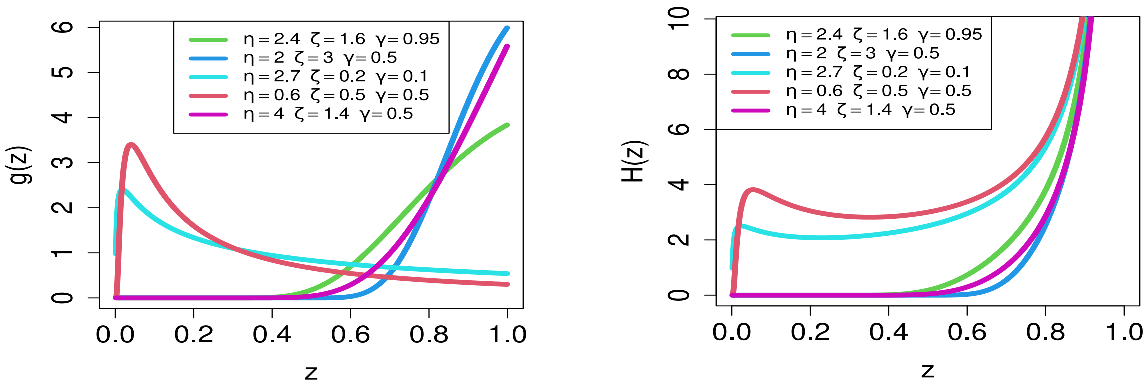


Figure 4: Plots of the UGB density function and hazard rate function

## 3. STATISTICAL PROPERTIES

We suggested a few of the fundamental statistical characteristics of the UGPS distribution in this section. We provide quantiles, moments, order statistics, and entropy as examples.

### 3.1. Quantiles and Moments

The quantile function of the UGPS distribution, $Q_z(q)$ can be defined as $G(Q_z(q))=q$.

$$Q_z(q;\eta,\zeta,\gamma)=\left[1-\frac{1}{\eta}\ln\left(1-\frac{C^{-1}\left((1-q)C(\gamma)\right)}{\gamma}\right)\right]^{\frac{-1}{\zeta}}. \quad (23)$$

where $C^{-1}(.)$ is the inverse function of $C(.)$. So, using (10), random numbers may be produced from the UGPS class of distributions. One can simulate a random sample of the UGG distribution through Q. Measures based on quantiles can be used to examine the skewness and kurtosis of the proposed distribution. The formulas for calculating the Galton skewness ($G_S$) and Moor's kurtosis ($M_k$), respectively, are given by

$$G_S = \frac{Q_3 - 2Q_2 + Q_1}{Q_3 - Q_1}, \qquad M_K = \frac{(\omega_2 - \omega_5) - (\omega_3 - \omega_1)}{(\omega_6 - \omega_2)},$$

where $Q_1 = Q\left(\frac{1}{4}; \eta, \zeta, \gamma\right)$, and $\omega_i$, is the $i^{th}$ octile $\omega_i = G^{-1}\left(\frac{i}{8}\right)$.

The necessity and importance of moments in any statistical analysis, particularly in applied work, cannot be overstated. Moments can be used to investigate some of the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness, and kurtosis).

The following expressions can be applied to determine the $r^{th}$ moment of the UGPS distribution using the infinite mixture representation in (10), we have:

$$\mu_r' = \sum_{m=1}^{\infty} \frac{m a_m \gamma^m \eta\zeta}{C(\gamma)} \int_0^1 z^{r-\zeta-1} e^{-\eta(z^{-\zeta}-1)} \left(1 - e^{-\eta(z^{-\zeta}-1)}\right)^{m-1} dz. \tag{24}$$

Using binomial expansion in (24)

$$\mu_r' = \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} \binom{m-1}{i} P(M = m)(-1)^i \, m\eta\zeta \int_0^1 z^{r-\zeta-1} e^{-\eta(z^{-\zeta}-1)(i+1)} dz.$$

Let $u = \eta(i+1) z^{-\zeta}$, then $\mu_r'$ of the UGPS distribution is

$$\mu_r' = \sum_{m=1}^{\infty} \sum_{i=0}^{m-1} \binom{m-1}{i} P(M = m)(-1)^i m \eta^{\frac{r}{\zeta}} (i+1)^{\frac{r}{\zeta}-1} e^{\eta(i+1)} \Gamma\left(1 - \frac{r}{\zeta}, \eta(i+1)\right), \tag{25}$$

where $\Gamma(., y)$ is the upper incomplete gamma function. Note that the moments exists only when $r < \zeta$.

Furthermore, the moment-generating function of the UGPS distributions is defined by

$$M_Z(t) = E\left(e^{tz}\right). \tag{26}$$

Using the infinite mixture representation in (10), we have

$$M_Z(t)=\sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\sum_{r=0}^{\infty}\frac{ma_m\gamma^m\eta\zeta}{C(\gamma)}\int_0^1\frac{(tz)^r}{r!}z^{-\zeta-1}e^{-\eta(z^{-\zeta}-1)}\left(1-e^{-\eta(z^{-\zeta}-1)}\right)^{m-1}dz$$

$$=\sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\sum_{r=0}^{\infty}\frac{t^r}{r!}\binom{m-1}{i}P(M=m)(-1)^i m\eta^{\frac{r}{\zeta}}(i+1)^{\frac{r}{\zeta}-1}e^{\eta(i+1)}\Gamma\left(1-\frac{r}{\zeta},\eta(i+1)\right). \quad (27)$$

### 3.2. Order Statistics and Entropy

Order statistics are important tools in nonparametric statistics and inference. Their significance is evident in a variety of estimation and hypothesis-testing problems. Their moments play an important role in quality control testing and reliability.

Let $Z_1, Z_2,\ldots,Z_n$ be a random sample of size $n$ from the UGPS class of distributions and suppose $Z_{1:n}, Z_{2:n},\ldots,Z_{n:n}$ denote the corresponding order statistics. The PDF of the $i^{th}$ order statistic can be expressed as

$$g_{k:n}(z;\eta,\zeta,\gamma)=\sum_{s=0}^{i-1}\sum_{j,t,u=0}^{\infty}\frac{\Omega_{s,j,t,u}\,g(z;\eta(u+1),\zeta)}{\mathrm{B}(i,n-i+1)}, \quad (28)$$

where B(.,.) is the complete beta function,

$$\Omega_{s,j,t,u}=\frac{b_j(j+1)\gamma^{n+s-i+t+j+1}(-1)^{u+s}}{(C(\gamma))^{n+s-i+1}(u+1)}\binom{i-1}{s}\binom{n+s-i+t+j}{u}a_1^{n+s-i+1}d_{n+s-i,m},$$

and

$$d_{n+s-i,m}=v^{-1}\sum_{t=1}^{v}(t(n+s-i+1)-v)b_t d_{n+s-i,v-t},v\geq 1,$$

$$d_{n+s-i,0}=1,\quad b_j=\frac{a_{j+1}}{a_j},\quad j=0,1,2,\ldots$$

In addition, (28) can be used to obtain the $\theta_{i:m}$ moment of ith order statistics from the UGPS distribution.

$$\mu'_{\theta_{i:m}}=\sum_{s=0}^{i-1}\sum_{j,t,u=0}^{\infty}\frac{\Omega_{s,j,t,u}\eta^{\frac{\theta}{\zeta}}(u+1)^{\frac{\theta}{\zeta}-1}}{\mathrm{B}(i,n-i+1)}e^{\eta(u+1)}\Gamma\left(1-\frac{\theta}{\zeta},\eta(u+1)\right). \quad (29)$$

Smaller samples are logically expected to contain less information than larger samples, but there haven't been many attempts to quantify the information lost as a result of using sub-samples rather than complete samples. In many fields, including statistics, economics, and physical, chemical, and biological phenomena, the concept of measuring entropy is essential.

Renyi entropy [27] of a random variable $Z$ following the UGPS distribution for $\varepsilon \neq 0, \varepsilon > 0,$ is given by

$$\Re_{\varepsilon}(\eta,\zeta,\gamma) = (1-\varepsilon)^{-1}\log\int_0^1 (g(z;\eta,\zeta,\gamma))^{\varepsilon} dz. \tag{30}$$

Substituting (6) for (30) yields

$$\Re_{\varepsilon}(\eta,\zeta,\gamma) = (1-\varepsilon)^{-1}\log\left[(\eta\gamma\zeta a_1)^{\varepsilon}\sum_{t=0}^{\infty}\sum_{j=0}^{t}\binom{t}{j}\frac{(-1)^j d_{\varepsilon,t}\zeta^{-1}\gamma^t e^{\eta(j+\varepsilon)}}{(\eta(j+\varepsilon))^{\frac{\varepsilon(\zeta+1)-1}{\zeta}}}\Gamma\left(\frac{\varepsilon(\zeta+1)-1}{\zeta},\eta(j+\varepsilon)\right)\right]. \tag{31}$$

### 3.3. Lorenz and Bonferroni Curves

The measure of distributional inequality of a random variable can be illustrated graphically by the Lorenz and Bonferroni curves. The Lorenz and Bonferroni curves for a random variable $Z$ are given by

$$L(\delta) = \frac{1}{\mu}\int_0^{\delta} zg(z;\eta,\zeta,\gamma)dz, \qquad Be(\delta) = \frac{1}{\mu G(z;\eta,\zeta,\gamma)}\int_0^{\delta} zg(z;\eta,\zeta,\gamma)dz,$$

where $\mu$ denotes the expected value of the random variable $Z$. Using the expectation (25) and PDF (10), the Lorenz curve for UGPS distributions is depicted as follows:

$$L(\delta) = \sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\binom{m-1}{i}P(M=m;\gamma)(-1)^i m\eta^{\frac{1}{\zeta}}(i+1)^{\frac{1}{\zeta}-1}e^{\eta(i+1)}\Gamma\left(1-\frac{1}{\zeta},\eta(i+1)\delta^{-\zeta}\right).$$

In a similar manner, the Bonferroni index of UGPS distribution is also calculated as

$$Be(\delta) = \frac{1}{\mu G(z;\eta,\zeta,\gamma)}\sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\binom{m-1}{i}P(M=m;\gamma)(-1)^i m\eta^{\frac{1}{\zeta}}(i+1)^{\frac{1}{\zeta}-1}e^{\eta(i+1)}\Gamma\left(1-\frac{1}{\zeta},\eta(i+1)\delta^{-\zeta}\right).$$

## 4. ESTIMATION AND INFERENCE OF THE MODEL PARAMETERS

This section presents the estimates of the three unknown parameters $(\eta,\zeta,\gamma)$ of the UGG distribution under the SRS and RSS using ML and the Bayesian methods.

### 4.1. Parameter estimation under SRS

### 4.1.1 Maximum likelihood estimator

Let $Z_1,\ldots,Z_n$ be a random sample from UGG distribution with the parameters $\eta,\zeta$ and $\gamma$. Then the likelihood function is obtained as

$$L_{MLS}(\eta,\zeta,\gamma)=(\eta\zeta)^n(1-\gamma)^n\prod_{i=1}^{n}\frac{z_i^{-\zeta-1}D_i}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}, \tag{32}$$

where $D_i=\dfrac{e^{-\eta(z_i^{-\zeta}-1)}}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}$. The natural logarithm of (32), can be given as

$$l_{MLS}(\eta,\zeta,\gamma)=n\ln(\eta\zeta)+n\ln(1-\gamma)-(\zeta+1)\sum_{i=1}^{n}\ln z_i+\sum_{i=1}^{n}\ln D_i$$
$$-\sum_{i=1}^{n}\ln\left[1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})\right]. \tag{33}$$

The first partial derivatives of (33) concerning $\eta,\zeta$ and $\gamma$ are given, respectively, by

$$\left.\begin{aligned}
\frac{\partial l_{MLS}(\eta,\zeta,\gamma)}{\partial\eta}&=\frac{n}{\eta}+\sum_{i=1}^{n}\left(2\gamma D_i-1\right)\left(z_i^{-\zeta}-1\right),\\
\frac{\partial l_{MLS}(\eta,\zeta,\gamma)}{\partial\zeta}&=\frac{n}{\zeta}-\sum_{i=1}^{n}\ln z_i-\sum_{i=1}^{n}\left(2\gamma D_i-1\right)\eta z_i^{\zeta}\ln z_i,\\
\frac{\partial l_{MLS}(\eta,\zeta,\gamma)}{\partial\gamma}&=\frac{n}{(1-\gamma)}+2\sum_{i=1}^{n}\left[\frac{1}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}-D_i\right].
\end{aligned}\right\} \tag{34}$$

The ML estimates (MLEs) of $\eta,\zeta$ and $\gamma$ denoted by $\hat{\eta}_{MLS},\hat{\zeta}_{MLS}$ and $\hat{\gamma}_{MLS}$ can be computed by equating (34) to zero and solve the three normal equations simultaneously. Because the MLE equations for $\hat{\eta}_{MLS},\hat{\zeta}_{MLS}$ and $\hat{\gamma}_{MLS}$ cannot be obtained theoretically, so analytical solutions are considered. Consequantely, the 'maxLik' package in R packages, which employs the Newton–Raphson method for maximization, provides a convenient alternative to find estimates of parameters.

### 4.1.2 Bayesian estimation

We now address the issue of parameter estimation for the generalized quadratic and linear exponential (LINEX) loss functions. Informative priors convey accurate and clear information about the parameters and are, as their name implies, more informative than non-informative priors. In our research, we suggested the assumption that the prior distributions are independent. In the Bayesian literature, such as Punt and Walker [28], Punt and Butterworth [29], and Kundu and Mitra [30], this assumption of independence is not new. The prior distributions for $\eta$ and $\zeta$, are assumed to be Gamma $(\theta_1, \lambda_1)$ and Gamma $(\theta_2, \lambda_2)$, respectively, while the prior distribution of $\gamma$ is assumed to be beta $(\theta_3, \lambda_3)$, with the following forms:

$$\pi(\eta) = \frac{\theta_1^{\lambda_1}}{\Gamma(\lambda_1)} \eta^{\lambda_1 - 1} e^{-\theta_1 \eta}, \quad \eta > 0, \theta_1, \lambda_1 > 0,$$

$$\pi(\zeta) = \frac{\theta_2^{\lambda_2}}{\Gamma(\lambda_2)} \zeta^{\lambda_2 - 1} e^{-\theta_2 \zeta}, \quad \zeta > 0, \theta_2, \lambda_2 > 0,$$

$$\pi(\gamma) = \frac{1}{\mathrm{B}(\theta_3, \lambda_3)} \gamma^{\theta_3 - 1} (1-\gamma)^{\lambda_3 - 1}, \quad 0 < \gamma < 1, \theta_3, \lambda_3 > 0,$$

where $\theta_1, \lambda_1, \theta_2, \lambda_2, \theta_3$ and $\lambda_3$ are referred to as hyper-parameters. Except for their flexibility, tractability, and status as natural conjugate priors for the exponential distributions, there is no objective reason to select the gamma family as prior distributions. Then the joint prior distribution of $\eta, \zeta$ and $\gamma$ is given by

$$\pi(\zeta) = \frac{\theta_1^{\lambda_1} \theta_2^{\lambda_2} \eta^{\lambda_1 - 1} \zeta^{\lambda_2 - 1} \gamma^{\theta_3 - 1}}{\Gamma(\lambda_1)\Gamma(\lambda_2)\mathrm{B}(\theta_3, \lambda_3)} e^{-(\theta_1 \eta + \theta_2 \zeta)} (1-\gamma)^{\lambda_3 - 1}. \tag{35}$$

From (32) and (35), the joint posterior of $\eta, \zeta$ and $\gamma$ is obtained as follows

$$\pi_1^*(\eta, \zeta, \gamma | \underline{z}) = h^{-1} \eta^{n+\lambda_1 - 1} \zeta^{n+\lambda_2 - 1} \gamma^{\theta_3 - 1} e^{-(\theta_1 \eta + \theta_2 \zeta)} (1-\gamma)^{n+\lambda_3 - 1} \prod_{i=1}^{n} \frac{z_i^{-\zeta - 1} D_i}{1 - \gamma(1 - e^{-\eta(z_i^{-\zeta} - 1)})}, \tag{36}$$

where,

$$h=\int_0^{\infty}\int_0^{\infty}\int_0^{1}\eta^{n+\lambda_1-1}\zeta^{n+\lambda_2-1}\gamma^{\theta_3-1}e^{-(\theta_1\eta+\theta_2\zeta)}(1-\gamma)^{n+\lambda_3-1}\prod_{i=1}^{n}\frac{z_i^{-\zeta-1}D_i}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}\,d\gamma\,d\eta\,d\zeta.$$

In a Bayesian analysis, the loss function must be carefully chosen. Thorough comparison of Bayes estimates, we take into account two different loss function types, namely the squared error (symmetric) and LINEX (asymmetric) loss functions. For a thorough comparison of Bayes estimates, the squared error loss function (SELF) for a parameter $\upsilon$ is define

$$L_1(\upsilon,\tilde{\upsilon})=(\tilde{\upsilon}-\upsilon)^2,$$

where $\tilde{\upsilon}$ is an estimator of $\upsilon$. The Bayesian estimator under SELF is given by:

$$\tilde{\upsilon}_{SE}=E(\upsilon|data). \tag{37}$$

However, we also take into consideration the LINEX loss function (LLF), which is denoted by

$$L_2(\upsilon,\tilde{\upsilon})=e^{c(\tilde{\upsilon}-\upsilon)}-c(\tilde{\upsilon}-\upsilon)-1,$$

where, $\tilde{\upsilon}$ is an estimator of $\upsilon$ and *c* is the sign presenting the asymmetry. In this case, the Bayesian estimator is obtained as:

$$\tilde{\upsilon}_L=\frac{-1}{c}\ln\left[E(e^{-c\upsilon}|data)\right]. \tag{38}$$

The Bayesian estimates of $S(\vartheta)=(\eta,\zeta,\gamma)$ under the SELF, and LLF can be calculated using (37) through (38) as follows:

$$\tilde{\upsilon}_{BESE}=E(S(\vartheta)|data)=h^{-1}\int_0^{\infty}\int_0^{\infty}\int_0^{1}S(\vartheta)\pi_1^*(\eta,\zeta,\gamma|\underline{z})\;d\gamma\,d\eta\;d\zeta.$$

$$\tilde{\upsilon}_{BLLS}=\frac{-1}{c}\ln\left[E(e^{-cS(\vartheta)}|data)\right]=\frac{-1}{c}\ln\left[h^{-1}\int_0^{\infty}\int_0^{\infty}\int_0^{1}e^{-cS(\vartheta)})\pi_1^*(\eta,\zeta,\gamma|\underline{z})\,d\gamma\,d\eta\,d\zeta\right].$$

The ratio of two integrals is part of the Bayes as mentioned earlier estimate of $S(\vartheta)$ and it is obvious that this ratio cannot be expressed in a more straightforward manner. The Markov chain Monte Carlo (MCMC) technique is used in the follow-up to generate approximations of the parameter Bayes estimates. The conditional posterior densities of $\eta, \zeta$ and $\gamma$ are produced as following

$$\left.\begin{aligned}
\pi_1^*(\eta|\zeta,\gamma,data) &\propto \eta^{n+\lambda_1-1} e^{-\theta_1\eta} \prod_{i=1}^{n} \frac{D_i}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}, \\
\pi_1^*(\zeta|\eta,\gamma,data) &\propto \zeta^{n+\lambda_2-1} e^{-\theta_2\zeta} \prod_{i=1}^{n} \frac{z^{-\zeta} D_i}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}, \\
\pi_1^*(\gamma|\eta,\zeta,data) &\propto \gamma^{\theta_3-1}(1-\gamma)^{n+\lambda_3-1} \prod_{i=1}^{n} \frac{D_i}{1-\gamma(1-e^{-\eta(z_i^{-\zeta}-1)})}.
\end{aligned}\right\} \quad (39)$$

The explicit forms of the Bayes estimates are not available due to the complexity of the posterior distribution in (39). To implement the MCMC method and obtain the Bayes estimate of $\eta, \zeta$ and $\gamma$ the Metropolis-Hastings algorithm is therefore proposed on the steps below.

**Step 1**: Start with $\eta^{(0)}, \zeta^{(0)}$ and $\gamma^{(0)}$ as the initial states of $\eta, \zeta$ and $\gamma$, respectively.

**Step 2**: Let $i$ = 0.

**Step 3**: Provide the transition probabilities $Q_\ell\left(\ell^{(*)}\middle|\ell^{(i)}\right)$ from $\ell^{(i)}$ to $\ell^{(*)}$ for $\ell = (\eta, \zeta, \gamma)$.

**Step 4**: Generate $\eta^{(*)} \sim Q\eta\left(\eta^{(*)}\middle|\eta^{(i)}\right)$ and $U$~(0,1), where $U$~(0,1) is the uniform distribution. Update $\eta^{(i+1)}$ by

$$\eta^{(i+1)} = \begin{cases} \eta^{(*)} & u \le \min\left\{1, \dfrac{\pi^*\left(\eta^{(*)}\middle|\zeta^{(i)},\gamma^{(i)}\right) Q\eta\left(\eta^{(*)}\middle|\eta^{(i)}\right)}{\pi^*\left(\eta^{(i)}\middle|\zeta^{(i)},\gamma^{(i)}\right) Q\eta\left(\eta^{(*)}\middle|\eta^{(i)}\right)}\right\} \\ \eta^{(i)} & \text{otherwise} \end{cases}$$

In the same way, we obtain $\zeta^{(i+1)}$ and $\gamma^{(i+1)}$.

**Step 5**: Put $i = i$ +1.

**Step 6**: Duplicate steps (2 - 5), $B$ times to obtain $(\eta^{(1)}, \zeta^{(1)}, \gamma^{(1)}), \ldots, (\eta^{(B)}, \zeta^{(B)}, \gamma^{(B)})$.

Thus, the Bayes estimates of $S(\vartheta)$ using MCMC under SELF, and LLF are respectively

$$\tilde{S}_{BESRS}=E\left(S(\vartheta)|\underline{z}\right)=\frac{1}{B-K}\sum_{i=K+1}^{B}S\left(\eta^{(i)},\zeta^{(i)},\gamma^{(i)}\right),$$

$$\tilde{S}_{LSRS}=E\left(e^{-cS(\vartheta)}|\underline{z}\right)=\frac{-1}{c}\ln\left[\frac{1}{B-K}\sum_{i=K+1}^{B}e^{-cS\left(\eta^{(i)},\zeta^{(i)},\gamma^{(i)}\right)}\right].$$

where $K$ is the burn-in period.

## 4.2. Parameter estimation under RSS

### 4.2.1 Maximum likelihood estimator

Suppose that $Z_{(i)ij}$ , $i$=1,…,$k$, $j$=1,…,$h$ denotes a selected RSS from the UGG distribution, where the sample size is $n= kh$, $k$ denotes the set size, and $h$ is the number of cycles. For the sake of simplicity, we use the notations $Z_{ij}$ instead of $Z_{(i)ij}$ in the remaining sections of the paper, then $Z_{ij}$ are independent with a density equal to the same density of the ith order statistic from a sample of size $kh$. The likelihood function the RSS of samples $Z_{1j},Z_{2j},\ldots,Z_{kj}$ can be represented as (Arnold et al. [31] ).

$$L_{MLR}(\eta,\zeta,\gamma)=(A\eta\zeta(1-\gamma))^{n}\prod_{j=1}^{h}\prod_{i=1}^{k}\frac{z_{ij}^{-\zeta-1}(D_{ij})^{i}(1-D_{ij})^{k-i}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})},\tag{40}$$

where $A=i\binom{k}{i},\ D_{ij}=\dfrac{e^{-\eta(z_{ij}^{-\zeta}-1)}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})}.$

The natural logarithm of (40), is obtained as follows:

$$l_{MLR}(\eta,\zeta,\gamma)=n\ln(\eta\zeta)+n\ln(1-\gamma)-(\zeta+1)\sum_{j=1}^{h}\sum_{i=1}^{k}\ln z_{ij}+\sum_{j=1}^{h}\sum_{i=1}^{k}i\ln D_{ij}$$
$$-\sum_{j=1}^{h}\sum_{i=1}^{k}\ln\left[1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})\right]+\sum_{j=1}^{h}\sum_{i=1}^{k}(k-i)\ln(1-D_{ij}).\tag{41}$$

Computing the first partial derivatives of (41) with respect to $\eta,\zeta$ and $\gamma$.

$$\left.\begin{aligned}
\frac{\partial l_{MLR}(\eta,\zeta,\gamma)}{\partial\eta} &= \frac{n}{\eta}+\sum_{j=1}^{h}\sum_{i=1}^{k}\gamma D_{ij}(z_{ij}^{-\zeta}-1)+\sum_{j=1}^{h}\sum_{i=1}^{k}\left[i-\frac{(k-i)D_{ij}}{1-D_{ij}}\right](\gamma D_{ij}-1)(z_{ij}^{-\zeta}-1),\\
\frac{\partial l_{MLR}(\eta,\zeta,\gamma)}{\partial\zeta} &= \frac{n}{\zeta}-\sum_{j=1}^{h}\sum_{i=1}^{k}\ln z_{ij}-\sum_{j=1}^{h}\sum_{i=1}^{k}\gamma\eta D_{ij}z_{ij}^{-\zeta}\ln z_{ij}+\sum_{j=1}^{h}\sum_{i=1}^{k}\left[i-\frac{(k-i)D_{ij}}{1-D_{ij}}\right](\gamma D_{ij}-1)\eta z_{ij}^{-\zeta}\ln z_{ij},\\
\frac{\partial l_{MLR}(\eta,\zeta,\gamma)}{\partial\gamma} &= \frac{n}{(1-\gamma)}+\sum_{i=1}^{n}\left[\frac{1}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})}-D_{ij}\right]\left[i+1-\frac{(k-i)D_{ij}}{1-D_{ij}}\right].
\end{aligned}\right\} \quad (42)$$

It is obvious that the likelihood equations (42) have no explicit solution. As a result, we employ iterative methods. Based on RSS, the ML estimators of $\eta,\zeta$ and $\gamma$ are designated are designated as $\hat{\eta}_{MLR},\hat{\zeta}_{MLR}$ and $\hat{\gamma}_{MLR}$, respectively.

### 4.2.2 Bayesian estimation

Using (35) and (40) the joint posterior density can be expressed as

$$\pi_1^*(\eta,\zeta,\gamma|\underline{z})=\aleph^{-1}\eta^{n+\lambda_1-1}\zeta^{n+\lambda_2-1}\gamma^{\theta_3-1}e^{-(\theta_1\eta+\theta_2\zeta)}(1-\gamma)^{n+\lambda_3-1}\prod_{j=1}^{h}\prod_{i=1}^{k}\frac{z_{ij}^{-\zeta-1}(D_{ij})^{i}(1-D_{ij})^{k-i}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})}, \quad (43)$$

where

$$\aleph=\int_0^\infty\int_0^\infty\int_0^1\eta^{n+\lambda_1-1}\zeta^{n+\lambda_2-1}\gamma^{\theta_3-1}e^{-(\theta_1\eta+\theta_2\zeta)}(1-\gamma)^{n+\lambda_3-1}\prod_{j=1}^{h}\prod_{i=1}^{k}\frac{z_{ij}^{-\zeta-1}(D_{ij})^{i}(1-D_{ij})^{k-i}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})}\,d\gamma d\eta d\gamma.$$

Therefore, the Bayes estimates of $\eta$; $\zeta$ and $\gamma$ under SELF, and LLF can be given as

$$\left.\begin{aligned}
\tilde{\upsilon}_{BSER} &= E(S(\vartheta)|data)=\aleph^{-1}\int_0^\infty\int_0^\infty\int_0^1 S(\vartheta)\pi_2^*(\eta,\zeta,\gamma|\underline{z})d\gamma d\eta d\zeta,\\
\tilde{\upsilon}_{BLLR} &= \frac{-1}{c}\ln\left[E(e^{-cS(\vartheta)}|data)\right]=\frac{-1}{c}\ln\left[h^{-1}\int_0^\infty\int_0^\infty\int_0^1 e^{-cS(\vartheta)}\pi_2^*(\eta,\zeta,\gamma|\underline{z})d\gamma d\eta d\zeta\right]
\end{aligned}\right\}. \quad (44)$$

The integrals given in (44) cannot be calculated analytically in a similar manner. As a result, we employ Metropolis-Hastings algorithm. The conditional posterior densities of $\eta,\zeta$ and $\gamma$ are produced as following

$$\left.\begin{aligned}
\pi_2^*(\eta|\zeta,\gamma,data) &\propto \eta^{n+\lambda_1-1}e^{-\theta_1\eta}\prod_{j=1}^{h}\prod_{i=1}^{k}\frac{(D_{ij})^i(1-D_{ij})^{k-i}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})},\\
\pi_2^*(\zeta|\eta,\gamma,data) &\propto \zeta^{n+\lambda_2-1}e^{-\theta_2\zeta}\prod_{j=1}^{h}\prod_{i=1}^{k}\frac{z_{ij}^{-\zeta-1}(D_{ij})^i(1-D_{ij})^{k-i}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})},\\
\pi_2^*(\gamma|\eta,\zeta,data) &\propto \gamma^{\theta_3-1}(1-\gamma)^{n+\lambda_3-1}\prod_{j=1}^{h}\prod_{i=1}^{k}\frac{(D_{ij})^i(1-D_{ij})^{k-i}}{1-\gamma(1-e^{-\eta(z_{ij}^{-\zeta}-1)})}.
\end{aligned}\right\} \tag{45}$$

Then, the Bayes estimator of $S(\vartheta)$ is evaluated by MCMC under SELF, and LLF respectively at the points $\tilde{S}_{SERSS}(\vartheta)$ and $\tilde{S}_{LRSS}(\vartheta)$.

## 5. SIMULATION

Several comparisons using Monte Carlo simulations are carried out to learn the associated performance of the provided estimators of $\eta,\zeta$ and $\gamma$ studied in the above sections. Therefore, we duplicated sample of the UGG distribution with 5000 times using the following procedures:

- Set the real $\eta,\zeta$ and $\gamma$ values as $(\eta=0.5,\zeta=0.5,\gamma=0.2)$, $(\eta=1.5,\zeta=0.5,\gamma=0.2)$, $(\eta=0.5,\zeta=1.5,\gamma=0.2)$, $(\eta=1.5,\zeta=1.5,\gamma=0.2)$, $(\eta=1.5,\zeta=1.5,\gamma=0.2)$, $(\eta=3,\zeta=2,\gamma=0.4)$, and $(\eta=3,\zeta=2,\gamma=0.8)$.
- In the case of given values for set size *k*=2,4,5,6 with cycle number *h*=5, then the sample size based on RSS is *n* = *kh* = 10,20,25,30. Also, the sample size *n* in case of SRS is *n* = 10,20,25,30.
- Create SRS and RSS from the UGG distribution sample using the R-4.3.0 program's "RSSampling" package.
- Find the ML and Bayesian estimates of the the UGG distribution parameters using the simulated data.
- 5000 times through the steps above.
- Mean squared errors (MSE), and biases are used to compare the suggested estimates for $\eta,\zeta$ and $\gamma$ using SRS and RSS.
- The width of credible confidence intervals (WCCI) for Bayesian estimation of parameters of the UGG distribution is computed.

All numerical assessments for both classical and Bayesian estimators were carried out using the 'maxLik' package for classical estimators and the 'coda' package for Bayesian estimators in R 4.1.2 software.

Tables 2, 3, 4 allow us to draw the following conclusions:

- In every calculation, as $n$ is raised, the bias, MSE, and WCCI decrease.
- As the number of cycles $h$ in the RSS sample increases, the Bias, MSE, and significance for UGG distribution parameters fall while efficiency increases.
- The relevance of the Bias, and MSE for UGG distribution parameters decreases as the sample size for each cycle $h$ in the RSS sampling rises, but the efficiency rises.
- When $\eta$ increases, then the MSE of $\eta$ and $\gamma$ estimates increases while the MSE of $\zeta$ estimates decreases.
- When $\gamma$ increases, then the MSE of all estimates increases.
- The estimates of LLF are better than estimates of SELF for Bayesian estimation.
- The Bayesian estimates of LLF with positive weight are better than the Bayesian estimates of LLF with negative weight value.

Table 2: The biases and MSEs for estimates under SRS and RSS (with 5 cycles) at $\zeta$ = 0.5, $\gamma$ = 0.2

| $\zeta$ =0.5, $\gamma$ = 0.2 | | | SRS | | | | | | | | | | | RSS | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | MLE | | SELF | | | LLF $c$ =-1.5 | | | LLF $c$ =1.5 | | | MLE | | SELF | | | LLF $c$ =-1.5 | | | LLF $c$ =1.5 | | |
| $\eta$ | $h$ | | Bias | MSE | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | WCCI |
| 0.5 | 2 | $\eta$ | -0.2454 | 0.1089 | 0.0095 | 0.0629 | 0.9136 | 0.0344 | 0.0725 | 0.9627 | -0.0149 | 0.0553 | 0.8566 | -0.2494 | 0.0994 | 0.0404 | 0.0608 | 0.8770 | 0.0630 | 0.0702 | 0.9086 | -0.2494 | 0.0530 | 0.8236 |
| | | $\zeta$ | 0.6091 | 0.4682 | 0.2331 | 0.0910 | 0.7086 | 0.2565 | 0.1084 | 0.7504 | 0.2092 | 0.0748 | 0.6597 | 0.5672 | 0.3949 | 0.1485 | 0.0513 | 0.6136 | 0.1633 | 0.0599 | 0.6686 | 0.5672 | 0.0436 | 0.5806 |
| | | $\gamma$ | 0.1701 | 0.2782 | -0.0021 | 0.0964 | 1.1535 | 0.0354 | 0.0928 | 1.1229 | -0.0400 | 0.1037 | 1.1756 | 0.1567 | 0.2031 | -0.0580 | 0.0920 | 1.1355 | -0.0201 | 0.0819 | 1.0885 | 0.1567 | 0.1059 | 1.1928 |
| | 4 | $\eta$ | -0.2628 | 0.0951 | 0.0119 | 0.0213 | 0.5236 | 0.0210 | 0.0221 | 0.5357 | 0.0029 | 0.0208 | 0.5263 | -0.2541 | 0.0786 | 0.0087 | 0.0201 | 0.5381 | 0.0153 | 0.0207 | 0.5477 | -0.2541 | 0.0195 | 0.5309 |
| | | $\zeta$ | 0.5603 | 0.3619 | 0.1697 | 0.0404 | 0.4106 | 0.1786 | 0.0449 | 0.4342 | 0.1607 | 0.0361 | 0.3849 | 0.5000 | 0.2826 | 0.1307 | 0.0277 | 0.3814 | 0.1356 | 0.0299 | 0.3866 | 0.5000 | 0.0255 | 0.3650 |
| | | $\gamma$ | 0.1924 | 0.1356 | 0.0077 | 0.0305 | 0.6560 | 0.0192 | 0.0313 | 0.6546 | -0.0039 | 0.0301 | 0.6513 | 0.1833 | 0.0689 | -0.0082 | 0.0288 | 0.6525 | 0.0024 | 0.0286 | 0.6623 | 0.1833 | 0.0292 | 0.6501 |
| | 5 | $\eta$ | -0.2627 | 0.0815 | 0.0006 | 0.0156 | 0.4889 | 0.0068 | 0.0158 | 0.4831 | -0.0056 | 0.0156 | 0.4894 | -0.2549 | 0.0732 | 0.0028 | 0.0130 | 0.4491 | 0.0075 | 0.0130 | 0.4520 | -0.2549 | 0.0130 | 0.4476 |
| | | $\zeta$ | 0.5338 | 0.3174 | 0.1677 | 0.0370 | 0.3405 | 0.1745 | 0.0401 | 0.3525 | 0.1607 | 0.0340 | 0.3284 | 0.4832 | 0.2572 | 0.1273 | 0.0231 | 0.2992 | 0.1310 | 0.0247 | 0.3160 | 0.4832 | 0.0216 | 0.2870 |
| | | $\gamma$ | 0.2144 | 0.1199 | 0.0057 | 0.0212 | 0.5855 | 0.0137 | 0.0214 | 0.5854 | -0.0024 | 0.0212 | 0.5881 | 0.1689 | 0.0679 | -0.0013 | 0.0193 | 0.5403 | 0.0057 | 0.0195 | 0.5405 | 0.1894 | 0.0193 | 0.5331 |
| | 6 | $\eta$ | -0.2555 | 0.0826 | 0.0224 | 0.0078 | 0.3325 | 0.0253 | 0.0080 | 0.3374 | 0.0195 | 0.0076 | 0.3316 | -0.2495 | 0.0686 | 0.0118 | 0.0058 | 0.2830 | 0.0141 | 0.0058 | 0.2793 | -0.2495 | 0.0058 | 0.2805 |
| | | $\zeta$ | 0.5242 | 0.3097 | 0.1147 | 0.0183 | 0.2602 | 0.1184 | 0.0195 | 0.2668 | 0.1108 | 0.0171 | 0.2548 | 0.4632 | 0.2324 | 0.1069 | 0.0138 | 0.1927 | 0.1086 | 0.0143 | 0.1968 | 0.4632 | 0.0133 | 0.1858 |
| | | $\gamma$ | 0.2251 | 0.0958 | -0.0070 | 0.0076 | 0.3430 | -0.0039 | 0.0076 | 0.3438 | -0.0100 | 0.0076 | 0.3416 | 0.1987 | 0.0543 | -0.0065 | 0.0065 | 0.2948 | -0.0039 | 0.0065 | 0.2947 | 0.1987 | 0.0065 | 0.2965 |
| 1.5 | 2 | $\eta$ | -0.2760 | 0.2328 | -0.0187 | 0.1102 | 1.2949 | 0.0246 | 0.1129 | 1.2856 | -0.0609 | 0.1127 | 1.2708 | -0.2183 | 0.1173 | -0.0294 | 0.0961 | 1.1556 | 0.0065 | 0.0967 | 1.1838 | -0.2183 | 0.0981 | 1.1468 |
| | | $\zeta$ | 0.5004 | 0.3215 | 0.1598 | 0.0495 | 0.5766 | 0.1808 | 0.0604 | 0.6137 | 0.1390 | 0.0402 | 0.5440 | 0.4539 | 0.2648 | 0.0865 | 0.0249 | 0.4608 | 0.0957 | 0.0283 | 0.4894 | 0.4539 | 0.0219 | 0.4412 |
| | | $\gamma$ | 0.2813 | 0.7113 | -0.0191 | 0.1041 | 1.1398 | 0.0221 | 0.0964 | 1.0812 | -0.0608 | 0.1161 | 1.1915 | 0.3991 | 0.4657 | -0.0511 | 0.0955 | 1.1239 | -0.0142 | 0.0854 | 1.0880 | 0.3991 | 0.1091 | 1.1712 |
| | 4 | $\eta$ | -0.2347 | 0.2242 | 0.0052 | 0.0286 | 0.6040 | 0.0165 | 0.0293 | 0.6005 | -0.0063 | 0.0283 | 0.6035 | -0.2031 | 0.1039 | -0.0125 | 0.0287 | 0.6565 | -0.0026 | 0.0282 | 0.6463 | -0.3124 | 0.0295 | 0.6534 |
| | | $\zeta$ | 0.4550 | 0.2397 | 0.1043 | 0.0188 | 0.3416 | 0.1115 | 0.0208 | 0.3456 | 0.0970 | 0.0169 | 0.3369 | 0.4061 | 0.1806 | 0.0633 | 0.0086 | 0.2599 | 0.0656 | 0.0090 | 0.2637 | 0.4061 | 0.0082 | 0.2576 |
| | | $\gamma$ | 0.3934 | 0.4240 | 0.0021 | 0.0333 | 0.7049 | 0.0136 | 0.0335 | 0.6990 | -0.0093 | 0.0333 | 0.7003 | 0.4631 | 0.3425 | -0.0233 | 0.0301 | 0.6515 | -0.0123 | 0.0293 | 0.6463 | 0.4631 | 0.0312 | 0.6516 |
| | 5 | $\eta$ | -0.2296 | 0.2050 | -0.0098 | 0.0199 | 0.5292 | -0.0024 | 0.0194 | 0.5287 | -0.0173 | 0.0205 | 0.5380 | -0.2002 | 0.0478 | -0.0083 | 0.0169 | 0.4746 | -0.0017 | 0.0165 | 0.4700 | -0.2002 | 0.0173 | 0.4840 |
| | | $\zeta$ | 0.4276 | 0.2070 | 0.1028 | 0.0166 | 0.2954 | 0.1084 | 0.0181 | 0.3040 | 0.0972 | 0.0151 | 0.2844 | 0.3325 | 0.1241 | 0.0645 | 0.0074 | 0.2220 | 0.0660 | 0.0077 | 0.2264 | 0.3325 | 0.0071 | 0.2182 |
| | | $\gamma$ | 0.2685 | 0.4058 | -0.0025 | 0.0199 | 0.5454 | 0.0054 | 0.0197 | 0.5438 | -0.0104 | 0.0203 | 0.5504 | 0.4967 | 0.3115 | -0.0039 | 0.0184 | 0.5271 | 0.0030 | 0.0185 | 0.5250 | 0.4967 | 0.0184 | 0.5278 |
| | 6 | $\eta$ | -0.2036 | 0.1925 | 0.0057 | 0.0082 | 0.3564 | 0.0088 | 0.0083 | 0.3567 | 0.0026 | 0.0082 | 0.3567 | -0.1926 | 0.0409 | 0.0044 | 0.0072 | 0.3257 | 0.0071 | 0.0071 | 0.3272 | -0.2619 | 0.0072 | 0.3289 |
| | | $\zeta$ | 0.3944 | 0.1923 | 0.0668 | 0.0087 | 0.2437 | 0.0696 | 0.0092 | 0.2493 | 0.0640 | 0.0081 | 0.2391 | 0.3154 | 0.1135 | 0.0551 | 0.0046 | 0.1486 | 0.0560 | 0.0047 | 0.1497 | 0.3535 | 0.0045 | 0.1481 |
| | | $\gamma$ | 0.2377 | 0.3958 | -0.0110 | 0.0085 | 0.3397 | -0.0078 | 0.0084 | 0.3456 | -0.0141 | 0.0087 | 0.3425 | 0.5190 | 0.3051 | -0.0068 | 0.0083 | 0.3503 | -0.0038 | 0.0083 | 0.3522 | 0.5190 | 0.0083 | 0.3492 |

Table 3: The biases and MSEs for estimates under SRS and RSS (with 5 cycles) at $\zeta$ = 1.5, $\gamma$ = 0.2

| $\zeta$ =1.5, $\gamma$ = 0.2 | | | SRS | | | | | | | | | | | | RSS | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | MLE | | SELF | | | LLF $c$ =-1.5 | | | LLF $c$ =1.5 | | | MLE | | SELF | | | LLF $c$ =-1.5 | | | LLF $c$=1.5 | | |
| $\eta$ | $h$ | | Bias | MSE | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | WCCI |
| 0.5 | 2 | $\eta$ | -0.0029 | 0.0558 | 0.0836 | 0.0516 | 0.7506 | 0.1100 | 0.0636 | 0.8094 | 0.0580 | 0.0421 | 0.7004 | 0.0265 | 0.0377 | 0.0269 | 0.0352 | 0.7286 | 0.1055 | 0.0612 | 0.7678 | 0.0265 | 0.0428 | 0.6845 |
| | | $\zeta$ | 0.7674 | 0.7211 | 0.1468 | 0.0959 | 1.1013 | 0.1830 | 0.1159 | 1.1558 | 0.1103 | 0.0794 | 0.9854 | 0.7452 | 0.6725 | 0.0618 | 0.0743 | 1.0351 | 0.0903 | 0.0829 | 1.0623 | 0.7452 | 0.0676 | 0.9941 |
| | | $\gamma$ | 0.3260 | 0.5975 | -0.0243 | 0.1117 | 1.1897 | 0.0157 | 0.1034 | 1.1413 | -0.0643 | 0.1237 | 1.2631 | 0.4165 | 0.4764 | -0.0601 | 0.1071 | 1.1592 | -0.0207 | 0.0947 | 1.1079 | 0.4165 | 0.1232 | 1.1949 |
| | 4 | $\eta$ | -0.0070 | 0.0311 | 0.0617 | 0.0171 | 0.4546 | 0.0698 | 0.0190 | 0.4663 | 0.0536 | 0.0155 | 0.4378 | -0.0400 | 0.0121 | 0.0399 | 0.0114 | 0.3869 | 0.0584 | 0.0147 | 0.3930 | -0.0400 | 0.0130 | 0.3786 |
| | | $\zeta$ | 0.6985 | 0.5634 | 0.0886 | 0.0293 | 0.5656 | 0.0995 | 0.0326 | 0.5897 | 0.0775 | 0.0263 | 0.5474 | 0.6355 | 0.4562 | 0.0551 | 0.0226 | 0.5503 | 0.0635 | 0.0243 | 0.5528 | 0.6355 | 0.0211 | 0.5420 |
| | | $\gamma$ | 0.4100 | 0.3600 | -0.0199 | 0.0319 | 0.7077 | -0.0081 | 0.0307 | 0.6974 | -0.0317 | 0.0334 | 0.7098 | 0.3893 | 0.3075 | -0.0254 | 0.0308 | 0.6434 | -0.0141 | 0.0303 | 0.6301 | 0.3893 | 0.0317 | 0.6548 |
| | 5 | $\eta$ | -0.0334 | 0.0229 | 0.0530 | 0.0127 | 0.3747 | 0.0584 | 0.0137 | 0.3819 | 0.0476 | 0.0117 | 0.3668 | -0.0414 | 0.0120 | 0.0425 | 0.0099 | 0.3521 | 0.0454 | 0.0104 | 0.3549 | -0.0414 | 0.0095 | 0.3506 |
| | | $\zeta$ | 0.7007 | 0.5502 | 0.0636 | 0.0200 | 0.4824 | 0.0712 | 0.0218 | 0.4858 | 0.0560 | 0.0184 | 0.4687 | 0.6196 | 0.4301 | 0.0702 | 0.0183 | 0.4261 | 0.0762 | 0.0198 | 0.4362 | 0.6196 | 0.0170 | 0.4154 |
| | | $\gamma$ | 0.4200 | 0.3529 | -0.0041 | 0.0211 | 0.5550 | 0.0034 | 0.0210 | 0.5583 | -0.0115 | 0.0214 | 0.5535 | 0.4405 | 0.2817 | -0.0091 | 0.0206 | 0.5779 | -0.0020 | 0.0204 | 0.5743 | 0.4405 | 0.0209 | 0.5775 |
| | 6 | $\eta$ | -0.0440 | 0.0214 | 0.0281 | 0.0062 | 0.2763 | 0.0307 | 0.0065 | 0.2771 | 0.0255 | 0.0059 | 0.2752 | -0.0524 | 0.0110 | 0.0436 | 0.0050 | 0.2079 | 0.0450 | 0.0052 | 0.2095 | -0.0524 | 0.0049 | 0.2042 |
| | | $\zeta$ | 0.6901 | 0.5214 | 0.0431 | 0.0083 | 0.3064 | 0.0462 | 0.0088 | 0.3080 | 0.0400 | 0.0078 | 0.3040 | 0.6075 | 0.4041 | 0.0478 | 0.0075 | 0.2768 | 0.0503 | 0.0079 | 0.2823 | 0.6075 | 0.0072 | 0.2721 |
| | | $\gamma$ | 0.3894 | 0.3309 | -0.0070 | 0.0073 | 0.3234 | -0.0041 | 0.0073 | 0.3229 | -0.0099 | 0.0074 | 0.3230 | 0.4440 | 0.2557 | -0.0101 | 0.0083 | 0.3502 | -0.0072 | 0.0082 | 0.3523 | 0.4440 | 0.0084 | 0.3503 |
| 1.5 | 2 | $\eta$ | 0.0817 | 0.2570 | 0.0573 | 0.0951 | 1.1525 | 0.0983 | 0.1072 | 1.1721 | 0.0162 | 0.0872 | 1.1120 | 0.1715 | 0.2544 | 0.0606 | 0.0924 | 1.1284 | 0.0946 | 0.1030 | 1.1795 | 0.1715 | 0.0845 | 1.0870 |
| | | $\zeta$ | 1.1254 | 1.4137 | 0.1448 | 0.1042 | 1.1794 | 0.1853 | 0.1284 | 1.2490 | 0.1041 | 0.0841 | 1.0978 | 1.0486 | 1.2615 | 0.1336 | 0.0795 | 0.9670 | 0.1633 | 0.0944 | 1.0147 | 1.0486 | 0.0669 | 0.9096 |
| | | $\gamma$ | 0.2603 | 1.1717 | -0.0504 | 0.1043 | 1.1588 | -0.0082 | 0.0920 | 1.1133 | -0.0923 | 0.1206 | 1.1960 | 0.4223 | 0.6716 | -0.0813 | 0.1074 | 1.1553 | -0.0415 | 0.0932 | 1.1089 | 0.4223 | 0.1262 | 1.1877 |
| | 4 | $\eta$ | -0.0523 | 0.1975 | 0.0353 | 0.0278 | 0.6204 | 0.0464 | 0.0291 | 0.6332 | 0.0243 | 0.0268 | 0.6220 | -0.0026 | 0.1226 | 0.0568 | 0.0266 | 0.5883 | 0.0664 | 0.0287 | 0.6171 | -0.0026 | 0.0248 | 0.5730 |
| | | $\zeta$ | 1.0700 | 1.2493 | 0.0887 | 0.0316 | 0.5847 | 0.1003 | 0.0348 | 0.6038 | 0.0772 | 0.0287 | 0.5773 | 0.9131 | 0.8883 | 0.0936 | 0.0256 | 0.5027 | 0.1019 | 0.0281 | 0.5112 | 0.9131 | 0.0232 | 0.4908 |
| | | $\gamma$ | 0.2380 | 0.6444 | -0.0024 | 0.0312 | 0.6906 | 0.0090 | 0.0310 | 0.6830 | -0.0139 | 0.0317 | 0.6950 | 0.4262 | 0.4624 | -0.0437 | 0.0296 | 0.6414 | -0.0331 | 0.0275 | 0.6293 | 0.4262 | 0.0319 | 0.6609 |
| | 5 | $\eta$ | -0.0633 | 0.1835 | 0.0156 | 0.0223 | 0.5882 | 0.0231 | 0.0232 | 0.5910 | 0.0080 | 0.0215 | 0.5836 | 0.0496 | 0.0727 | 0.0439 | 0.0170 | 0.4674 | 0.0497 | 0.0180 | 0.4704 | 0.0496 | 0.0162 | 0.4616 |
| | | $\zeta$ | 1.0012 | 1.1492 | 0.0909 | 0.0258 | 0.5207 | 0.0993 | 0.0282 | 0.5364 | 0.0824 | 0.0235 | 0.5011 | 0.8497 | 0.7744 | 0.0993 | 0.0213 | 0.4265 | 0.1051 | 0.0232 | 0.4332 | 0.8497 | 0.0196 | 0.4168 |
| | | $\gamma$ | 0.2037 | 0.6071 | -0.0137 | 0.0200 | 0.5341 | -0.0065 | 0.0198 | 0.5278 | -0.0209 | 0.0203 | 0.5328 | 0.5397 | 0.3826 | -0.0348 | 0.0193 | 0.5069 | -0.0277 | 0.0186 | 0.5020 | 0.5397 | 0.0201 | 0.5099 |
| | 6 | $\eta$ | -0.0891 | 0.1726 | 0.0145 | 0.0080 | 0.3661 | 0.0175 | 0.0082 | 0.3653 | 0.0114 | 0.0079 | 0.3629 | -0.0019 | 0.0699 | 0.0391 | 0.0078 | 0.3161 | 0.0417 | 0.0082 | 0.3165 | -0.0019 | 0.0076 | 0.3147 |
| | | $\zeta$ | 0.8092 | 0.9355 | 0.0331 | 0.0086 | 0.3151 | 0.0361 | 0.0090 | 0.3233 | 0.0301 | 0.0083 | 0.3121 | 0.8067 | 0.7280 | 0.0776 | 0.0113 | 0.2725 | 0.0804 | 0.0119 | 0.2778 | 0.8671 | 0.0106 | 0.2668 |
| | | $\gamma$ | 0.1911 | 0.5739 | -0.0082 | 0.0080 | 0.3412 | -0.0052 | 0.0079 | 0.3395 | -0.0111 | 0.0081 | 0.3433 | 0.5069 | 0.3795 | -0.0289 | 0.0088 | 0.3458 | -0.0259 | 0.0085 | 0.3426 | 0.5069 | 0.0091 | 0.3495 |

Table 4: The biases and MSEs for estimates under SRS and RSS (with 5 cycles) at $\eta = 3, \zeta = 2$

| $\eta=3$, $\zeta=2$ | | | SRS | | | | | | | | | | | | | RSS | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | **MLE** | | **SELF** | | | **LLF $c$=-1.5** | | | **LLF $c$=1.5** | | | **MLE** | | **SELF** | | | **LLF $c$=-1.5** | | | **LLF $c$=1.5** | | |
| $\gamma$ | $h$ | | Bias | MSE | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | Bias | MSE | WCCI | Bias | MSE | WCCI | Bias | MSE | WCCI |
| 0.4 | 2 | $\eta$ | 0.4904 | 0.6371 | 0.0433 | 0.1265 | 1.3854 | 0.0882 | 0.1336 | 1.3662 | -0.0030 | 0.1241 | 1.3527 | 0.4232 | 0.5526 | 0.0435 | 0.1127 | 1.2843 | 0.0861 | 0.1234 | 1.3105 | 0.4232 | 0.1075 | 1.2552 |
| | | $\zeta$ | 1.5890 | 2.9196 | 0.1406 | 0.1189 | 1.1542 | 0.1840 | 0.1414 | 1.2125 | 0.0966 | 0.1001 | 1.1125 | 1.4752 | 2.4784 | 0.1755 | 0.1107 | 1.0812 | 0.2175 | 0.1389 | 1.1615 | 1.4752 | 0.0877 | 1.0102 |
| | | $\gamma$ | -0.0209 | 0.6856 | -0.2007 | 0.0829 | 0.7205 | -0.1680 | 0.0604 | 0.6312 | -0.2343 | 0.1098 | 0.8188 | 0.0024 | 0.3753 | -0.2308 | 0.0986 | 0.7559 | -0.1982 | 0.0744 | 0.6776 | 0.0024 | 0.1264 | 0.8316 |
| | 4 | $\eta$ | 0.4362 | 0.5344 | 0.0273 | 0.0296 | 0.6656 | 0.0390 | 0.0311 | 0.6775 | 0.0157 | 0.0285 | 0.6458 | 0.4916 | 0.2952 | 0.0495 | 0.0343 | 0.6743 | 0.0616 | 0.0369 | 0.6918 | 0.4916 | 0.0322 | 0.6597 |
| | | $\zeta$ | 1.4966 | 2.4664 | 0.0825 | 0.0378 | 0.6972 | 0.0953 | 0.0420 | 0.7185 | 0.0697 | 0.0340 | 0.6721 | 1.2762 | 1.7046 | 0.1352 | 0.0416 | 0.5986 | 0.1476 | 0.0472 | 0.6240 | 1.2762 | 0.0364 | 0.5742 |
| | | $\gamma$ | 0.0525 | 0.3089 | -0.0888 | 0.0212 | 0.4161 | -0.0792 | 0.0184 | 0.3986 | -0.0984 | 0.0243 | 0.4371 | 0.1471 | 0.0406 | -0.1441 | 0.0360 | 0.4581 | -0.1347 | 0.0318 | 0.4324 | 0.1471 | 0.0405 | 0.4763 |
| | 5 | $\eta$ | 0.4691 | 0.4157 | 0.0147 | 0.0209 | 0.5614 | 0.0220 | 0.0213 | 0.5555 | 0.0074 | 0.0206 | 0.5653 | 0.4809 | 0.2830 | 0.0462 | 0.0210 | 0.5262 | 0.0539 | 0.0222 | 0.5272 | 0.4809 | 0.0199 | 0.5113 |
| | | $\zeta$ | 1.4029 | 2.0843 | 0.0916 | 0.0281 | 0.5431 | 0.1004 | 0.0311 | 0.5535 | 0.0827 | 0.0252 | 0.5175 | 1.2716 | 1.6817 | 0.1488 | 0.0354 | 0.4496 | 0.1582 | 0.0395 | 0.4726 | 1.2716 | 0.0316 | 0.4352 |
| | | $\gamma$ | 0.1105 | 0.1341 | -0.0679 | 0.0155 | 0.3482 | -0.0615 | 0.0138 | 0.3459 | -0.0743 | 0.0173 | 0.3542 | 0.1470 | 0.0394 | -0.1300 | 0.0279 | 0.3915 | -0.1233 | 0.0251 | 0.3730 | 0.1470 | 0.0307 | 0.4026 |
| | 6 | $\eta$ | 0.4704 | 0.3919 | 0.0138 | 0.0079 | 0.3409 | 0.0168 | 0.0080 | 0.3388 | 0.0108 | 0.0078 | 0.3397 | 0.4993 | 0.2794 | 0.0377 | 0.0091 | 0.3560 | 0.0408 | 0.0095 | 0.3646 | 0.4993 | 0.0087 | 0.3495 |
| | | $\zeta$ | 1.4109 | 2.1009 | 0.0437 | 0.0105 | 0.3659 | 0.0473 | 0.0111 | 0.3746 | 0.0401 | 0.0098 | 0.3584 | 1.2397 | 1.5884 | 0.0998 | 0.0167 | 0.3149 | 0.1044 | 0.0181 | 0.3268 | 1.2397 | 0.0154 | 0.3097 |
| | | $\gamma$ | 0.1015 | 0.1242 | -0.0304 | 0.0060 | 0.2604 | -0.0278 | 0.0056 | 0.2548 | -0.0331 | 0.0063 | 0.2642 | 0.1566 | 0.0326 | -0.1015 | 0.0144 | 0.2355 | -0.0982 | 0.0135 | 0.2253 | 0.1566 | 0.0154 | 0.2439 |
| 0.8 | 2 | $\eta$ | 0.7057 | 1.1770 | 0.0357 | 0.1149 | 1.3097 | 0.0809 | 0.1255 | 1.3261 | -0.0101 | 0.1101 | 1.2799 | 0.6667 | 1.0187 | 0.0792 | 0.1126 | 1.2821 | 0.1213 | 0.1267 | 1.2917 | 0.6667 | 0.1036 | 1.2425 |
| | | $\zeta$ | 2.1605 | 5.2411 | 0.1915 | 0.1447 | 1.2741 | 0.2421 | 0.1825 | 1.3544 | 0.1412 | 0.1156 | 1.2068 | 2.0626 | 4.6799 | 0.2364 | 0.1377 | 1.0649 | 0.2840 | 0.1745 | 1.1263 | 2.0626 | 0.1065 | 1.0138 |
| | | $\gamma$ | 0.2506 | 1.1855 | -0.0843 | 0.0825 | 0.9673 | -0.0479 | 0.0690 | 0.9107 | -0.1212 | 0.1001 | 1.0422 | 0.2632 | 0.9857 | -0.1326 | 0.0921 | 0.9900 | -0.0941 | 0.0725 | 0.9198 | 0.2632 | 0.1159 | 1.0684 |
| | 4 | $\eta$ | 0.6058 | 0.9273 | 0.0302 | 0.0315 | 0.6420 | 0.0424 | 0.0330 | 0.6495 | 0.0180 | 0.0304 | 0.6361 | 0.6629 | 0.7471 | 0.0898 | 0.0342 | 0.6116 | 0.1024 | 0.0383 | 0.6190 | 0.6629 | 0.0306 | 0.5890 |
| | | $\zeta$ | 2.0876 | 4.6181 | 0.0970 | 0.0396 | 0.6866 | 0.1112 | 0.0450 | 0.7140 | 0.0828 | 0.0348 | 0.6349 | 1.8529 | 3.5966 | 0.1802 | 0.0546 | 0.5909 | 0.1957 | 0.0629 | 0.6276 | 1.8529 | 0.0469 | 0.5522 |
| | | $\gamma$ | 0.3567 | 0.5826 | -0.0153 | 0.0303 | 0.6582 | -0.0041 | 0.0293 | 0.6467 | -0.0266 | 0.0316 | 0.6741 | 0.4333 | 0.3717 | -0.0931 | 0.0323 | 0.5870 | -0.0812 | 0.0282 | 0.5679 | 0.4333 | 0.0369 | 0.6015 |
| | 5 | $\eta$ | 0.6275 | 0.9305 | 0.0249 | 0.0228 | 0.5701 | 0.0329 | 0.0235 | 0.5663 | 0.0169 | 0.0222 | 0.5738 | 0.6970 | 0.6216 | 0.0766 | 0.0235 | 0.5176 | 0.0849 | 0.0259 | 0.5347 | 0.6970 | 0.0214 | 0.5051 |
| | | $\zeta$ | 2.0839 | 4.6825 | 0.1019 | 0.0291 | 0.5230 | 0.1116 | 0.0326 | 0.5373 | 0.0921 | 0.0259 | 0.5076 | 1.7582 | 3.2088 | 0.1714 | 0.0438 | 0.4570 | 0.1830 | 0.0495 | 0.4832 | 1.7582 | 0.0383 | 0.4281 |
| | | $\gamma$ | 0.3849 | 0.5448 | -0.0275 | 0.0215 | 0.5359 | -0.0198 | 0.0205 | 0.5277 | -0.0352 | 0.0226 | 0.5454 | 0.4780 | 0.3354 | -0.0914 | 0.0242 | 0.4782 | -0.0836 | 0.0219 | 0.4628 | 0.4780 | 0.0265 | 0.4905 |
| | 6 | $\eta$ | 0.6191 | 0.8751 | 0.0105 | 0.0084 | 0.3532 | 0.0137 | 0.0085 | 0.3474 | 0.0072 | 0.0083 | 0.3517 | 0.6882 | 0.5874 | 0.0454 | 0.0092 | 0.3177 | 0.0486 | 0.0097 | 0.3218 | 0.6882 | 0.0087 | 0.3092 |
| | | $\zeta$ | 2.0355 | 4.4244 | 0.0504 | 0.0107 | 0.3383 | 0.0541 | 0.0114 | 0.3419 | 0.0467 | 0.0101 | 0.3352 | 1.7416 | 3.1182 | 0.1079 | 0.0179 | 0.2984 | 0.1129 | 0.0194 | 0.3104 | 1.7416 | 0.0164 | 0.2895 |
| | | $\gamma$ | 0.3943 | 0.5257 | -0.0072 | 0.0078 | 0.3397 | -0.0043 | 0.0078 | 0.3374 | -0.0101 | 0.0079 | 0.3422 | 0.5010 | 0.2836 | -0.0554 | 0.0091 | 0.3092 | -0.0521 | 0.0086 | 0.3039 | 0.5010 | 0.0098 | 0.3168 |

## 6. APPLICATION

We take into account two well-known and frequently used datasets to demonstrate the capability of the suggested models. These datasets are referred to as unit interval data, and one of them, household food expenditures, was included in the betareg R package by Zeileis et al. [32]. The household food expenditure data, in particular, focus on the percentage of income spent on food for 38 families in a sizable American city and include details on the perceived in- come and the number of occupants. The second data set, which was reported by [19, 33], consists of 20 observations of the Susquehanna River's greatest flood level near Harrisburg, Pennsylvania (measured in millions of cubic feet per second).

To evaluate the adequacy of the proposed models in describing the considered data, the MLEs of the parameters for the proposed densities models and comparative models are obtained, along with the corresponding standard errors (SE), Kolmogorov–Smirnov (KS) with P-value (KSPV), Cramér–von Mises criterion (CVM), Anderson–Darling (AD) and the values for the Akaike information criterion (AIC), Bayesian IC (BIC), consistent AIC (CAIC), and Hannan-Quinn IC (HQIC).

In this part, we take into account both the ratings for reading accuracy and the percentage of revenue spent on food as follows: 0.2560663, 0.2023231, 0.2911260, 0.1898036, 0.1619337, 0.3682923, 0.2800173, 0.2067752, 0.1604955, 0.2280656, 0.1921144, 0.2541947, 0.3015883, 0.2570303, 0.2914370, 0.3624967, 0.2265521, 0.3086045, 0.3705066, 0.1075258, 0.3306025, 0.2590826, 0.2501853, 0.2387817, 0.4144203, 0.1782736, 0.2250664, 0.2630519, 0.3652334, 0.5612430, 0.2423906, 0.3418765, 0.3485698, 0.3284759, 0.3508731, 0.2353782, 0.5140399, 0.5429749.

The Data II are as follows: 0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74.

The UGPS distributions are compared to the other eight competing models listed below to show the validity and superiority of the proposed models, which are based on the first and second data sets: unit exponentiated half logistic (UEHL) by Hassan et al. [34], Type-II power Topp-Leone exponential (TIIPTLE) by Bantan et al. [35], unit Weibull (UW) by Mazucheli et al. [36], Marshall-Olkin Kumaraswamy (MOK), Kumaraswamy Kumaraswamy (KwKw) by George and Thobias [37], Kumaraswamy (Kw), and Beta distributions. Figures 5 and 6 display the corresponding Box-plot, TTT plot and HRF plot, to ensure that this data is valid to represent this data initially only for each data sets, respectively.

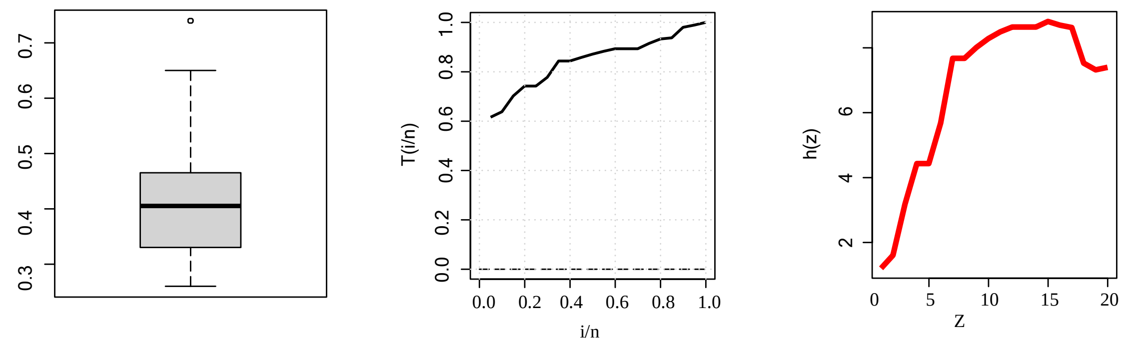


Figure 5: Box, TTT and HRF plots for UGG distribution: Data I

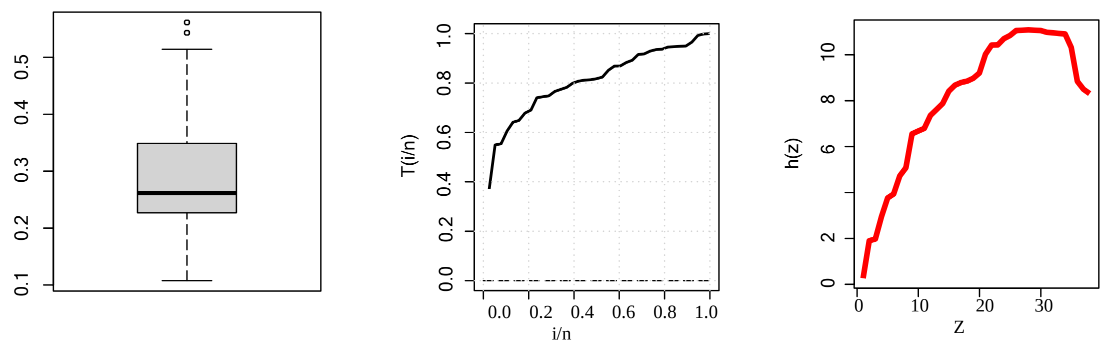


Figure 6: Box, TTT and HRF plots for UGG distribution: Data II

Figures 7, 8 and 9 display the profile log-likelihood of MLE for the UGP, UGG, and UGB parameters with data set I. Table 7 displays the Bayesian estimation for each data sets. They both demonstrate the presence and uniqueness of the MLE. The convergence line of the MCMC results for the UGP, UGG, and UGB parameters using data set I are shown in Figures 10, 11, 12 along with a trace plot of these results. Also, Figures 10, 11, 12 display the density map of the posterior MCMC findings for the UGP, UGG, and UGB parameters for data set I to support the normality plot for MCMC estimation. Furthermore, the estimated PDF, CDF and PP plots for the UGG distribution for data I are provided in Figure 13.

Table 5: MLE with different measure of goodness of fit

| | | estimates | SE | KS | PVKS | AIC | BIC | CAIC | HQIC | CVM | AD |
|---|---|---|---|---|---|---|---|---|---|---|---|
| UGP | $\eta$ | 0.2122 | 0.4240 | 0.1344 | 0.8628 | -26.8062 | -23.8190 | -25.3062 | -26.2231 | 0.0456 | 0.2628 |
| | $\zeta$ | 2.3576 | 1.3973 | | | | | | | | |
| | $\gamma$ | 3.5361 | 3.5506 | | | | | | | | |
| UGG | $\eta$ | 1.8538 | 5.0469 | 0.1305 | 0.8855 | -26.6635 | -23.6763 | -25.1635 | -26.0804 | 0.0437 | 0.2554 |
| | $\zeta$ | 1.1619 | 1.4440 | | | | | | | | |
| | $\gamma$ | 0.9713 | 0.0741 | | | | | | | | |
| UGB | $\eta$ | 0.1918 | 0.3768 | 0.1346 | 0.8619 | -26.8114 | -23.8242 | -25.3114 | -26.2283 | 0.0457 | 0.2634 |
| | $\zeta$ | 2.4188 | 1.3862 | | | | | | | | |
| | $\gamma$ | 0.1960 | 0.2212 | | | | | | | | |
| UEHL | $\delta$ | 3.5190 | 0.5763 | 0.2089 | 0.3472 | -22.6239 | -20.6324 | -21.9180 | -22.2351 | 0.1520 | 0.8943 |
| | $\varphi$ | 7.1457 | 3.0300 | | | | | | | | |
| TLPTLE | $\alpha$ | 0.3018 | 0.7170 | 0.1458 | 0.7886 | -26.2467 | -23.2595 | -24.7467 | -25.6636 | 0.0524 | 0.3164 |
| | $\theta$ | 1.3447 | 0.4483 | | | | | | | | |
| | $\beta$ | 80.6994 | 23.3897 | | | | | | | | |
| Kw | $a$ | 3.3787 | 0.6043 | 0.2176 | 0.2998 | -21.9465 | -19.9551 | -21.2407 | -21.5578 | 0.1673 | 0.9748 |
| | $b$ | 12.0126 | 5.4759 | | | | | | | | |
| Beta | $a$ | 6.8315 | 2.1181 | 0.2063 | 0.3625 | -24.3671 | -22.3757 | -23.6613 | -23.9784 | 0.1267 | 0.7514 |
| | $b$ | 9.2373 | 2.8915 | | | | | | | | |
| UW | $\alpha$ | 1.0249 | 0.2387 | 0.1448 | 0.7956 | -28.3430 | -26.3515 | -27.6371 | -27.9543 | 0.0581 | 0.3441 |
| | $\beta$ | 3.9037 | 0.7077 | | | | | | | | |
| MOK | $\alpha$ | 69.3288 | 18.5832 | 0.1678 | 0.6268 | -18.9886 | -16.0014 | -17.4886 | -18.4055 | 0.1670 | 0.9923 |
| | $\beta$ | 0.8517 | 0.4573 | | | | | | | | |
| | $\theta$ | 6.7450 | 2.1179 | | | | | | | | |
| KwKw | $\alpha$ | 12.1697 | 6.1200 | 0.1865 | 0.4897 | -23.4429 | -19.4599 | -20.7762 | -22.6654 | 0.0797 | 0.4625 |
| | $\beta$ | 2.3118 | 0.0149 | | | | | | | | |
| | $\theta$ | 47.8991 | 0.0138 | | | | | | | | |
| | $\lambda$ | 0.1626 | 0.0381 | | | | | | | | |

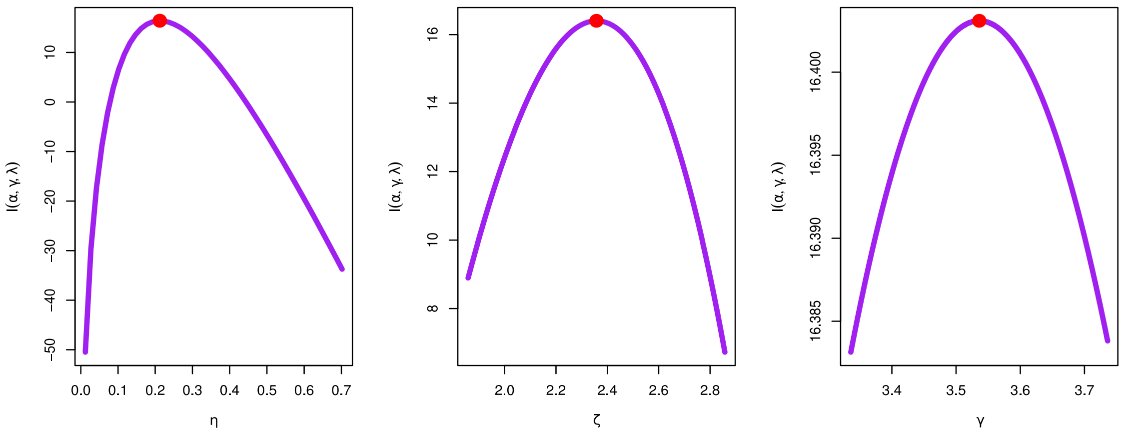


Figure 7: Profile likelihood for UGP distribution: Data I

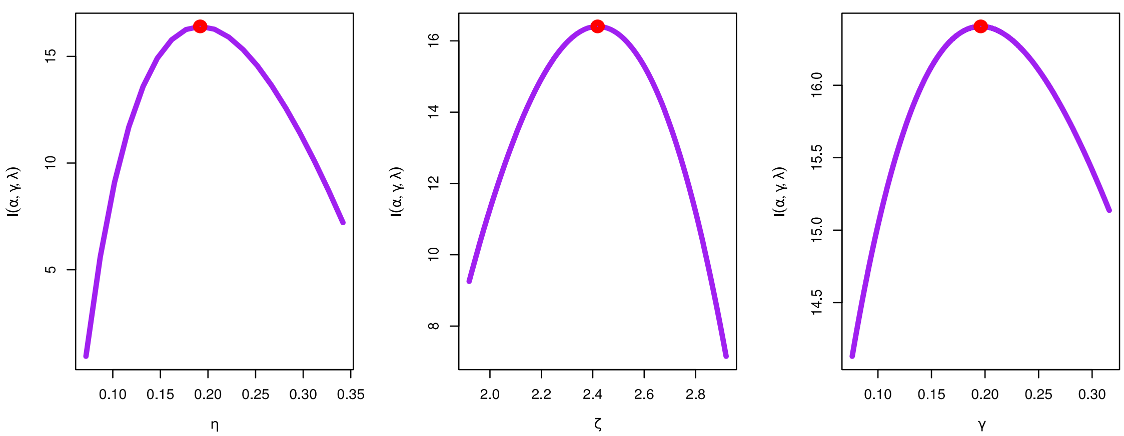


Figure 8: Profile likelihood for UGB distribution: Data I

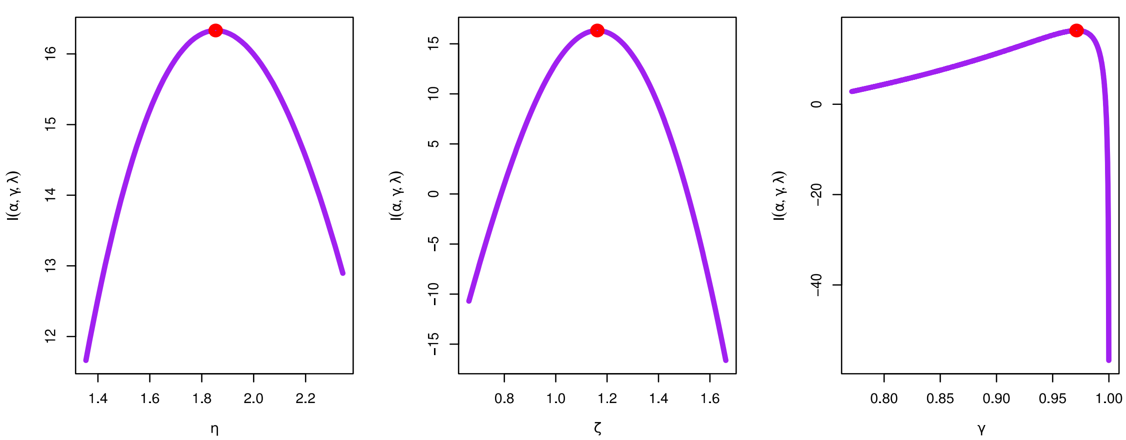


Figure 9: Profile likelihood for UGG distribution: Data I

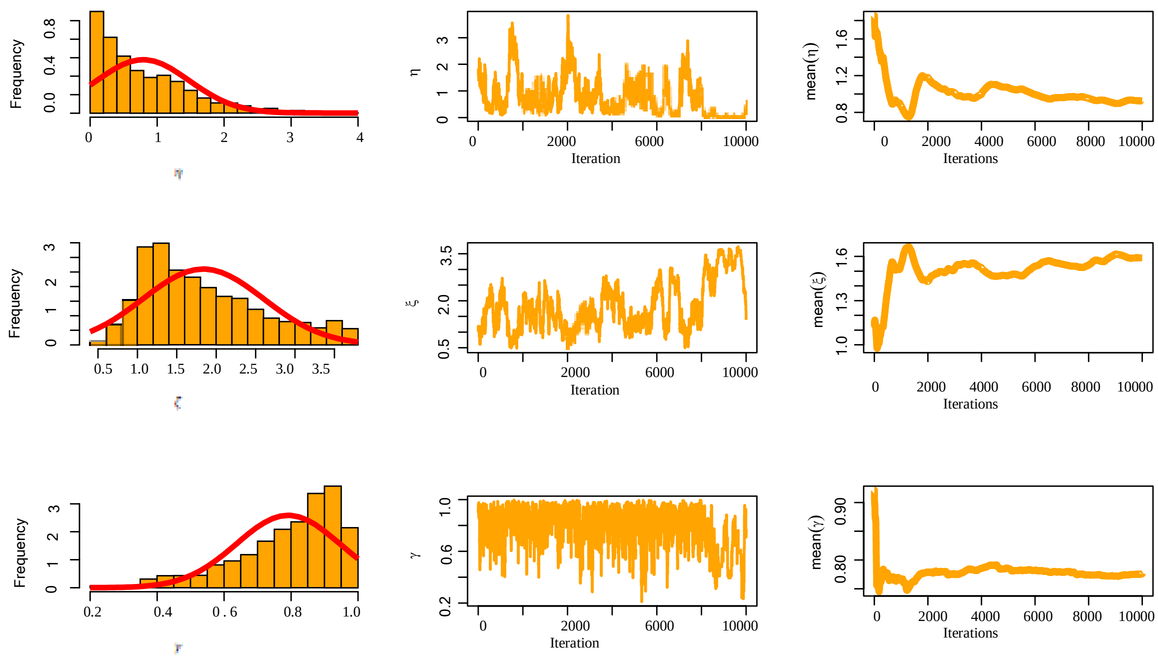

Figure 10: MCMC plots for parameters UGG distribution: Data I

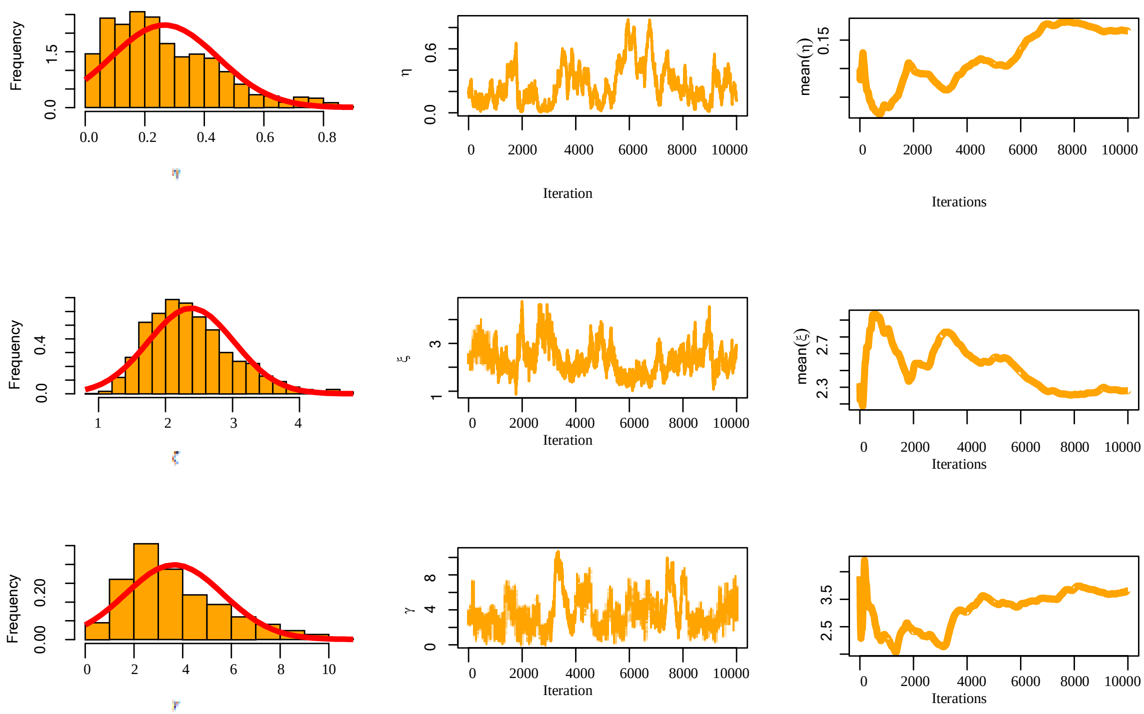

Figure 11: MCMC plots for parameters UGP distribution: Data I

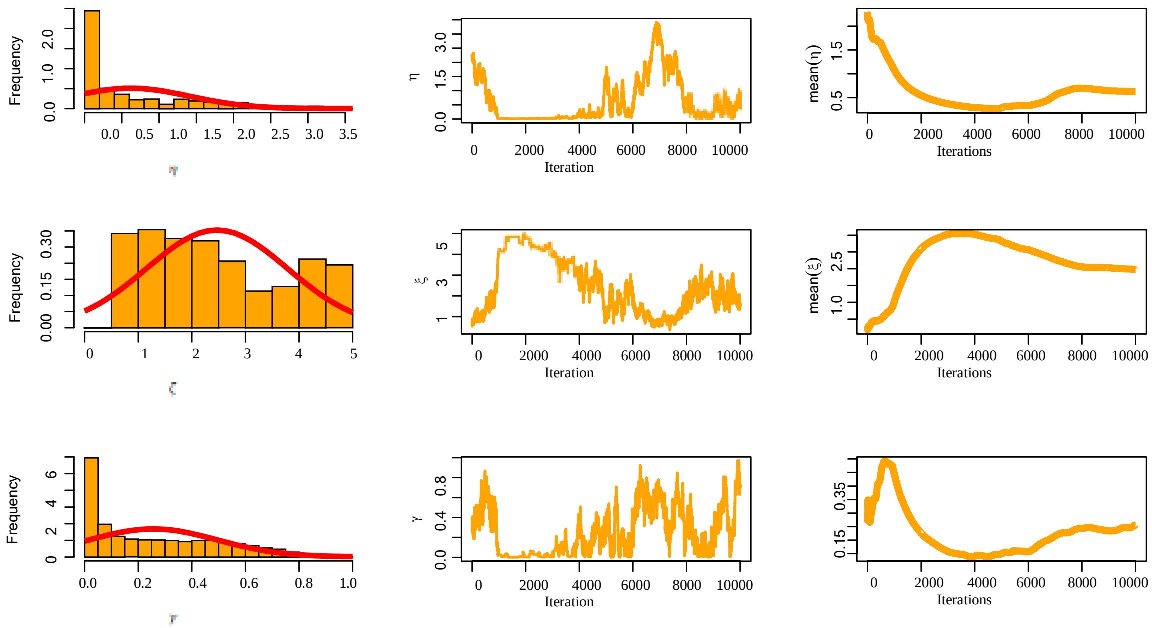


Figure 12: MCMC plots for parameters UGB distribution: Data I

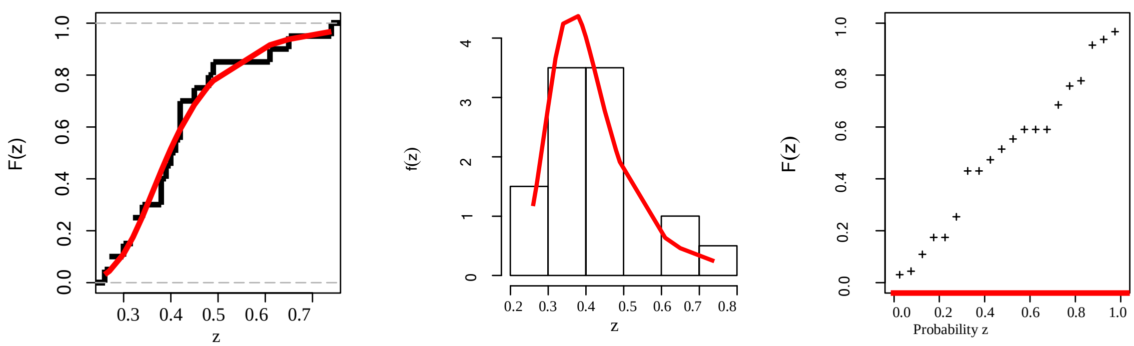


Figure 13: Estimated CDF, PDF, PP plots for the UGG distribution:Data I

Table 6: MLE with different measure of goodness of fit: data 2

| | | estimates | SE | KS | PVKS | AIC | BIC | CAIC | HQIC | CVM | AD |
|---|---|---|---|---|---|---|---|---|---|---|---|
| UGP | $\eta$ | 1.1034 | 1.0631 | 0.0788 | 0.9576 | -66.6675 | -61.7547 | -65.9616 | -64.9196 | 0.0319 | 0.2458 |
| | $\zeta$ | 0.9313 | 0.4148 | | | | | | | | |
| | $\gamma$ | 9.0620 | 4.5785 | | | | | | | | |
| UGG | $\eta$ | 14.8700 | 8.5472 | 0.0693 | 0.9872 | -66.7074 | -61.7946 | -66.0015 | -64.9595 | 0.0264 | 0.1904 |
| | $\zeta$ | 0.2518 | 0.1184 | | | | | | | | |
| | $\gamma$ | 0.9969 | 0.0011 | | | | | | | | |
| UGB | $\eta$ | 1.0146 | 0.9540 | 0.0797 | 0.9536 | -66.6525 | -61.7397 | -65.9466 | -64.9046 | 0.0324 | 0.2500 |
| | $\zeta$ | 0.9613 | 0.4114 | | | | | | | | |
| | $\gamma$ | 0.2814 | 0.1717 | | | | | | | | |
| UEHL | $\delta$ | 3.0100 | 0.3559 | 0.1192 | 0.6103 | -63.5101 | -60.2349 | -63.1672 | -62.3448 | 0.1100 | 0.7991 |
| | $\varphi$ | 14.8214 | 5.6451 | | | | | | | | |
| TLPTLE | $\alpha$ | 15.6653 | 84.6908 | 0.0796 | 0.9540 | -66.6341 | -61.7213 | -65.9282 | -64.8862 | 0.0324 | 0.2504 |
| | $\theta$ | 0.4642 | 0.1691 | | | | | | | | |
| | $\beta$ | 1.1446 | 4.1674 | | | | | | | | |
| Kw | $a$ | 2.9534 | 0.3691 | 0.1237 | 0.5642 | -62.9782 | -59.7030 | -62.6353 | -61.8129 | 0.1192 | 0.8624 |
| | $b$ | 26.9319 | 10.8102 | | | | | | | | |
| Beta | $a$ | 6.0716 | 1.3586 | 0.1101 | 0.7056 | -66.6929 | -61.4177 | -65.3500 | -63.5276 | 0.0695 | 0.5196 |
| | $b$ | 14.8221 | 3.3988 | | | | | | | | |
| UW | $\alpha$ | 0.2320 | 0.0656 | 0.0964 | 0.8384 | -67.8687 | -61.5935 | -65.5259 | -63.7034 | 0.0389 | 0.2964 |
| | $\beta$ | 4.1286 | 0.4995 | | | | | | | | |
| MOK | $\alpha$ | 0.0271 | 0.0443 | 0.0777 | 0.9623 | -67.4351 | -61.5224 | -65.7292 | -64.6872 | 0.0259 | 0.2120 |
| | $\beta$ | 5.1054 | 0.7375 | | | | | | | | |
| | $\theta$ | 19.9298 | 28.8811 | | | | | | | | |
| KwKw | $\alpha$ | 13.8915 | 28.0193 | 0.0875 | 0.9085 | -64.6741 | -58.1238 | -63.4620 | -62.3435 | 0.0372 | 0.2883 |
| | $\beta$ | 0.9971 | 1.5199 | | | | | | | | |
| | $\theta$ | 10.2994 | 25.6245 | | | | | | | | |
| | $\lambda$ | 0.7460 | 1.4955 | | | | | | | | |

Table 7: Bayesian by MCMC results based on SELF for each data

| Data | | UGP | | UGG | | UGB | |
|---|---|---|---|---|---|---|---|
| | | Estimates | SE | Estimates | SE | Estimates | SE |
| I | $\eta$ | 0.2660 | 0.1801 | 0.7894 | 0.6914 | 0.6203 | 0.7888 |
| | $\zeta$ | 2.3803 | 0.6418 | 1.8363 | 0.7672 | 2.4700 | 1.3180 |
| | $\gamma$ | 3.6534 | 2.0106 | 0.7914 | 0.1541 | 0.2554 | 0.2387 |
| II | $\eta$ | 2.1846 | 1.0563 | 15.1700 | 3.3687 | 1.3324 | 0.6916 |
| | $\zeta$ | 0.7052 | 0.2429 | 0.2141 | 0.0460 | 0.9255 | 0.2771 |
| | $\gamma$ | 10.1993 | 2.7434 | 0.9860 | 0.0113 | 0.3446 | 0.1645 |

## 7. CONCLUSION

A novel class of bounded distributions known as UGPS distribution has motivated the introduction in this paper which is produced by compounding the unit Gompertz with the power series distributions. We propose explicit expressions for UGPS's density, cumulative distribution functions, quantiles, moments, entropy, and order statistics. Our work's main contribution relates to the estimation techniques and their comparisons via simulation studies. The mathematical expressions of the UGPS density function and hazard rate function are identified and displayed with graphs. The maximum likelihood and Bayesian estimation techniques based on SRS, and RSS are used to estimate parameters, and the results are applied to the two data sets (household food expenditures, and Susquehanna River's greatest flood level near Harrisburg, Pennsylvania). From the simulation study, it is observed that the method of Bayesian is the better than maximum likelihood method since it has a minimum value of MSE.

## REFERENCES

[1] K. Adamidis, S. Loukas, A lifetime distribution with decreasing failure rate. Stat. Probab. Lett., 39(1), (1998), 35-42.

[2] R. B. Silva, M. Bourguignon, C. R. B. Dias, G.M. Cordeiro, The compound class of extended Weibull power series distributions. Comput. Stat. Data.

Anal., 58, (2013), 352–367.

[3] R. B. Silva, G. M. Corderio, The Burr XII power series distributions: A newcompounding family. Braz. J. Probab. Stat., 29 (3), ( 2015), 565-589.

[4] G. Warahena-Liyanage, M. Pararai, The Lindley power series class of distributions: model, properties and applications. Journal of Computations and Modelling,5(3), (2015), 35-80.

[5] A. A. Jafari, S. Tahmasebi, Gompertz-power series distributions. Commun. Stat. Theory Methods, 45 (13), (2016), 3761–3781

[6] B. Oluyede, P. Mdlongwa, B. Makubate, S. Huang, The Burr-Weibull power series class of distributions. Austrian J. Stat., 48(1), (2018), 1–13

[7] M. Alizadeh, S. F. Bagheri, E. Bahrami-Samani, S. Ghobadi, S. Nadarajah, Exponentiated power Lindley power series class of distributions: Theory and applications. Commun. Stat. Simul. Comput., 47, (2018), 2499–2531.

[8] M. A. Aldahlan, F. Jamal, C. Chesneau, I. Elbatal, M. Elgarhy, Exponentiated power generalized Weibull power series family of distributions: Properties, estimation and applications. PLoS ONE, 15(3), (2020).

[9] P. Osatohanmwen, F. O. Oyegue, S. M. Ogbonmwan, The T–R{Y} power series family of probability distributions. J. Egyptian Math. Soc., 28, 1-18, 2020.

[10] A. S. Hassan, S.M. Assar, A new class of power function distribution: Properties and applications. Ann. Data Sci. 8, (2021), 205–225.

[11] P.A. Rivera, E. Calderı́n-Ojeda, D. I. Gallardo, H. W. Gómez, A compound class of the inverse gamma and power series distributions. Symmetry, 13, (2021), 1328.

[12] A. S. Hassan, E. M. Almetwally, S. C. Gamoura, A. S. M. Metwally, Inverse exponentiated Lomax power series distribution: Model, estimation and application. J. Math., (2022),1998653.

[13] N. Khojastehbakht, A. Ghatari, E. B. Samani, The beta exponential power series distribution. Ann. Data Sci., (2022), 1-22.

[14] M. K. Shakhatreh, S. Dey, D. Kumar, Inverse Lindley power series distributions: a new compounding family and regression model with censored data. J. Appl. Stat., 49(13), (2022), 3451-3476.

[15] S. M. Alghamdi, M. Shrahili, A. S. Hassan, R. E. Mohamed, I. Elbatal, M. Elgarhy, Analysis of milk Production and failure data: Using unit exponentiated half logistic power series class of distributions. Symmetry, 15, (2023), 714.

[16] A. S. Hassan, M. Abd-Allah, Power quasi Lindley power series class of distributions: Theory and applications. Thailand Statistician, 21(2), (2023), 314-336.

[17] M. A. Zayed, A. S. Hassan, E. M. Almetwally, A. M. Aboalkhair, A. H. Al-Nefaie, H. M. Almonry, A compound class of unit Burr XII model: Theory, estimation, fuzzy, and application, Sci. Program., (2023) 17 pages.

[18] F. Willekens, Gompertz in Context: the Gompertz and Related Distributions (pp.105-126). Springer Netherlands, 2001.

[19] J. Mazucheli, A. F. Menezes, S. Dey, Unit-Gompertz distribution with applications. Statistica, 79(1), (2019), 25-43.

[20] D. Kumar, S. Dey, E. Ormoz, S. M. T. K. MirMostafaee, Inference for the unit-Gompertz model based on record values and inter-record times with an application. Rendiconti del Circolo Mat. di Palermo, 2, 69, (2020), 1295-1319.

[21] M. K. Jha, S. Dey, R. M. Alotaibi, Y. M. Tripathi, Reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive Type II censoring. Qual. Reliab. Eng. Int., 36(3), (2020), 965-987.

[22] M. Arshad, Q. J. Azhad, N. Gupta, A. K. Pathak, Bayesian inference of Unit Gompertz distribution based on dual generalized order statistics. Commun. Stat. Simul. Comput., (2021),1-19.

[23] A. Ahmed, N. Aftab, Inference for the unit-Gompertz distribution based on record data. Punjab Univ. J.math., 55(2), (2023).

[24] I. U. Akata, F. C. Opone, F. E. Osagiede, The Kumaraswamy unit-Gompertz distribution and its application to lifetime datasets. Earthline J. Math. Sci., 11(1), (2023), 1-22.

[25] N. Alsadat, A. S. Hassan, M. Elgarhy, C. Chesneau, R. E. Mohamed, An efficient stress–strength reliability estimate of the unit Gompertz distribution using ranked set sampling. Symmetry, 15, (2023), 1121.

[26] A. Noack, A class of random variables with discrete distributions. Ann. Math. Stat., 21(1), (1950), 127-132

[27] A. Renyi, On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics (Vol. 4, pp. 547-562). University of California Press. 1961.

[28] A. E. Punt, T. I. Walker, Stock assessment and risk analysis for the school shark (Galeorhinus galeus) off southern Australia. Marine and Freshwater Research, 49(7), 719-731, 1998

[29] A. E. Punt, D. S. Butterworth, Why do Bayesian and maximum likelihood assessments of the Bering-Chukchi-Beaufort Seas stock of bowhead whales differ?, J. Cetacean Res. Manag., 2(2) , (2000), 125-133.

[30] D. Kundu, D. Mitra, Bayesian inference of Weibull distribution based on left truncated and right censored data. Comput. Stat. Data Anal., 99, (2016), 38-50.

[31] B. C. Arnold, N. Balakrishnan, H. N. Nagaraja, A First Course in Order Statistics. Society for Industrial and Applied Mathematics. 2008.

[32] A. Zeileis, F. Cribari-Neto, B. Gruen, I. Kosmidis, A. B. Simas, A. V. Rocha, M. A. Zeileis, Package ‘betareg’. R package, 3(2), 2016.

[33] R. Dumonceaux, C. E. Antle, Discrimination between the log-normal and the Weibull distributions. Technometrics, 15(4), (1973), 923-926.

[34] A. S. Hassan, A. Fayomi, A. Algarni, E. M. Almetwally, Bayesian and non-Bayesian inference for unit-exponentiated half-logistic distribution with data analysis. Appl. Sci., 12(21), (2022), 11253.

[36] R. A. Bantan, F. Jamal, C. Chesneau, M. Elgarhy, Type II power Topp-Leone generated family of distributions with statistical inference and applications. Symmetry, 12(1), (2020), 75.

[37] J. Mazucheli, A. F. B. Menezes, M. E. Ghitany, The unit-Weibull distribution and associated inference. J. Appl. Probab. Stat, 13(2), (2018), 1-22.

[38] R. George, S. Thobias, Marshall-Olkin Kumaraswamy distribution, Int. Math. Forum, 12 (2), (2017), 47-69.

[39] E. S. A. El-Sherpieny, M. A. Ahmed, On the Kumaraswamy Kumaraswamy distribution. Int. J. Basic Appl. Sci., 3(4), (2014), 372-381.