KUMARASWAMY WEIBULL-GENERATED FAMILY OF DISTRIBUTIONS WITH APPLICATIONS

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Abstract

In this paper, we introduce and study a new family of continuous distributions called Kumaraswamy Weibull-generated (\(KwW-G\)) family of distributions which is an extension of the Weibull-\(G\) family of distributions proposed by Bourguignon in [3]. The new family includes several known models. We obtain general explicit expressions for the quantile function, moments, probability weighted moments, generating function, mean deviation and order statistics. Making use of maximum likelihood method, we discuss the model...
parameters. Furthermore, besides showing the usefulness of our proposed family of distributions by considering a three real data set, evidence of this family to outperform other classes of lifetime models has been noticed.

1. Introduction

Over the past decades, several statistical distributions have been extensively used and applied in several areas such as engineering, actuarial, medical sciences, demography, etc. However, in many situations, the classical distributions are not suitable for describing and predicting real world phenomena. Recently, there has been an increased interest in defining new generators or generalized classes of univariate continuous distributions by introducing additional shape parameter(s) to baseline model. The new generating families extended the well-known distributions and at the same time provide great flexibility in modeling data in practice.

Eugene et al. [8] introduced a new class of generated distributions based on beta distribution, called the beta-generated family of distributions. Zografos and Balakrishnan [14] proposed gamma generated family. Cordeiro and de Castro [4], introduced Kumaraswamy generalized distributions. From an arbitrary parent cumulative distribution function (cdf) $G(x)$, Ristic and Balakrishnan [13] proposed an alternative gamma generator for any parent distribution $G(x)$. Pescim et al. [11] proposed the Kummer beta-generated family of distributions for any parent cdf. Cordeiro et al. [5], proposed the exponentiated generalized class of distributions. Alzaatreh et al. [1], extended the beta-generated family of distributions by using any non-negative continuous random variable $T$ as the generator, in place of the beta random variable. They defined this class as follows:

$$F(x) = \int_0^{W(G(x))} r(t)dt, \quad (1)$$

where $r(t)$ is the probability density function (pdf) of a non-negative
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continuous random variable $T$. In this new class, the distribution of the random variable $T$ is the generator and the upper limit is the transformation $W(G(x)) = -\log[1 - G(x)]$. This generated family of distributions is called “$T$-$X$ distribution”. It is clear that one can define a different upper limit for generating different types of $T$-$X$ distributions. The transformation $W(G(x))$ satisfies the following two conditions: $W(G(x)) \in [0, \infty)$ and it is a monotonic non-decreasing function.

Recently, Bourguignon et al. [3], proposed the Weibull-$G(W-G)$ family of distributions. They considered a continuous distribution $G(\cdot)$ with density function $g(\cdot)$ and the Weibull cdf $R(t) = 1 - e^{-\alpha t^\beta}$ (for $t > 0$) with positive parameters $\alpha$ and $\beta$. Based on this density, by replacing $W(G(x))$ in (1) with $G(x; \xi)/\overline{G}(x; \xi), [\overline{G}(x; \xi) = 1 - G(x; \xi)]$, they defined cdf of their family by

$$F(x; \alpha, \beta, \xi) = \int_0^{G(x; \xi)} \frac{G(x; \xi)}{1 - G(x; \xi)} \alpha \beta t^{\beta-1} e^{-\alpha t^\beta} dt = 1 - e^{-\alpha \left[ \frac{G(x; \xi)}{1 - G(x; \xi)} \right]^\beta},$$

where $G(x; \xi)$ is a baseline cdf, which depends on a parameter vector $\xi$.

In this paper, a new extension of the $W$-$G$ family of distributions called Kumaraswamy Weibull-$G$ family of distributions is proposed. The rest of this article is organized as follows: In Section 2, the Kumaraswamy Weibull-$G$ family of distributions is defined. Some general mathematical properties of the family are discussed in Section 3. Some members of the proposed family are discussed in Section 4. In Section 5, the estimation of the model parameters is performed by the method of maximum likelihood. An illustrative application based on real data is examined in Section 6. Finally, concluding remarks are presented in Section 7.
2. The Kumaraswamy Weibull-generated Family

In this section, the new class of distributions, called $KwW-G$ family of distributions is introduced.

Cordeiro et al. [6] presented the $KwW$ distribution with the following cdf and pdf:

\[
R(t) = 1 - \left[ 1 - \left( 1 - e^{-\alpha t^\beta} \right)^a \right]^b; \quad t > 0; \ a, b, \alpha, \beta > 0,
\]

and

\[
r(t) = ab\alpha\beta t^{\beta-1} e^{-\alpha t^\beta} \left[ 1 - e^{-\alpha t^\beta} \right]^{a-1} \left[ 1 - \left( 1 - e^{-\alpha t^\beta} \right)^a \right]^{b-1},
\]

\[t > 0; \ a, b, \alpha, \beta > 0. \quad (2)\]

We obtain the new family by replacing the generator $r(t)$ defined in (1) by the pdf generator defined in (2) as follows:

\[
F(x) = \int_0^{G(x;\xi)} ab\alpha\beta t^{\beta-1} e^{-\alpha t^\beta} \left[ 1 - e^{-\alpha t^\beta} \right]^{a-1} \left[ 1 - \left( 1 - e^{-\alpha t^\beta} \right)^a \right]^{b-1} dt.
\]

Then, the distribution function of $KwW-G$ family takes the following form:

\[
F(x) = 1 - \left[ 1 - (1 - e^{-\alpha \left[ G(x;\xi) \right]^\beta}) \right]^{a} \quad ; x > 0; \ a, b, \alpha, \beta > 0, \quad (3)
\]

where $a, b, \beta > 0$ are the three shape parameters and $\alpha > 0$ is the scale parameter. The cdf (3) provides a wider family of continuous distributions. The pdf corresponding to (3) is given by

\[
f(x) = ab\alpha\beta \left[ \frac{G(x;\xi)}{1-G(x;\xi)} \right]^{\beta-1} \frac{G(x;\xi)}{1-G(x;\xi)} e^{-\alpha \left[ G(x;\xi) \right]^\beta} \left[ 1 - e^{-\alpha \left[ G(x;\xi) \right]^\beta} \right]^{a-1}
\]

\[
\times \left[ 1 - (1 - e^{-\alpha \left[ G(x;\xi) \right]^\beta}) \right]^{a} \quad ; x > 0; \ a, b, \alpha, \beta > 0. \quad (4)
\]

Hereafter, a random variable $X$ with pdf (4) is denoted by $X \sim KwW-G$. 
Note that.

(i) For \( b = 1 \), the KwW-G distribution reduces to a new family, called exponentiated Weibull-generated (EW-G) family of distributions.

(ii) For \( a = b = 1 \) the KwW-G distribution reduces to the family which is obtained by Bourguignon et al. [3].

The survival, hazard, reversed hazard and cumulative hazard rate functions are obtained, respectively, as follows:

\[
F(x) = \left[ 1 - \left( 1 - e^{-\frac{G(x; \xi)}{1-G(x; \xi)}} \right)^{\frac{1}{a}} \right]^{\frac{1}{b}}; \quad x > 0; \quad a, b, \alpha, \beta > 0,
\]

\[
h(x) = \frac{aba\beta G(x; \xi)^{\beta-1} g(x; \xi) e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \left[ 1 - e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \right]^{a-1}}{(1 - G(x; \xi))^{\beta+1} \left[ 1 - \left( 1 - e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \right)^{a} \right]},
\]

\[
\tau(x) = \frac{\alpha \beta G(x; \xi)^{\beta-1} g(x; \xi) e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \left[ 1 - e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \right]^{a-1}}{(1 - G(x; \xi))^{\beta+1} \left\{ 1 - \left[ 1 - \left( 1 - e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \right)^{a} \right]^{b} \right\}},
\]

and

\[
H(x) = -\ln \left[ 1 - F(x) \right] = -\ln \left[ 1 - \left( 1 - e^{-\alpha \left[ \frac{G(x; \xi)}{1-G(x; \xi)} \right]^{\beta}} \right)^{a} \right]^{b}.
\]

3. Statistical Properties

In this section, some properties of the KwW-G family will be obtained as follows:

3.1. Quantile and median

Quantile functions are used in theoretical aspects of probability theory,
statistical applications and simulations. Simulation methods utilize quantile
function to produce simulated random variables for classical and new
continuous distributions. The quantile function, say \( Q(u) = F^{-1}(u) \) of \( X \) is
given by

\[
u = 1 - [1 - (1 - e^{-\alpha \left\{ \frac{G(Q(u))}{1 - G(Q(u))} \right\}^{\beta}})\\n\]

after some simplifications, it reduces to the following form:

\[
Q(u) = G^{-1}\left[ \frac{\ln[1 - (1 - u)^{\frac{1}{\beta}}]}{1 + \ln[1 - (1 - u)^{\frac{1}{\beta}}]} \right], \tag{5}\n\]

where \( u \) is considered as a uniform random variable on the unit interval \((0, 1)\)
and \( G^{-1}(\cdot) \) is the inverse function of \( G(\cdot) \).

In particular, the median can be derived from (5) by setting \( u = 0.5 \), i.e.,
the median is given by

\[
\text{Median} = G^{-1}\left[ \frac{\ln[1 - (0.5)^{\frac{1}{\beta}}]}{1 + \ln[1 - (0.5)^{\frac{1}{\beta}}]} \right]. \tag{5}\n\]

3.2. Expansions for distribution and density functions

In this section, some representations of the cdf and pdf for
Kumaraswamy Weibull-\( G \) family of distributions will be presented. The
mathematical relation given below will be useful in this section.

It is well-known that, if \( \beta > 0 \) is real non integer and \( |z| < 1 \), the
generalized binomial theorem is written as follows:

\[
(1 - z)^{\beta - 1} = \sum_{i=0}^{\infty} (-1)^i \binom{\beta - 1}{i} z^i. \tag{6}\n\]

Then, by applying the binomial theorem (6) in (4), the probability density
function of \( KwW - G \) family, where \( b \) is real non integer becomes
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\[
f(x) = \frac{ab\alpha \beta G(x; \beta)^{-1} g(x; \xi)}{(1 - G(x; \xi))^{\beta + 1}} \times \sum_{i=0}^{\infty} (-1)^i \binom{b - 1}{i} [1 - e^{-\alpha \frac{G(x; \xi)}{1 - G(x; \xi)} e^{a(i + 1) - 1}}. \]

Then, using binomial expansion again in the previous equation, where \(a\) is real non integer, leads to:

\[
f(x) = \frac{ab\alpha \beta G(x; \beta)^{-1} g(x; \xi)}{(1 - G(x; \xi))^{\beta + 1}} \times \sum_{i, j=0}^{\infty} (-1)^{i+j} \binom{b - 1}{i,j} \left(a(i + 1) - 1\right) \frac{G(x; \xi)^{\beta}}{(1 - G(x; \xi))^{\beta+1}}. \tag{7}
\]

by using the power series for the exponential function, we obtain

\[
e^{-\alpha(j+1)} \left[\frac{G(x; \xi)}{1 - G(x; \xi)}\right]^{\beta} = \sum_{k=0}^{\infty} \left(-1\right)^{k} \frac{a \alpha^{k}(j + 1)^{k}}{k!} \left[\frac{G(x; \xi)}{1 - G(x; \xi)}\right]^{\beta k}. \tag{8}
\]

Inserting expansion (8) in (7) we have

\[
f(x) = ab\alpha \beta g(x; \xi) \sum_{i, j, k=0}^{\infty} (-1)^{i+j+k} \frac{\alpha^{k}(j + 1)^{k}}{k!} \binom{b - 1}{i,j} \left(a(i + 1) - 1\right) \times [G(x; \xi)^{\beta (k+1) - 1}[1 - G(x; \xi)]^{-[\beta (k+1) + 1]]. \tag{9}
\]

Now, using the generalized binomial theorem, we can write

\[
[1 - G(x; \xi)]^{-[\beta (k+1) + 1]} = \sum_{\ell=0}^{\infty} \binom{\beta (k + 1) + \ell}{\ell} (G(x; \xi))^\ell. \tag{10}
\]

Inserting (10) in (9), the probability density function of \(KwW - G\) can
be expressed as an infinite linear combination of exponentiated-\( G \) (exp-\( G \) for short) density functions, i.e.,

\[
f(x) = \sum_{i, j, k, \ell=0}^{\infty} \eta_{i, j, k, \ell} g(x; \xi) G(x; \xi)^{\beta(k+1)+\ell-1},
\]

(11)

then

\[
f(x) = \sum_{i, j, k, \ell=0}^{\infty} W_{i, j, k, \ell} h_{\beta(k+\xi)+\ell}(x; \xi),
\]

(12)

where

\[
W_{i, j, k, \ell} = \frac{a b \alpha^{k+1} (-1)^{i+j+k} (j+1)^{k}}{\beta(k+1) + \frac{h}{k}!} \left( \begin{array}{c} b-1 \\ i \end{array} \right) \left( \begin{array}{c} a(i+1)-1 \\ j \end{array} \right) \left( \begin{array}{c} \beta(k+1) + \ell \\ \ell \end{array} \right),
\]

and

\[
W_{i, j, k, \ell} = \frac{\eta_{i, j, k, \ell}}{\beta(k+1) + \ell}, \quad h_{\alpha}(x; \xi) = a g(x; \xi) G(x; \xi)^{a-1}.
\]

If \( a, b \) and \( \beta \) are integers, the index \( i \) stops at \( b-1 \), \( j \) stops at \( a(i+1)-1 \) and \( \ell \) stops at \( \beta(k+1) + \ell \).

Another form can be yielded when \( \beta \) is real non integer by adding and subtracting 1 to \( G(x; \xi)^{\beta(k+1)+\ell-1} \) into (10) as follows:

\[
f(x) = \sum_{i, j, k, \ell, u=0}^{\infty} \eta_{i, j, k, \ell} g(x; \xi) [1 - [1 - G(x; \xi)]^{\beta(k+1)+\ell-1}].
\]

Using binomial expansion yields:

\[
f(x) = \sum_{i, j, k, \ell, u=0}^{\infty} (-1)^{u} \eta_{i, j, k, \ell} \left( \begin{array}{c} \beta(k+1) + \ell \\ u \end{array} \right) g(x; \xi) [1 - G(x; \xi)]^{u}.
\]

Using binomial expansion more time yields:
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\[
f(x) = \sum_{i, j, k, \ell, u=0}^{\infty} \sum_{m=0}^{u} (-1)^{u+m} \eta_{i, j, k, \ell} \left( \frac{\beta (k+1) + \ell}{u} \right) \left( \frac{u}{m} \right) g(x) (G(x))^m.
\]

So the expansion form of the pdf is as follows:

\[
f(x) = \sum_{i, j, k, \ell, u=0}^{\infty} \sum_{m=0}^{u} \eta_{i, j, k, \ell, u, m} g(x) (G(x))^m,
\]

where

\[
\eta_{i, j, k, \ell, u, m} = (-1)^{u+m} \eta_{i, j, k, \ell} \left( \frac{\beta (k+1) + \ell}{u} \right) \left( \frac{u}{m} \right).
\]

Furthermore, an expansion for the cumulative function is derived as follows:

Using binomial expansion for \([F(x)]^h\), where \(h\) is an integer, leads to:

\[
[F(x)]^h = \sum_{g=0}^{h} (-1)^g \binom{h}{g} [1 - (1 - e^{-\alpha \left[ \frac{G(x; \xi) - 1}{1 - G(x; \xi)} \right]^\beta})^b g].
\]

Using binomial expansion another time, where \(b\) is a real non integer, leads to

\[
[F(x)]^h = \sum_{g=0}^{h} \sum_{p=0}^{\infty} (-1)^g \binom{h}{g} \binom{bg}{p} (1 - e^{-\alpha \left[ \frac{G(x; \xi) - 1}{1 - G(x; \xi)} \right]^\beta})^p.
\]

Using binomial expansion again, where \(a\) is a real non integer, leads to

\[
[F(x)]^h = \sum_{g=0}^{h} \sum_{p, q=0}^{\infty} (-1)^g \binom{h}{g} \binom{bg}{p} \binom{ap}{q} e^{-\alpha q \left[ \frac{G(x; \xi) - 1}{1 - G(x; \xi)} \right]^\beta},
\]

by using the power series for the exponential function (8) in the previous equation, we obtain
$[F(x)]^h = \sum_{g=0}^{\infty} \sum_{p,q,t=0}^{\infty} \frac{(-1)^{g+p+q+t}}{t!} \left( \begin{array}{c} h \\ g \\ p \\ q \\ t \\ u \end{array} \right) (\beta t + u)^{\beta t + u} \left( G(x; \xi) \right)^{\beta t}$,

by using the relation (10) in the previous equation where $\beta$ is real non integer, we obtain

$[F(x)]^h = \sum_{g=0}^{\infty} \sum_{p,q,u,t=0}^{\infty} \frac{(-1)^{g+p+q+t} \left( \begin{array}{c} h \\ g \\ p \\ q \\ t \\ u \end{array} \right) (\beta t)^{\beta t + u} (1 - G(x; \xi))^{\beta t + u}}{t!}$.

Using binomial expansion again, leads to

$[F(x)]^h = \sum_{g=0}^{\infty} \sum_{p,q,u,t=0}^{\infty} \frac{(-1)^{g+p+q+t}}{t!} \left( \begin{array}{c} h \\ g \\ p \\ q \\ t \\ u \end{array} \right) \left( \beta t + u \right)^{\beta t + u} (1 - G(x; \xi))^{\beta t + u}.$

Using binomial expansion again, where $l$ is a real integer, leads to

$[F(x)]^h = \sum_{g=0}^{\infty} \sum_{p,q,t,u,=0}^{\infty} \frac{(-1)^{g+p+q+t+l} \left( \begin{array}{c} h \\ g \\ p \\ q \\ t \\ u \end{array} \right) (\beta t)^{\beta t + u + l}}{l!} (1 - G(x; \xi))^{\beta t + u + l}.$

Replacing $\sum_{l=0}^{\infty} \sum_{z=0}^{\infty} \sum_{l=0}^{\infty} \sum_{z=0}^{\infty}$ with $\sum_{l=0}^{\infty} \sum_{z=0}^{\infty}$ yielding

$[F(x)]^h = \sum_{g=0}^{\infty} \sum_{p,q,t,u=0}^{\infty} \sum_{l=0}^{\infty} \sum_{z=0}^{\infty} \frac{(-1)^{g+p+q+t+l+z} \left( \begin{array}{c} h \\ g \\ p \\ q \\ t \\ u \end{array} \right) (\beta t)^{\beta t + u + l}}{l!} (1 - G(x; \xi))^{\beta t + u + l}.$

Finally,
where

\[ s_z = \sum_{g=0}^{h} \sum_{p,q,l,u=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{g+p+q+l+u} (aq)^t h g b p a q u \beta t \beta t + u \ l \ z. \]

3.3. Moments

If \( X \) has the pdf (11), then its \( r \)th moment can be obtained through the following relation:

\[ \mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx. \quad (15) \]

First, when \( \beta \) is an integer, then substituting (11) into (15) yields:

\[ \mu_r' = E(X^r) = \sum_{i,j,k,\ell=0}^{\infty} \eta_{i,j,k,\ell} \int_{-\infty}^{\infty} x^r g(x; \xi)(G(x; \xi))^{\beta(k+1)+\ell-1} dx. \]

Then

\[ \mu_r' = \sum_{i,j,k,\ell=0}^{\infty} \eta_{i,j,k,\ell} \tau_{r,\beta(k+1)+\ell-1}, \]

where \( \tau_{r,\beta(k+1)+\ell-1} \) is the probability weighted moments (PWMs) of the baseline distribution. Second, when \( \beta \) is a real non integer, then substituting (13) into (15) yields:

\[ \mu_r' = E(X^r) = \sum_{i,j,k,\ell,u,m=0}^{\infty} \sum_{i,j,k,\ell,u,m}^{u} \eta_{i,j,k,\ell,u,m} \int_{-\infty}^{\infty} x^r g(x; \xi)(G(x; \xi))^m dx. \]

Then

\[ \mu_r' = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,\ell,u,m} \tau_{r,m}. \]
Another form based on the parent quantile function:

First, when $\beta$ is an integer yields

$$\mu'_r = \sum_{i, j, k, \ell=0}^{\infty} \eta_{i, j, k, \ell} \int_0^1 (Q_G(u))^\ell u^{\beta(k+1)+\ell-1} du.$$ 

Second, when $\beta$ is a real non integer yields

$$\mu'_r = \sum_{i, j, k, \ell, u=0}^{\infty} \sum_{m=0}^u \eta_{i, j, k, \ell, u, m} \int_0^1 (Q_G(u))^\ell u^m du.$$ 

### 3.4. The probability weighted moments

A general theory for PWMs covers the summarization and description of the theoretical probability distributions, estimation of parameters, quantiles of probability distributions and hypothesis testing for probability distributions. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The probability weighted moments can be obtained from the following relation:

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x) (F(x))^s dx. \quad (16)$$

First, when $\beta$ is an integer, by substituting (11) and (14) into (16), replacing $h$ with $s$, leads to:

$$\tau_{r,s} = \sum_{i, j, k, \ell, u=0}^{\infty} \sum_{z=0}^{\infty} s z \eta_{i, j, k, \ell} \int_{-\infty}^{\infty} x^r g(x; \xi) (G(x; \xi))^{z+\beta(k+1)+\ell-1} dx.$$ 

Then,

$$\tau_{r,s} = \sum_{i, j, k, \ell=0}^{\infty} \sum_{z=0}^{\infty} s z \eta_{i, j, k, \ell} \tau_{r,z+\beta(k+1)+\ell-1}.$$
Second, when $\beta$ is real non integer, by substituting (13) and (14) into (16), replacing $h$ with $s$, leads to:

$$
\tau_{r,s} = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{\infty} s^2 \eta_i, j, k, \ell, u, m \int_{-\infty}^{\infty} x^r g(x; \xi) G(x; \xi)^{m+z} dx.
$$

Then

$$
\tau_{r,s} = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{\infty} s^2 \eta_i, j, k, \ell, u, m \tau_{r,z+m}.
$$

Another form based on the parent quantile function:

First, when $\beta$ is an integer yields

$$
\tau_{r,s} = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{\infty} s^2 \eta_i, j, k, \ell, u, m \int_{0}^{1} (Q_G(u))^r u^{\beta(k+1)+\ell-1} du.
$$

Second, when $\beta$ is real non integer yields

$$
\tau_{r,s} = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{\infty} s^2 \eta_i, j, k, \ell, u, m \int_{0}^{1} (Q_G(u))^r u^{\beta(k+1)+\ell-1} du.
$$

3.5. The moment generating function

Generally, the moment generating function is $M_X(t) = E(e^{tX})$, by using the exponential expansion, it can written as $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$.

Another form based on the parent quantile function: first, when $\beta$ is an integer yields

$$
M_u(t) = \sum_{i,j,k,\ell=0}^{\infty} \eta_i, j, k, \ell \int_{0}^{1} e^{t(Q_G(u)u^\beta(k+1)+\ell-1} du.
$$
Second, when $\beta$ is real non integer yields
\[
M_w(t) = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,\ell,u} \int_{0}^{1} e^{(tQ_G(u))} u^m du.
\]

3.6. The mean deviation

Basically, the mean deviation is a measure for the amount of scatter in $X$.
Generally, the mean deviation is expressed by:
\[
\delta_1(X) = 2\mu F(\mu) - 2T(\mu) \quad \text{and} \quad \delta_2(X) = \mu - 2T(M),
\]
where $T(q) = \int_{-\infty}^{q} xf(x) dx$ which is the first incomplete moment.

Another form based on the parent quantile function:

First, when $\beta$ is an integer yields
\[
T(q) = \sum_{i,j,k,\ell=0}^{\infty} \eta_{i,j,k,\ell} \int_{0}^{q} Q_G(u) u^{\beta(k+1)+\ell+1} du.
\]

Second, when $\beta$ is real non integer yields
\[
T(q) = \sum_{i,j,k,\ell,u=0}^{\infty} \sum_{m=0}^{u} \eta_{i,j,k,\ell,u} \int_{0}^{1} Q_G(u) u^m du.
\]

3.7. Order statistics

Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ be the order statistics of a random sample of size $n$ following the Kumaraswamy Weibull-$G$ family of distributions, with parameters $a$, $b$, $\alpha$, and $\beta$. The pdf of the $k$th order statistic, as mentioned in David [7], can be written as follows:
\[
f_{X(k)}(x(k)) = \frac{f(x(k))}{B(k, n-k+1)} \sum_{v=0}^{n-k} (-1)^v \binom{n-k}{v} F(x(k))^{v+k-1}.
\]
First, when $\beta$ is an integer substituting (11) and (14) in (17), replacing $h$ with $\nu + k - 1$, leads to

$$f_{X(k)}(x(k)) = \frac{g(x(k); \xi)}{B(k, n-k+1)} \times \sum_{\nu=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k,\ell=0}^{\infty} \eta_{i,j,k,\ell} P_{\nu,\ell} G(x(k); \xi)^{z+\beta(k+1)+\ell-1}, \quad (18)$$

where

$$P_{\nu,\ell} = (-1)^{\nu} \binom{n-k}{\nu} s\nu.$$

Second, when $\beta$ is real non integer substituting (13) and (14) in (17), replacing $h$ with $\nu + k - 1$, leads to

$$f_{X(k)}(x(k)) = \frac{g(x(k); \xi)}{B(k, n-k-1)} \sum_{\nu=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k,\ell,u,m=0}^{\infty} \eta_{i,j,k,\ell,u,m} P_{\nu,\ell} G(x(k); \xi)^{m+z}, \quad (19)$$

where $g(\cdot)$ and $G(\cdot)$ are the density and cumulative functions of the KwW-G distributions, respectively.

Moreover, $r$th moment of $k$th order statistics is obtained as follows:

$$E(X_{(k)}^r) = \int_{-\infty}^{\infty} x_{(k)}^r f(x(k)) dx(k). \quad (20)$$

First, when $\beta$ is an integer substituting (18) in (20), leads to

$$E(X_{(k)}^r) = \frac{1}{B(k, n-k+1)} \sum_{\nu=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j,k,\ell=0}^{\infty} \eta_{i,j,k,\ell} P_{\nu,\ell} G(x(k); \xi)^{z+\beta(k+1)+\ell-1}.$$
Second, when $\beta$ is real non integer substituting (19) in (20), leads to

$$E(X_{(k)}^\tau) = \frac{1}{B(k, n - k + 1)} \sum_{v=0}^{\infty} \sum_{z=0}^{\infty} \sum_{i,j,k,\ell,u,m} \eta_{i,j,k,\ell,u,m} p_z v^{\tau} r_z m.$$  

4. Some Special Sub-models

Here, we discuss a few examples of $KwW$-G family of distributions.

4.1. KwW-uniform distribution

As a first example, suppose that the parent distribution is uniform in the interval $0 < x < \theta < \infty$ and $g(x; \theta) = \frac{1}{\theta}$. Therefore, the KwW-uniform (KwU) distribution has the following cdf, pdf by direct substituting $G(x; \theta) = \frac{x}{\theta}$, in (3) and (4) as follows:

$$F(x) = 1 - \left[ 1 - e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \right]^a \psi, \quad a, b, \alpha, \beta > 0, 0 < x < \theta,$$

$$f(x) = \frac{ab\alpha_1 \theta x^{b-1} e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \left[ 1 - e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \right]^{a-1}}{(\theta - x)^{b+1}} \left[ 1 - e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \right]^{\psi-1},$$

$a, b, \alpha, \beta > 0, 0 < x < \theta$.

Furthermore, the survival, hazard rate and reversed hazard rate functions are obtained, respectively, as follows:

$$\bar{F}(x) = \left[ 1 - e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \right]^a \psi,$$

$$h(x) = \frac{ab\alpha_1 \theta x^{b-1} e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \left[ 1 - e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \right]^{a-1}}{(\theta - x)^{b+1}} \left[ 1 - e^{-\alpha_1 \left( \frac{x}{\theta-x} \right)^\beta} \right]^{\psi-1}.$$
and
\[
\tau(x) = a b \alpha \beta x^{\beta - 1} e^{-\alpha \left( \frac{x}{\theta - x} \right)^{\beta}} \left[ 1 - e^{-\alpha \left( \frac{x}{\theta - x} \right)^{\beta}} \right]^{\theta - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\theta - x} \right)^{\beta}} \right]^{\theta} \right]^{\psi - 1}.
\]

4.2. KwW-BurrXII distribution

Let us consider the parent BurrXII distribution with pdf and cdf given by
\[
g(x; c, \sigma, \mu) = c \sigma \mu^{-c} x^{c-1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1}, \quad c, \sigma, \mu > 0 \quad \text{and} \quad G(x; c, \sigma, \mu) = 1 - \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma}, \quad \text{respectively. Then the KwW-BurrXII (KwWBurrXII)}
\]
distribution, has the following cdf, pdf, survival, hazard rate and reversed hazard rate functions:
\[
F(x) = 1 - \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi}, \quad a, b, \alpha, \beta, c, \mu, \sigma > 0, x > 0,
\]
\[
f(x) = \frac{\alpha c \sigma \mu^{-c} x^{c-1} e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-1} - \alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma} \right]^{\beta - 1}}{\left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{\psi - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi - 1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{\psi - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi - 1}},
\]
\[
S(x) = \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi},
\]
\[
H(x) = \frac{\alpha c \sigma \mu^{-c} x^{c-1} e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-1} - \alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma} \right]^{\beta - 1}}{\left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{\psi - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi - 1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{\psi - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi - 1}},
\]
\[
R(x) = \frac{\alpha c \sigma \mu^{-c} x^{c-1} e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-1} - \alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma} \right]^{\beta - 1}}{\left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{\psi - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi - 1} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{\psi - 1} \left[ 1 - \left[ 1 - e^{-\alpha \left( \frac{x}{\mu} \right)^{c} \left[ 1 + \left( \frac{x}{\mu} \right)^{c} \right]^{-\sigma - 1} \right]^{\beta} \right]^{\sigma} \right]^{\psi - 1}}.$$
\[ abca\beta\mu^{-c}x^{-c-1}e^{-a\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-1}}\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\beta-1} \]

\[ h(x) = \frac{x[1-e^{-a\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-1}}]}{\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{1-\sigma}\left[1-\left[1-e^{-a\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-1}}\right]^{\beta}\right]} \]

and

\[ abca\beta\mu^{-c}x^{-c-1}e^{-a\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-1}}\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\beta-1} \]

\[ \tau(x) = \frac{\left[1-\left[1-e^{-a\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-1}}\right]^{\beta}\right]}{\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{1-\sigma}\left[1-\left[1-e^{-a\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-1}}\right]^{\beta}\right]} \]

where \( x > 0, \mu > 0 \) is scale parameter, \( a, b, \alpha, \beta, c, \sigma > 0 \) are shape parameters.

### 4.3. KwW-Weibull distribution

If the random variable \( X \) follows the Weibull distribution with cdf given by

\[ G(x; \lambda, \gamma) = 1 - e^{-\lambda x^{\gamma}}, \quad x, \lambda, \gamma > 0, \] where \( \lambda \) and \( \gamma \) are scale and shape parameters. The pdf and cdf of KwW-Weibull (KwWW) is derived from the KwW-G family of distributions as follows:

\[ F(x) = 1 - \left[1 - (1 - e^{-a(e^{\lambda x^{\gamma}} - 1)^\beta})^\alpha\right]^\beta \quad a, b, \alpha, \beta, \lambda, \gamma > 0, \ x > 0, \]

\[ f(x) = ab\alpha\beta\lambda\gamma^{-a-1}\left[e^{\lambda x^{\gamma}} - 1\right]^\beta \left[1 - (1 - e^{-a(e^{\lambda x^{\gamma}} - 1)^\beta})^\alpha\right]^{\beta-1} \]

\[ \times \left[1 - (1 - e^{-a(e^{\lambda x^{\gamma}} - 1)^\beta})^\alpha\right]^{\beta-1}. \]
Additionally, survival, hazard rate and reversed hazard rate functions of \( KwWW \) take the following forms:

\[
F(x) = \left[ 1 - \left( 1 - e^{-\alpha(e^{\lambda x^\gamma} - 1)^\beta} \right)^a \right]^b,
\]

\[
h(x) = \frac{ab\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^\gamma} - 1]^{\beta-1}e^{-\{\alpha(e^{\lambda x^\gamma} - 1)^\beta - \lambda x^\gamma\}}(1 - e^{-\alpha(e^{\lambda x^\gamma} - 1)^\beta})^{a-1}}{1 - (1 - e^{-\alpha(e^{\lambda x^\gamma} - 1)^\beta})^a},
\]

and

\[
\tau(x) = \frac{ab\alpha\beta\lambda\gamma x^{\gamma-1}[e^{\lambda x^\gamma} - 1]^{\beta-1}e^{-\{\alpha(e^{\lambda x^\gamma} - 1)^\beta - \lambda x^\gamma\}}}{1 - [1 - (1 - e^{-\alpha(e^{\lambda x^\gamma} - 1)^\beta})^a]^b},
\]

respectively. Note that for \( \gamma = 1 \) the \( KwWW \)-Weibull distribution reduces to \( KwWW \)-exponential distribution.

### 4.4. KwW-quasi Lindley distribution

Quasi Lindley distribution is introduced by Shanker and Mishra [12]. They defined the cdf and pdf of the quasi Lindley with two parameters as follows:

\[
G(x; \theta, p) = 1 - e^{-\theta x \left[ 1 + \frac{\theta x}{p + 1} \right]},
\]

\[
g(x; \theta, p) = \frac{\theta}{p + 1} (p + \theta x) e^{-\theta x}.
\]

The cdf and pdf for Kumaraswamy Weibull quasi Lindley (\( KwWQL \)) distribution are derived from (3) and (4), respectively, as follows:

\[
F(x) = 1 - \left[ 1 - \left( 1 - e^{-\alpha\left(1 + \frac{\theta x}{p+1}\right)^{-1}}\right)^\beta \right]^a, a, b, \alpha, \beta, \theta > 0, p > -1, x > 0,
\]
Furthermore, the survival, hazard rate and reversed hazard rate functions are obtained as follows:

\[
\begin{align*}
F(x) &= \left[ 1 - (1 - e^{-\alpha e^{\frac{\theta x}{p+1}}})^{-1} \right]^{-\beta} \\
h(x) &= \frac{\alpha \left( e^{\frac{\theta x}{p+1}} - 1 \right) - \beta \left( e^{\frac{\theta x}{p+1}} - 1 \right)}{(p+1)\left(1 + \frac{\theta x}{p+1}\right)^2 \left[1 - (1 - e^{-\alpha e^{\frac{\theta x}{p+1}}})^{-1} \right]^{-\beta} } \\
\tau(x) &= \frac{\alpha \left( e^{\frac{\theta x}{p+1}} - 1 \right) - \beta \left( e^{\frac{\theta x}{p+1}} - 1 \right)}{(p+1)\left(1 + \frac{\theta x}{p+1}\right)^2 \left[1 - (1 - e^{-\alpha e^{\frac{\theta x}{p+1}}})^{-1} \right]^{-\beta} } \times \left[1 - (1 - e^{-\alpha e^{\frac{\theta x}{p+1}}})^{-1} \right]^{-\beta} \\
\end{align*}
\]

respectively. For \( p = 0 \) the \( KwW \)-Lindley distribution will be obtained.
Plots of the pdf and hazard rate function for the new four special different distributions deduced from \( KwW-G \) family for some parameter values are displayed in Figures 1 and 2, respectively.

**Figure 1.** Density plots (a) \( KwWU \), (b) \( KwWBurrXII \), (c) \( KwWW \) (d) \( KwWQL \).
Figure 2. Hazard plots (a) $K_{wU}$, (b) $K_{WBurrXII}$, (c) $K_{wW}$, (d) $K_{wWQL}$.
5. Maximum Likelihood Estimation

The maximum likelihood estimators of the unknown parameters of the KwW-G family from complete samples are determined. Let \(X_1, ..., X_n\) be observed values from the KwW-G distribution with parameters \(a, b, \alpha, \beta\) and \(\xi\). Let \(\Theta = (a, b, \alpha, \beta, \xi)^T\) be the \(p \times 1\) parameter vector. The total log-likelihood function for the vector of parameters \(\Theta\) can be expressed as

\[
\ln L(\Theta) = n \ln a + n \ln b + n \ln \alpha + n \ln \beta + (\beta - 1) \sum_{i=1}^{n} \ln[G(x_i, \xi)]
\]

\[
- (\beta + 1) \sum_{i=1}^{n} \ln[1 - G(x_i, \xi)] + \sum_{i=1}^{n} \ln[g(x_i, \xi)] - \alpha \sum_{i=1}^{n} [H(x_i, \xi)]^\beta
\]

\[
+ (a - 1) \sum_{i=1}^{n} \ln[1 - e^{-\alpha[H(x_i, \xi)]^\beta}] + (b - 1) \sum_{i=1}^{n} \ln[1 - (1 - e^{-\alpha[H(x_i, \xi)]^\beta})^a],
\]

where \(H(x_i, \xi) = \frac{G(x_i, \xi)}{1 - G(x_i, \xi)}\). The elements of the score function \(U(\Theta) = (U_a, U_b, U_\alpha, U_\beta, U_\xi)\) are given by

\[
U_a = \frac{n}{a} + \sum_{i=1}^{n} \ln[1 - e^{-\alpha[H(x_i, \xi)]^\beta}] - (b - 1) \sum_{i=1}^{n} \frac{(1 - e^{-\alpha[H(x_i, \xi)]^\beta})^a \ln(1 - e^{-\alpha[H(x_i, \xi)]^\beta})}{1 - (1 - e^{-\alpha[H(x_i, \xi)]^\beta})^a},
\]

\[
U_b = \frac{n}{b} + \sum_{i=1}^{n} \ln[1 - (1 - e^{-\alpha[H(x_i, \xi)]^\beta})^a],
\]

\[
U_\alpha = \frac{n}{\alpha} - \sum_{i=1}^{n} [H(x_i, \xi)]^\beta + (a - 1) \sum_{i=1}^{n} \frac{[H(x_i, \xi)]^\beta e^{-\alpha[H(x_i, \xi)]^\beta}}{1 - e^{-\alpha[H(x_i, \xi)]^\beta}}
\]

\[- a(b - 1) \sum_{i=1}^{n} \frac{[H(x_i, \xi)]^\beta e^{-\alpha[H(x_i, \xi)]^\beta} [1 - e^{-\alpha[H(x_i, \xi)]^\beta}]^a - 1}{1 - [e^{-\alpha[H(x_i, \xi)]^\beta}]^a},\]
\[ U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \ln[H(x_i, \xi)] - \alpha \sum_{i=1}^{n} [H(x_i, \xi)]^\beta \ln[H(x_i, \xi)] \]

\[ + \alpha(a - 1) \sum_{i=1}^{n} \frac{e^{-\alpha[H(x_i, \xi)]^\beta}}{1 - e^{-\alpha[H(x_i, \xi)]^\beta}} [H(x_i, \xi)]^\beta \ln[H(x_i, \xi)] \]

\[ - a\alpha(b - 1) \sum_{i=1}^{n} [H(x_i, \xi)]^\beta \ln[H(x_i, \xi)] e^{-\alpha[H(x_i, \xi)]^\beta} \left[ 1 - e^{-\alpha[H(x_i, \xi)]^\beta} \right]^{a-1} \]

and

\[ U_{\xi_k} = (\beta - 1) \sum_{i=1}^{n} \frac{\partial G(x_i, \xi)}{G(x_i, \xi)} \frac{\partial \xi_k}{\partial \xi_k}\]

\[ + \sum_{i=1}^{n} \frac{\partial g(x_i, \xi)}{g(x_i, \xi)} - \alpha \beta \sum_{i=1}^{n} [H(x_i, \xi)]^{\beta-1} \frac{\partial H(x_i, \xi)}{\partial \xi_k} \]

\[ + \alpha\beta(a - 1) \sum_{i=1}^{n} \frac{[H(x_i, \xi)]^{\beta-1} e^{-\alpha[H(x_i, \xi)]^\beta}}{1 - e^{-\alpha[H(x_i, \xi)]^\beta}} \left[ H(x_i, \xi) \right]^{\beta-1} e^{-\alpha[H(x_i, \xi)]^\beta} \]

\[ - a\beta\alpha(b - 1) \sum_{i=1}^{n} \frac{\left[ 1 - e^{-\alpha[H(x_i, \xi)]^\beta} \right]^{a-1} \frac{\partial H(x_i, \xi)}{\partial \xi_k}}{1 - e^{-\alpha[H(x_i, \xi)]^\beta}} \]

Setting \( U_a, U_b, U_\alpha, U_\beta \) and \( U_\xi \) equal to zero and solving the equations simultaneously yields the maximum likelihood estimator \( \hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\alpha}, \hat{\beta}, \hat{\xi}) \) of \( \Theta = (a, b, \alpha, \beta, \xi)^T \). These equations cannot be solved analytically and statistical software can be used to solve them numerically.

For interval estimation of the parameters, the \( 5 \times 5 \) observed information matrix \( I(\Theta) = \{I_{u,v}\} \) for \( (u, v = a, b, \alpha, \beta, \xi) \), whose elements are the
second derivatives of the log-likelihood (\(\ln L\) function. Under the regularity conditions, the known asymptotic properties of the maximum likelihood method ensure that: \(\sqrt{n}(\hat{\Theta} - \Theta) \overset{d}{\to} N(0, I^{-1}(\Theta))\) as \(n \to \infty\), where \(\overset{d}{\to}\) means the convergence in distribution, with mean zero and covariance matrix \(I^{-1}(\Theta)\) then, the \(100(1 - \omega)\%\) confidence interval for \(\Theta = (a, b, \alpha, \beta, \xi)\) is given as follows:

\[
\hat{\Theta} \pm Z_{\omega/2} \sqrt{\text{var}(\Theta)},
\]

where \(Z_{\omega/2}\) is the upper \(100(\omega/2)\) percentile of the standard normal distribution and \(\text{var}(\cdot)\) denote the diagonal elements of \(I^{-1}(\Theta)\) corresponding to the model parameters.

6. Applications

In this section, we explain the flexibility and superiority of new KwWE as special distribution from the Kumaraswamy Weibull-\(G\) family proposed here. We examine three real data sets to compare the fits of KwWE with exponentiated Weibull exponential (EWE) distribution and Weibull exponential (WE) distribution with the following corresponding densities:

\[
f_{\text{KwWE}}(x) = ab\alpha\beta\lambda[e^{\lambda x} - 1]^{\beta-1}e^{-[\alpha(e^{\lambda x} - 1)\beta - \lambda x]}(1 - e^{-\alpha(e^{\lambda x} - 1)\beta})a^{-1}
\]

\[
\times [1 - (1 - e^{-\alpha(e^{\lambda x} - 1)\beta})a]^{\beta-1}, \quad a, b, \alpha, \beta, \lambda > 0, x > 0,
\]

\[
f_{\text{EWE}}(x) = a\alpha\beta\lambda[e^{\lambda x} - 1]^{\beta-1}e^{-[\alpha(e^{\lambda x} - 1)\beta - \lambda x]}(1 - e^{-\alpha(e^{\lambda x} - 1)\beta})a^{-1},
\]

\[
a, \alpha, \beta, \lambda > 0, x > 0,
\]

and

\[
f_{\text{WE}}(x) = \alpha\beta\lambda[e^{\lambda x} - 1]^{\beta-1}e^{-[\alpha(e^{\lambda x} - 1)\beta - \lambda x]}, \quad \alpha, \beta, \lambda > 0, x > 0.
\]
In each real data set, the parameters are estimated by maximum likelihood method. Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan information criterion (HQIC), $-2 \ln L$, the Kolmogorov-Smirnov (K-S) and $p$-value statistics are considered to compare the three models. The formula for these criteria is as follows:

$$ AIC = 2k - 2 \ln L, \quad CAIC = AIC + \frac{2k(k + 1)}{n - k - 1}, $$

$$ BIC = k \ln(n) - 2 \ln L \quad \text{and} \quad HQIC = 2k \ln[\ln(n)] - 2 \ln L, $$

where $k$ is the number of parameters in the statistical model, $n$ is the sample size and $\ln L$ is the maximized value of the log-likelihood function under the considered model. Also, $k - s = \sup_y [F_n(y) - F(y)]$, where $F_n(y) = \frac{1}{n}$ (number of observation $\leq y$), and $F(y)$ denotes the cdf.

In general, the best distribution corresponds to the smallest values of $-2 \ln L, AIC, BIC, CAIC, HQIC, K-S$ and the biggest value of $p$-value criteria.

6.1. The first real data set

The first data set was taken from Hinkley [9], which consists of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul. The data set is as follows:

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05.

Table 1 provides the maximum likelihood estimates (MLEs) of the model parameters. Also, the numerical values of statistics, $AIC, BIC, CAIC, HQIC$ and Kolmogorov-Smirnov values as well as the $p$-value statistics are listed in Table 1. Figures 3 and 4 provide the plots of estimated cumulative and estimated densities of the fitted KwWE, EWE and WE models for the first data set.
Table 1. The MLEs, AIC, CAIC, BIC, HQIC, K-S and p-value of the models based on the first data set

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>KwWE</th>
<th>EWE</th>
<th>WE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\alpha} = 3.901 )</td>
<td>( \hat{\alpha} = 12.402 )</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( \hat{\beta} = 0.578 )</td>
<td>( \hat{\beta} = 0.581 )</td>
<td>( \hat{\beta} = 1.572 )</td>
<td></td>
</tr>
<tr>
<td>( \hat{\lambda} = 4.328 \times 10^{-3} )</td>
<td>( \hat{\lambda} = 8.066 \times 10^{-3} )</td>
<td>( \hat{\lambda} = 0.02 )</td>
<td></td>
</tr>
</tbody>
</table>

| \(-2 \ln L\) | 108.217 | 115.109 | 117.798 |
| AIC | 118.217 | 123.109 | 123.798 |
| CAIC | 120.717 | 124.709 | 124.721 |
| BIC | 115.603 | 121.017 | 122.229 |
| HQIC | 120.458 | 124.902 | 125.142 |
| K-S | 0.06 | 0.089 | 0.0796 |
| p-value | 0.9999 | 0.971 | 0.991 |

Based on the values of AIC, CAIC, BIC, HQIC, K-S and p-value in Table 1, the KwWE model provides better fit than the EWE and WE models.
Figure 3. Estimated cumulative densities for the first data.

Figure 4. Estimated densities of models for the first data.
It is clear from Figure 4 that the fitted density for KwWE model is closer to the empirical histogram than EWE and WE models.

6.2. The second real data set

The second data set was taken from Murthy et al. [10]. The corresponding data is referring to the time between failures for a repairable item. The data set is as follows:

1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

The following Table 2 provides MLEs, the AIC, BIC, CAIC, HQIC and K-S values as well as the p-value statistics for the second data set.

**Table 2.** The MLEs, AIC, CAIC, BIC, HQIC, K-S and p-value of the models based on second data set

<table>
<thead>
<tr>
<th>Model</th>
<th>KwWE</th>
<th>EWE</th>
<th>WE</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \hat{\alpha} ) = 2.797</td>
<td>( \hat{\alpha} ) = 6.355</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>( \hat{\beta} ) = 29.55</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>( \hat{\alpha} ) = 6.521</td>
<td>( \hat{\alpha} ) = 30.227</td>
<td>( \hat{\alpha} ) = 127.933</td>
<td></td>
</tr>
<tr>
<td>( \hat{\beta} ) = 0.593</td>
<td>( \hat{\beta} ) = 0.58</td>
<td>( \hat{\beta} ) = 1.318</td>
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</tr>
<tr>
<td>( \hat{\lambda} ) = 4.326 \times 10^{-3}</td>
<td>( \hat{\lambda} ) = 9.347 \times 10^{-3}</td>
<td>( \hat{\lambda} ) = 0.015</td>
<td></td>
</tr>
<tr>
<td>-2 ln L</td>
<td>106.964</td>
<td>113.344</td>
<td>111.901</td>
</tr>
<tr>
<td>AIC</td>
<td>116.964</td>
<td>121.344</td>
<td>117.901</td>
</tr>
<tr>
<td>CAIC</td>
<td>119.464</td>
<td>122.944</td>
<td>118.824</td>
</tr>
<tr>
<td>BIC</td>
<td>114.35</td>
<td>119.252</td>
<td>116.332</td>
</tr>
<tr>
<td>HQIC</td>
<td>119.205</td>
<td>123.137</td>
<td>119.246</td>
</tr>
<tr>
<td>K-S</td>
<td>0.062</td>
<td>0.078</td>
<td>0.083</td>
</tr>
<tr>
<td>p-value</td>
<td>0.9998</td>
<td>0.993</td>
<td>0.987</td>
</tr>
</tbody>
</table>
In order to assess whether the KwWE model is appropriate, Figures 5 and 6 provide some plots of the estimated cdf as well as the estimated probability densities of the fitted KwWE, EWE and WE models for the second data set.

**Figure 5.** Estimated cumulative densities for the second data.

**Figure 6.** Estimated densities of models for the second data.
From Figure 6, it is concluded that the KwWE distribution is quite suitable for this data set.

6.3. The third real data set

The following data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [2]. The data is as follows:

0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

The following Table 3 gives the MLEs, AIC, BIC, CAIC, HQIC and K-S values as well as the $p$-value statistics for the third data set.

**Table 3.** The MLEs, AIC, CAIC, BIC, HQIC, K-S and $p$-value of the models based on third data set

<table>
<thead>
<tr>
<th>Parameter Estimates</th>
<th>KwWE</th>
<th>EWE</th>
<th>WE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\alpha} = 4.494$</td>
<td>$\hat{\alpha} = 10.973$</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$\hat{b} = 22.374$</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$\hat{\alpha} = 9.739$</td>
<td>$\hat{\alpha} = 32.229$</td>
<td>$\hat{\alpha} = 163.738$</td>
<td></td>
</tr>
<tr>
<td>$\hat{\beta} = 0.556$</td>
<td>$\hat{\beta} = 0.601$</td>
<td>$\hat{\beta} = 1.584$</td>
<td></td>
</tr>
<tr>
<td>$\hat{\lambda} = 4.37 \times 10^{-3}$</td>
<td>$\hat{\lambda} = 0.011$</td>
<td>$\hat{\lambda} = 0.02$</td>
<td></td>
</tr>
<tr>
<td>$-2 \ln L$</td>
<td>265.787</td>
<td>283.242</td>
<td>292.659</td>
</tr>
<tr>
<td><strong>AIC</strong></td>
<td>275.787</td>
<td>291.242</td>
<td>298.659</td>
</tr>
</tbody>
</table>
The values in Table 3, indicate that the KwWE model is a strong competitor among other distributions that was used here to fit this data set. Figures 7 and 8 provide some plots of the estimated cdf and pdf of the fitted KwWE, EWE and WE models for the third data set.

![Figure 7](image)

**Figure 7.** Estimated cumulative densities for the third data.
Based on the plots of Figure 8 one can notice that the fitted density of the KwWE model is closer to the empirical histogram than the corresponding densities of the EWE and WE models.

7. Conclusion

In view of an idea of the family of Weibull-$G$ distributions and Kumaraswamy distribution, we introduce a new family of distributions, called Kumaraswamy Weibull-$G$. The Kumaraswamy Weibull-$G$ family extends several widely known distributions. An expansion for the density function and explicit expressions for the moments, probability weighted moments, generating function, mean deviation, quantile function and order statistics are obtained. The estimation of parameters is approached by the method of maximum likelihood. Some applications of the new family to three real data are given to demonstrate the importance and usefulness of the suggested family. In this study, we wish that this new generalization will
attract a broadly applications in several areas such as engineering, survival and lifetime data, hydrology, etc.

**Acknowledgment**

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**References**


