Truncated Weibull Fréchet Distribution: Statistical Inference and Applications

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A new four-parameter lifetime model, called the Truncated Weibull Fréchet (TWFr) distribution, is proposed. We provide forms for the density, distribution function, hazard function, quantile function, rth moment, incomplete moments, Lorenz and Bonferroni curves, Rényi and q entropies. Different methods to estimate the TWFr distribution parameters are studied. The maximum likelihood, least squares and weighted least squares estimators are obtained. An empirical study is conducted to compare among these estimators. Two real data sets are applied to illustrate the flexibility of the proposed model compared with some models.

Keywords: Fréchet Distribution, Truncated Weibull-G Family, Entropies, Moments, Maximum Likelihood Estimation.

1. INTRODUCTION

The Fréchet (Fr) distribution is useful for modeling and analysis several of extreme events. It has applications in many areas such as; accelerated life testing, queues in supermarkets, earthquakes, floods, horse racing, rainfall, sea waves and wind speeds. Furthermore, applications of this model in various fields are given in Harlow.9

Some extensions of the Fr distribution are available in the literature, such as; exponentiated Fr,17 beta Fr,8 transmuted Fr (Mahmoud and Mandouh), Marshall–Olkin Fr,12 gamma extended Fr,7 transmuted exponentiated Fr,4 Kumaraswamy Fr,15 transmuted Marshall–Olkin Fr,16 Weibull Fr,2 Burr X exponentiated Fréchet.19

The cumulative distribution function (cdf) and probability density function (pdf) of the Fr distribution are given by

\[ G_{Fr}(x; \mu, \delta) = e^{-(\mu/x)^\delta}, \quad x, \mu, \delta > 0 \]  
and,

\[ g_{Fr}(x; \mu, \delta) = \delta \mu x^{-\delta-1} e^{-(\mu/x)^\delta} \]  

where \( \mu \) and \( \delta \) are a scale parameter and a shape parameter, respectively.

Many researchers are interested to expand family of distributions in order to obtain better fit for data analyzing. In the last few years, numerous distributions have been proposed based on an extension of known distributions. So, several ways for generating new distributions from classic ones were developed. Najarzadegan et al.18 proposed a new truncated Weibull-G (TW-G) family of distributions as an alternative to beta-G distribution with flexible hazard rate and greater reliability. The TW-G family has the following cdf and pdf

\[ F_{TW-G}(x; \alpha, \beta) = A(1 - e^{-\alpha(G(x))^{\beta}}), \quad x \in R, \alpha, \beta > 0 \]  

and,

\[ f_{TW-G}(x; \alpha, \beta) = A\alpha\beta g(x)(G(x))^{\beta-1} e^{-\alpha(G(x))^{\beta}}, \quad x \in R, \alpha, \beta > 0 \]  

where \( A = (1 - e^{-\alpha})^{-1} \), \( \alpha \) is scale parameter, \( \beta \) is the shape parameter and \( G(.) \) is cdf of any baseline distribution.

The purpose of this paper is to introduce a new lifetime model based on the TW-G family. The new distribution provides more flexible model compared with cdf (1). We hope that the new model will attract wider applications in some areas. The statistical properties and parameter estimation by three different methods are derived. This paper can be outlined as follows. In the next section, the TWFr distribution is introduced. Section 3 gives some main properties of the TWFr distribution. The maximum likelihood, least squares and weighted least squares methods

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are implemented to obtain the estimators of the population parameters as well as the simulation study is employed in Section 4. Application to a real data illustrating the performance of TWFr distribution is given in Section 5. Concluding remarks appear in the last section.

2. TRUNCATED WEIBULL-FRÈCHET MODEL

A random variable is said to has the TWFr distribution with vector parameters \( \Xi = (\alpha, \beta, \mu, \delta) \), if its cdf is defined by

\[
F_{\text{TWFr}}(x; \Xi) = A(1 - \exp(-ae^{-\beta(x/\alpha)^{\mu}})),
\]

\[
\alpha, \beta, \delta, \mu, x > 0 \quad (5)
\]

The pdf corresponding to (5) is given by

\[
f_{\text{TWFr}}(x; \Xi) = A\alpha\beta\delta x^{-\delta-1}e^{-\beta(x/\alpha)^{\mu}}\exp(-ae^{-\beta(x/\alpha)^{\mu}}),
\]

\[
\alpha, \beta, \delta, \mu, x > 0 \quad (6)
\]

where \( \alpha, \beta \) and \( \mu \) are three scale parameters and \( \delta \) is a shape parameter. A random variable \( X \) that follows the distribution in (6) is denoted by \( X \sim \text{TWFr} (\Xi) \). Two new special sub-models from (6) can be obtained as follows

- For \( \delta = 2 \), the pdf (6) reduces to truncated Weibull inverse Rayleigh distribution (new).
- For \( \delta = 1 \), the pdf (6) reduces to truncated Weibull inverse exponential distribution (new).

The reliability function and the hazard rate function (hrf) are, respectively, given by

\[
\bar{F}_{\text{TWFr}}(x; \Xi) = 1 - A(1 - \exp(-ae^{-\beta(x/\alpha)^{\mu}}))
\]

and,

\[
H_{\text{TWFr}}(x; \Xi) = \frac{A\alpha\beta\delta x^{-\delta-1}e^{-\beta(x/\alpha)^{\mu}}\exp(-ae^{-\beta(x/\alpha)^{\mu}})}{1 - A(1 - \exp(-ae^{-\beta(x/\alpha)^{\mu}}))}
\]

Some descriptive pdf and hrf plots of \( X \sim \text{TWFr} (\Xi) \) are illustrated below for specific parameter choices of \( \Xi \) (see Fig. 1).

As seen from Figure 1, that the pdf and hrf take different forms according to different values of parameters. Furthermore, the reversed hazard rate function and cumulative hazard rate function of \( X \) take the following forms

\[
\tau_{\text{TWFr}}(x; \Xi) = \alpha\beta\delta x^{-\delta-1}e^{-\beta(x/\alpha)^{\mu}}\exp(-ae^{-\beta(x/\alpha)^{\mu}})
\]

\[
\times (1 - \exp(-ae^{-\beta(x/\alpha)^{\mu}}))^{-1}
\]

and,

\[
H_{\text{TWFr}}(x; \alpha, \beta, \delta, \mu) = -\ln(1 - A(1 - \exp(-ae^{-\beta(x/\alpha)^{\mu}})))
\]

Additionally; the quantile function of the TWFr can be generated by inverting cdf (5) as follows

\[
Q(u) = \frac{\mu}{\ln[\ln(1 - u/A) - 1]}/(1/\beta)^{1/\alpha}
\]

Simulating the TWFr random variable is straightforward. If is \( U \) a uniform variate in the unit interval \((0, 1)\), then the random variable \( X = Q(u) \) follows (6).

3. MAIN PROPERTIES

This section studies some mathematical properties of TWFr model.

3.1. The Probability Weighted Moments

The probability weighted moments (PWMs) can be used to estimate the parameters of a distribution do not have a closed form or maximum likelihood estimates are unavailable or difficult to compute. The PWMs are expectations of multiplication of two certain functions of a random variable \( X \) defined as

\[
\tau_{r, s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^s \, dx \quad (7)
\]

Fig. 1. Plots of the (a) pdf and (b) hrf of the TWFr distribution for some parameter values.
We now obtain the

\[ \tau_{r,s} = \frac{\sum_{k=0}^{\infty} \binom{s}{k} \int_0^\infty x^k A^{s+k} \alpha \beta \delta \mu^3 \lambda^{s-k-1} e^{-\beta(x/\lambda)^3} \exp(-\alpha e^{-\beta(x/\lambda)^3})}{k!} dx \]

(8)

For \( s \) is an integer and by using binomial expansion, then (8) will be

\[ \tau_{r,s} = \sum_{k=0}^{\infty} \frac{(-1)^k \binom{s}{k} \int_0^\infty x^k A^{s+k} \alpha \beta \delta \mu^3 \lambda^{s-k-1} e^{-\beta(x/\lambda)^3}}{k!} \]

(9)

Also, using exponential expansion in (9), then

\[ \tau_{r,s} = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+i} \alpha^i (k+1)^i}{i!} \frac{x^s}{k!} A^{k+i} \lambda^{s-k-1} e^{-\beta(x/\lambda)^3} \]

Setting \( y = (\mu/x)^3 \), and after simplification, the \( E(X') \) of TWFr will be

\[ E(X') = A \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i}{i!} \frac{\beta^i}{(i+1)^{-(r/\delta)}} \Gamma \left( 1 - r \frac{1}{\delta} \right), \]

\[ r < \delta, \ r = 1, 2, \ldots, (11) \]

Setting \( r = 1, 2, 3 \) and 4 in (11), we obtain the first four moments about zero. Thus, numerical values of the first four moments, variance \( (\sigma^2) \), coefficient of variation (CV), CS and CK of the TWFr distribution for some values of parameters are displayed in Table 1.

Next, we derive a simple formula for the \( r \)th incomplete moment of \( X \), say \( m_r(z) = P(X > z) \). From (6), we obtain

\[ m_r(z) = A \alpha \beta \delta \mu^3 \int_0^z x^{r-\delta-1} e^{-\beta(x/\lambda)^3} \exp(-\alpha e^{-\beta(x/\lambda)^3}) \]

Using the exponential expansion and and setting \( y = \beta x^3, \) and after simplification, then \( m_r(z) \) of TWFr distribution is

\[ m_r(z) = A \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i \beta^i}{i!} \Gamma \left( 1 - r \frac{1}{\delta} \right), \]

\[ r = 1, 2, \ldots, (12) \]

where, \( \Gamma \left( 1 - \frac{r}{\delta} \right) \) is the incomplete gamma function. The first incomplete moments is derived by substituting \( r = 1 \) in (12). The Lorenz curve; say \( L_r(z) \), and Bonferroni curve; say \( B_r(z) \), are the main applications of the first incomplete moment. The Lorenz and Bonferroni curves are obtained as follows

\[ L_r(z) = \frac{m_r(z)}{E(Z)} \]

\[ B_r(z) = \frac{L_r(z)}{F(Z)} \]

Using the exponential expansion, then (9) will be

\[ E(X') = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} A \alpha^i \beta \int_0^\infty x^{r-\delta-1} e^{-\beta(x/\lambda)^3} \]

(10)

Table 1. Summary statistics of moments of TWFr distribution.

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \delta )</th>
<th>( \mu' )</th>
<th>( \alpha' )</th>
<th>( \beta' )</th>
<th>( \delta' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(X) )</td>
<td>0.666</td>
<td>1.844</td>
<td>1.114</td>
<td>1.505</td>
<td>0.674</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E(X') )</td>
<td>0.498</td>
<td>1.994</td>
<td>1.287</td>
<td>2.315</td>
<td>0.496</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E(X') )</td>
<td>0.458</td>
<td>3.563</td>
<td>1.562</td>
<td>3.658</td>
<td>0.422</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( E(X') )</td>
<td>0.792</td>
<td>17.231</td>
<td>2.04</td>
<td>6.008</td>
<td>0.494</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>0.054</td>
<td>0.162</td>
<td>0.046</td>
<td>0.048</td>
<td>0.042</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( CV )</td>
<td>0.348</td>
<td>0.218</td>
<td>0.192</td>
<td>0.146</td>
<td>0.303</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( CS )</td>
<td>4.394</td>
<td>3.2</td>
<td>2.658</td>
<td>2.644</td>
<td>3.672</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( CK )</td>
<td>105.421</td>
<td>31.161</td>
<td>20.31</td>
<td>19.566</td>
<td>51.365</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.3. Rényi and q Entropies

Entropy has been used in various situations in science and engineering. The entropy of a random variable $X$ is a measure of variation of the uncertainty. The Rényi entropy is defined by

$$ I_q(X) = \frac{1}{1-q} \log \left( \int_{-\infty}^{\infty} f(x)^q \, dx \right), \quad Q > 0 \quad \text{and} \quad R \neq 1 $$

To obtain the Rényi entropy of TWFr distribution, firstly we obtain $(f_{\text{TWFr}}(x; \Xi))^R$, by using exponential expansion as follows

$$(f_{\text{TWFr}}(x; \Xi))^R = \sum_{i=0}^{\infty} (\alpha \beta \delta \mu)^i \frac{(-1)^i}{i!} x^{-R(i+1)} e^{-\beta i(x/\mu)^R}$$

where,

Therefore, the Rényi entropy of TWFr distribution is given by

$$ I_q(X) = \frac{1}{1-q} \log \left( \int_{-\infty}^{\infty} f(x)^q \, dx \right), \quad Q > 0 \quad \text{and} \quad R \neq 1 $$

Hence, the q entropy of TWFr distribution is given by

$$ I_q(X) = \frac{1}{1-q} \log \left( \int_{-\infty}^{\infty} f(x)^q \, dx \right), \quad Q > 0 \quad \text{and} \quad R \neq 1 $$

3.4. Order Statistics

The density of the rth order statistic, for $r = 1, \ldots, n$ from independent and identically distributed random variables $X_1, X_2, \ldots, X_n$, is given by

$$ f_{X_{(r)}}(x) = \frac{1}{B(r, n-r+1)} f(x) F(x)^{r-1} [1 - F(x)]^{n-r} $$

The binomial expansion yields

$$ f_{X_{(r)}}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} F(x)^{r+i-1} [1 - F(x)]^{n-r-i} $$

Substituting (5) and (6) in (13) and using binomial expansion, then

$$ f_{X_{(r)}}(x) = \frac{1}{B(r, n-r+1)} \sum_{i=0}^{n-r} \binom{n-r}{i} \sum_{k=0}^{r+i} \binom{r+i}{k} \mu^{-k} e^{-\beta \mu x^k}$$

Hence the pdf of rth of TWFr distribution can be written as follows

$$ f_{X_{(r)}}(x) = \sum_{i=0}^{n-r} \sum_{k=0}^{i} \rho_{i,k} f_{\text{TWFr}}(x; \alpha(k+1), \beta, \mu, \delta) $$

4. PARAMETER ESTIMATION

In this section, the estimators of the TWFr model parameters are obtained using maximum likelihood (ML), least squares (LS) and weighted least squares methods. Simulation study is performed to compare the behavior of different estimates.

4.1. ML Estimators

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the TWFr distribution with set of parameters $\Xi = (\alpha, \beta, \mu, \delta)$. The likelihood function based on the observed random sample of size $n$ from density (6) is given by:

$$ L(\Xi|x) = ((1 - e^{-\alpha})^{-1} \alpha \beta \delta \mu^{\delta}) \prod_{i=1}^{n} x_i^{-\delta - 1}$$

$$ \times \exp\left(-\beta \left(\frac{\mu}{x_i}\right)^\delta - \alpha e^{-\beta (\mu/x_i)^\delta}\right)$$

The partial derivatives of the log-likelihood function, say $\ln \ell$, with respect to the population parameters are given by:

$$ \frac{\partial \ln \ell}{\partial \alpha} = \frac{n}{\alpha} \frac{ne^{-\alpha}}{1 - e^{-\alpha}} + \sum_{i=1}^{n} e^{-\beta (\mu/x_i)^\delta}$$

$$ \frac{\partial \ln \ell}{\partial \mu} = \frac{n \delta}{\mu} - \beta \delta \mu^{\delta-1} \sum_{i=1}^{n} x_i^{-\delta} + \alpha \beta \delta \mu^{\delta-1} \sum_{i=1}^{n} x_i^{-\delta} e^{-\beta (\mu/x_i)^\delta}$$

$$ \frac{\partial \ln \ell}{\partial \delta} = \frac{n}{\delta} + n \ln \mu - \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} \left(\frac{\mu}{x_i}\right)^\delta \ln \left(\frac{\mu}{x_i}\right)$$

$$ + \alpha \beta \sum_{i=1}^{n} \left(\frac{\mu}{x_i}\right)^\delta \ln \left(\frac{\mu}{x_i}\right) e^{-\beta (\mu/x_i)^\delta}$$
and,
\[
\frac{\partial \ln \ell}{\partial \beta} = \frac{n}{\beta} - \frac{n}{\beta} \sum_{i=1}^{n} \left( \frac{\mu}{\mu_i} \right)^{\frac{\beta}{\mu}} x_i^{\frac{\beta}{\mu}}
\]

The ML estimators of the model parameters are determined by solving numerically the non-linear equations \( \frac{\partial \ln \ell}{\partial \alpha} = 0, \frac{\partial \ln \ell}{\partial \mu} = 0, \frac{\partial \ln \ell}{\partial \delta} = 0, \) and \( \frac{\partial \ln \ell}{\partial \beta} = 0, \) simultaneously.

4.2. Least Squares and Weighted Least Squares Estimators
Suppose \( X_1, X_2, \ldots, X_n \) is a random sample of size \( n \) from TWFr distribution and suppose \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) denotes the corresponding ordered sample. The expectation and the variance of distribution are independent of the unknown parameter and are given by
\[
E(F(X_{(i)})) = \frac{i}{n+1}, \quad \text{and} \quad \text{var}(F(X_{(i)})) = \frac{(i(n-i+1))}{(n+1)^2(n+2)}
\]

where \( F(X_{(i)}) \) is cdf for any distribution and \( X_{(i)} \) is the \( i \)th order statistic. Then LS estimators can be obtained by minimizing the sum of squares errors,
\[
\sum_{i=1}^{n} \left[ F_i(x) - \frac{i}{n+1} \right]^2
\]

with respect to the model parameters. So, the LS estimators of the population parameters \( \alpha, \beta, \mu \) and \( \delta \) of the TWFr model are obtained by minimizing the following
\[
\sum_{i=1}^{n} \left[ \frac{1}{1 - e^{-\alpha \left(1 - \exp(-\alpha e^{-\beta \mu / x_i})\right)}} - \frac{i}{n+1} \right]^2
\]

with respect to \( \Xi = (\alpha, \beta, \mu, \delta) \). WLS estimators can be obtained by minimizing the following sum of squares errors
\[
\sum_{i=1}^{n} \left[ \frac{(n+1)^2(n+2)}{i(n-i+1)} \left( \frac{1}{1 - e^{-\alpha \left(1 - \exp(-\alpha e^{-\beta \mu / x_i})\right)}} - \frac{i}{n+1} \right)^2 \right]
\]

with respect to the parameters \( \Xi = (\alpha, \beta, \mu, \delta) \).

4.3. Simulation Study
Since, the ML, LS and WLS estimators are very hard to obtain analytically; hence a numerical study is achieved to compare the different estimates. The performances of the different estimates are compared in terms of their mean square error (MSE) and standard error (SE). We generate

<table>
<thead>
<tr>
<th>Case I</th>
<th>Case II</th>
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<tbody>
<tr>
<td>( n )</td>
<td>Method</td>
</tr>
<tr>
<td>10</td>
<td>ML</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
<tr>
<td></td>
<td>LS</td>
</tr>
<tr>
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<td></td>
<td>WLS</td>
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<tr>
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</tr>
<tr>
<td>20</td>
<td>ML</td>
</tr>
<tr>
<td></td>
<td>SE</td>
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<td></td>
<td>LS</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
<tr>
<td></td>
<td>WLS</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
<tr>
<td>30</td>
<td>ML</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
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</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
<tr>
<td></td>
<td>WLS</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
<tr>
<td>50</td>
<td>ML</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
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<tr>
<td></td>
<td>SE</td>
</tr>
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<td>WLS</td>
</tr>
<tr>
<td></td>
<td>SE</td>
</tr>
</tbody>
</table>

Note: *Indicate that the value multiply \( 10^{-3} \).

5
WLS estimates of population parameters are obtained. The
ML estimates of the fitted distributions using data 1.

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Table III. MSEs and SEs of TWFr estimates for Case III and Case IV.

<table>
<thead>
<tr>
<th>n</th>
<th>Method</th>
<th>Properties</th>
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<th>Case IV</th>
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<tr>
<td></td>
<td></td>
<td></td>
<td>$\alpha = 0.75$</td>
<td>$\beta = 1.75$</td>
</tr>
<tr>
<td>10</td>
<td>ML</td>
<td>MSE</td>
<td>0.774</td>
<td>0.080</td>
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<td></td>
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<td>SE</td>
<td>0.046</td>
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<td>MSE</td>
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<td>0.169</td>
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<td>0.913†</td>
<td>5.924†</td>
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<td>WLS</td>
<td>0.597†</td>
<td>5.250†</td>
</tr>
</tbody>
</table>

Note: †Indicate that the value multiply by 10⁻¹.

10000 random samples of sizes 10, 20, 30 and 50 from the TWFr distribution. We select four cases of parameters as; Case I $\equiv (\alpha = 0.25$, $\mu = 0.8$, $\delta = 0.5$, $\beta = 1.5$), Case II $\equiv (\alpha = 0.25$, $\mu = 1.25$, $\delta = 1.5$, $\beta = 1.5$), Case III $\equiv (\alpha = 0.75$, $\mu = 2$, $\delta = 2$, $\beta = 1.75$) and Case IV $\equiv (\alpha = 0.2$, $\mu = 0.5$, $\delta = 0.25$, $\beta = 0.5$). Then, the ML, LS and WLS estimates of population parameters are obtained. The MSEs and SEs of different estimates are computed and listed in Tables II and III. From these tables we conclude the following
- The MSEs and SEs decrease as sample size increases for all estimates (see Tables II and III).
- In the four cases, the MSEs of ML estimates of $\mu$ and $\beta$ are better than MSEs of LS and WLS estimates. While the MSEs of LS estimates of $\alpha$ are better than MSEs of ML and WLS estimates (see Tables II and III).
- In the four cases, the SEs of ML estimates of $\mu$, $\delta$ and $\beta$ are better than SEs of LS and WLS estimates. Whereas, the SEs of LS estimate of $\alpha$ are better than SEs of ML and WLS estimates (see Tables II and III).
- For the case IV: $(\alpha = 0.2$, $\mu = 0.5$, $\delta = 0.25$, $\beta = 0.5$), the WLS estimates have good statistical properties then the other cases of parameters (see Tables I and II).
- For fixed value of $\alpha = 0.25$, $\beta = 1.5$ and as the value of parameters $\mu$, $\delta$ increases, the MSEs and SEs of ML, LS and WLS for estimates of $\mu$, $\delta$, $\beta$ are increasing but decreasing for $\alpha$ estimates (see Table II).

Table IV. ML estimates of the fitted distributions using data 1.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$\hat{\gamma}$</th>
<th>$\hat{\delta}$</th>
<th>$\hat{\mu}$</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TWFr</td>
<td>0.075</td>
<td>1.971</td>
<td>1.943</td>
<td>0.743</td>
<td></td>
</tr>
<tr>
<td>WFr</td>
<td>0.056</td>
<td>0.774</td>
<td>2.877</td>
<td>0.629</td>
<td></td>
</tr>
<tr>
<td>MOFr</td>
<td>0.609</td>
<td>1.608</td>
<td>1.870</td>
<td></td>
<td></td>
</tr>
<tr>
<td>EFr</td>
<td>0.988</td>
<td>1.803</td>
<td>1.094</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ku-W</td>
<td>0.641</td>
<td>1.062</td>
<td>2.310</td>
<td>1.432</td>
<td></td>
</tr>
<tr>
<td>ZW</td>
<td>0.056</td>
<td>0.774</td>
<td>5.877</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table V. The statistics of the fitted models using data 1.

<table>
<thead>
<tr>
<th>Distribution</th>
<th>KS</th>
<th>CM</th>
<th>AD</th>
<th>AIC</th>
<th>BIC</th>
<th>CIAC</th>
<th>HQIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>TWFr</td>
<td>0.093</td>
<td>0.089</td>
<td>0.561</td>
<td>208.70</td>
<td>214.53</td>
<td>208.05</td>
<td>210.42</td>
</tr>
<tr>
<td>WFr</td>
<td>0.096</td>
<td>0.092</td>
<td>0.599</td>
<td>218.14</td>
<td>217.07</td>
<td>210.95</td>
<td>212.90</td>
</tr>
<tr>
<td>MOFr</td>
<td>0.106</td>
<td>0.116</td>
<td>0.780</td>
<td>213.45</td>
<td>218.28</td>
<td>213.80</td>
<td>214.17</td>
</tr>
<tr>
<td>EFr</td>
<td>0.102</td>
<td>0.120</td>
<td>0.807</td>
<td>214.62</td>
<td>219.05</td>
<td>214.07</td>
<td>215.93</td>
</tr>
<tr>
<td>Ku-W</td>
<td>0.097</td>
<td>0.107</td>
<td>0.714</td>
<td>213.63</td>
<td>222.73</td>
<td>214.22</td>
<td>217.25</td>
</tr>
<tr>
<td>ZW</td>
<td>0.098</td>
<td>0.095</td>
<td>0.601</td>
<td>210.98</td>
<td>217.87</td>
<td>211.65</td>
<td>213.09</td>
</tr>
</tbody>
</table>

As the values of $\alpha, \mu, \delta$ and $\beta$ decreases, the MSEs and SEs for estimates of $\alpha \mu, \delta$ and $\beta$ are decreasing based on ML, LS and WLS (see Table III). As the value of $\alpha, \mu, \delta$ and $\beta$ increases, the MSEs of ML estimates of $\alpha \delta$ and $\beta$ are increasing but the MSEs for ML estimate of $\mu$ are decreasing. While, the MSEs for estimates $\alpha, \mu$ and $\delta$ are increasing under LS and WLS except the MSE for estimate $\beta$ is decreasing (see Tables II and III).

5. DATA ANALYSIS
Two real data sets are analyzed to examine the flexibility of TWFr distribution. The goodness of fits of the TWFr distribution are compared with the other lifetime models including Weibull Fréchet (WFr), exponentiated Fréchet (EFr), Marshall-Olkin Fréchet (MOFr), Kumaraswamy Weibull (Ku-W) and Zubair Weibull (ZW) distributions. The analytical measures of goodness of fit including the Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), Kolmogorov-Smirnov (KS), Cramer–von Mises (CM) and Anderson-Darling (AD) statistics are considered. In general, a model with smaller values of these analytical measures indicates better fit to the data. From Tables IV and V, it is clear that the proposed model provides best fit. All the required computations are carried out in the R-language using “Nelder-Mead” algorithm. The cdfs of the competing distributions are as follows:

- The exponentiated Fréchet\(^{17}\) is

$$G_{EFr}(x; \alpha, \mu, \sigma) = 1 - (1 - e^{-(\mu/x)^\alpha})^\sigma, \quad x, \alpha, \sigma, \mu > 0$$
The Kumaraswamy Weibull is
\[ G_{KW}(x; \alpha, \beta, \gamma) = 1 - [1 - (1 - e^{-\gamma x})^\alpha]^{\beta}, \quad x, \alpha, \beta, \gamma > 0 \]

The Marshall-Olkin Fréchet is
\[ G_{MOFr}(x; \mu, \alpha, \sigma) = e^{-\frac{\mu}{\sigma}(1 - \alpha)(1 - e^{-\frac{\mu}{\sigma}x})}, \quad x, \alpha, \sigma, \mu > 0 \]

The Weibull Fréchet is
\[ G_{WF}(x; \alpha, \beta, \mu, \sigma) = 1 - \exp[-\alpha(1 - e^{-\frac{\mu}{\sigma}x})^{\beta}], \quad x, \alpha, \beta, \mu, \sigma > 0 \]

The Zubair-Weibull is
\[ G_{ZW}(x; \alpha, \beta, \mu, \sigma) = e^{\frac{\alpha(1 - e^{-\frac{\mu}{\sigma}x})^{2}}{e^{\alpha} - 1}}, \quad x, \alpha, \beta, \mu, \sigma > 0. \]

Data 1: The first data set represents survival times of guinea pigs injected with the different amount of tubercle bacilli studied by Bjerkedal. The ML estimates and the considered statistics are shown in Tables IV and V.

Figure 2 displays the estimated pdfs and cdfs of the fitted models. Figure 3 displays the fitted PP and Kaplan-Meir survival plots of TWFr distribution. Figures 2 and 3 prove that TWFr distribution provides the superior fit to this data set.

Data 2: The second data set representing the remission times (in months) of a random sample of 128 bladder cancer patients taken from Lee and Wang. The ML estimates and the proposed measures are shown in Tables VI and VII, respectively.

Figures 4 and 5 display the estimated pdfs, cdfs, PP and Kaplan-Meir survival plots of the fitted models. From these figures we conclude that the TWFr distribution provide the superior fit to the considered data set.

6. CONCLUDING REMARKS
In this article, we propose a new four-parameter model, called the truncated Weibull Fréchet distribution. We derive explicit expressions for the ordinary and incomplete moments, probability weighted moments, quantile and generating functions, and order statistics. We discuss the maximum likelihood, least squares and weighted least squares estimation of the model parameters. A simulation study is carried out to investigate the behavior of different estimators. Two applications illustrate that the truncated Weibull Fréchet distribution provides consistently better fit than other models.
Truncated Weibull Frèchet Distribution: Statistical Inference and Applications

References


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