International Journal of Algebra, Vol. 6, 2012, no. 8, 365 - 373

Applications of a Theorem of Gonshor

A. Y. Abdelwanis

Department of Mathematics, Faculty of Science
Cairo University, Giza, Egypt
ahmedyones2@yahoo.com

Abstract. In this paper, a new proof of the following result is given. For any abstract affine near ring A and any natural number n,
(i) there is a bijection between the $r-$ideals of A and those of $M_n(A)$,
(ii) there is no 1 to 1 correspondence between the ideals of A and those of $M_n(A)$.The result is derived with the aid of a Theorem of Gonshor.

If the additive group $(A,\ +)$ is torsion-free, we show that
(iii) there is a bijection between the pure $r-$ideals of A and those of $M_n(A)$,
(iv) there is a bijection between the strongly pure $r-$ideals of A and those of $M_n(A)$.

Mathematics Subject Classification: 16Y30

Keywords: abstract affine near ring, matrix near ring, r-ideal, pure r-ideal, strongly pure r-ideal

1. Introduction

The theory of near-rings is presented in[5]. We recall some concepts of this theory. Let $A = (A, +, \cdot)$ be an abstract affine near-ring, i.e.$(A, +)$ is an abelian group, $(A, \cdot)$ is a semigroup, $(a + b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in A$, and $A_0 = A_d$, where $A_0 = \{a \in A : a \cdot 0 = 0\}$ is the zero symmetric part and $A_d = \{a \in A : a \cdot (x + y) = a \cdot x + a \cdot y$, for all $x, y \in A\}$ is the distributive part. Let R be a ring and M a left module$R$. By [5 ,Prop.4] there is exactly one way to extend the multiplication $\cdot : R \times M \to M$ to a multiplication "$\circ$" in $(A, +) = (R, +) \oplus (M, +)$ such that $(A, +, \circ)$ is a near-ring with $A_d = A_0 = R \oplus (0)$ and $A_c = (0) \oplus M$, namely $(r, m) \circ (s, n) = (r \cdot s, r \cdot n + m)$.In fact, $A = (A, +, \circ)$ is an abstract affine near-ring and it will be denoted by $R * M$. Moreover all abstract affine near-rings arise in this way. Conversely, for any abstract affine near-ring A, the zero symmetric part A_0 is a ring and the constant part $A_c = \{a \in A : a.x = a$,for all $x \in A\}$ is a left module over A_0. For any natural number $n \geq 1$,let $M_n(A)$ be the $n \times n$ matrix near-ring over A, in the sense of Meldrum and Van der Walt in [4]. In this paper we prove (with the aid of a Theorem of Gonshor [1]) that :

(i) There is a one to one correspondence between the $r-$ideals of A and those of $M_n(A)$.

(ii)There is no one to one correspondence between the ideals of A and those of $M_n(A)$. If the additive group $(A,+)$ is torsion-free, then

(iii) There is a bijection between the pure $r-$ideals of A and those of $M_n(A)$.

(iii) There is a bijection between the strongly pure $r-$ideals of A and those of $M_n(A)$.

Recall that an $r-$ideal of a right near ring A is a subgroup of $(A,+)$ closed under multiplication on the left and right by arbitrary elements of A.

An ideal I of a right near ring A is a normal subgroup of $(A,+)$ closed under right multiplication by elements of A and which furthermore satisfies $y(x+a) - yx \in I$ for all $a \in I, x, y \in A$.

The results we seek follow with the aid of the following facts:

Lemma 1.1

Let R be a ring and let $n > 1$be a natural number. For each ideal I of R , $I \mapsto M_n(I)$,defines an order preserving bijection η from the set of ideals of R onto the set of ideals of $M_n(R)$.

Lemma 1.2 [7,Lemma 2.1].

Let $A = R * M$,be an abstract affine near-ring and let $n > 1$be a natural number. Then $M_n(R * M) \simeq M_n(R) * {}^nM$.

Lemma 1.3 [1, Theorem]

Let $A = R * M$ be an abstract affine near-ring. The $r-$ideals of A are exactly the sets of the form $I * M$ where I is an ideal of R $(I \blacktriangleleft R)$.The ideals of A are exactly the sets of the form $I * N$ where I is an ideal of R and N is a submodule of M $(N \leq M)$ containing IM.

Lemma 1.4

Let $A = R * M$ be an abstract affine near ring, $I, J \blacktriangleleft R$ and $N, L \leq M$ then

(i) $I * N = J * L$ implies $I = J$ and $N = L$

(i) ${}^nN = {}^nL$ implies $N = L$.

Proof

Let $I * N = J * L$

(i) First we show that $I = J$.

For any $a \in I$, $(a, 0_M) \in I * N = J * L$. Then $a \in J$. Conversely, for any $a \in J$, $(a, 0_M) \in J * L = I * N$, and so $a \in I$. Hence, $I = J$.

To show that $N = L$, let $n \in N$ and so $(0_R, n) \in I * N = J * L$. Thus $n \in L$.

Conversely, for any $n \in L$, $(0_R, n) \in J * L = I * N$. Thus $n \in N$,and $N = L$.
(ii) Let ${}^nN = {}^nL$ To show that $N = L$,let $n \in N$ and so, $(n, 0, 0, ..., 0) \in {}^nN .= {}^nL$.Hence, $n \in L$. Conversely, for any $k \in L$, $(k, 0, 0, ..., 0) \in {}^nL = {}^nN$.It follows that $k \in N$, and so, $N = L$.■

2. Main Results

In this section, we study the relation between the $r-$ideals of A and those of $M_n(A)$.Let $S(A)$ (resp.$S(M_n(A))$) be the set of r -ideals of A (resp. of $M_n(A)$)
Proposition 2.1 Let $A = R * M$ be an abstract affine near ring. There is a one to one correspondence between $S(A))$ and $S(M_n(A))$.
Proof
By Lemma 1.2,

$$\begin{aligned} M_n(A) &= M_n(R * M)) \\ &\simeq M_n(R) * {}^nM \end{aligned}$$

By Lemma 1.3, $S(A) = \{I * M : I \blacktriangleleft R\}$.
Define

$$\begin{aligned} \Phi &: S(A) \to S(M_n(R) * {}^nM) \\ \Phi(I * M) &= \eta(I) * {}^nM \ , I \blacktriangleleft R \end{aligned}$$

[a] Φ is one to one: let $I, J \blacktriangleleft R$ be such that $\Phi(I * M) = \Phi(J * M)$.
Thus

$$\eta(I) * {}^nM = \eta(J) * {}^nM.$$

It follows from lemma 1.4(i) that $\eta(I) = \eta(J)$,and so $I = J$.
[b] Φ is onto: let K be an $r-$ideal of $M_n(A)$. By lemma 1.3, $K = J * {}^nM$, where $J \blacktriangleleft M_n(A)$. By lemma 1.1, $J = \eta(I)$, for some $I \blacktriangleleft R$. Now

$$\begin{aligned} K &= J * {}^nM \\ &= \eta(I) * {}^nM \\ &= \Phi(I * M).■ \end{aligned}$$

To study the relation between the ideals of A and those of $M_n(A)$, let $D(A)$ (resp.$D(M_n(A))$ be the set of ideals of A (resp. of $M_n(A)$).

Proposition 2.2
The function $\Phi : D(A) \to D(M_n(A))$,defined by $\Phi(I * N) = \eta(I) * {}^nN$, where $I \blacktriangleleft R, N \leq M$ and $IM \subseteq N$, is one-to-one.
Proof
By lemma 1. 2, $M_n(A) = M_n(R * M)) \simeq M_n(R) * {}^nM$.

By lemma 1. 3, $D(A) = D(R*M) = \{I*N \ : I \blacktriangleleft R, N \leq M$ and $IM \subseteq N\}$. Observe that $\eta(I) \blacktriangleleft M_n(R)$, ${}^nN \leq {}^nM$ and $\eta(I)\ {}^nM \subseteq^n N$. Hence $\eta(I)*\ {}^nN \in D(M_n(R)*\ {}^nM)$.
To show that Φ is one to one , let $I*N, J*L \in D(R*M)$ be such that $\Phi(I*N) = \Phi(J*L)$.Then $\eta(I)*\ {}^nN = \eta(J)*\ {}^nL$.
So from Lemma 1.4(i) $\eta(I) = \eta(J)$, ${}^nN =^n L$. Since η is one to one so $I = J$ and from Lemma 1.4(i) we have $N = L$. Now $I*N = J*L$.■

Now, we give an example which explains that there is no surjective map between the set of ideals of A and the set of ideals of $M_n(A)$.

Example 2.3

Let $A = R*M$ be an abstract affine near ring such that, $R = Z_3$, $M = Z_3$ where the action of R on M is the trivial action i.e

$$\begin{aligned} . \quad : \quad & R \times M \to M \\ (a,b) \quad \to \quad & a.b = 0, \forall a, b \in Z_3. \end{aligned}$$

It is clearly that R, M are simple. For $n = 2$, the matrix ring $M_2(R)$ is simple. Let $L = Z_3 \oplus Z_3$ the direct sum of 2 copies of Z_3.Clearly L is a left $M_2(R)-$module in a natural way by writing the elements of $Z_3 \oplus Z_3$ as columns. Note that K is a submodule of $L = Z_3 \oplus Z_3$ iff K is a subgroup of $Z_3 \oplus Z_3$. Now the set of ideals of $R*M$ contains only four element : $D(R*M) = \{0*0, 0*Z_3, Z_3*0, Z_3*Z_3\}$. But the set of ideals of $M_2(R)*L$ contains seven elements : $D(M_2(R)*L) = \{0*0, 0*N, 0*P, 0*H, 0*L, M_2(R)*0, M_2(R)*L\}$ where $N = \{(0,0),(1,0),(2,0)\}, P = \{(0,0),(0,1),(0,2)\}, H = \{(0,0),(1,1),(2,2)\}$ are submodules of $L = Z_3 \oplus Z_3$. It follows that there is no one to one correspondence between the ideals of A and those of $M_n(A)$.

3. Pure r-ideals

In this section , we study the relation between the pure $r-$ideals of A and those of $M_n(A)$,where A is an abstract affine near ring. For any $r-$ideal L of A, let

$$L^* := \{B \in \mathrm{M}_n(A) : B\alpha \in L^n, \text{ for all } \alpha \in A^n\}$$

and for any $r-$ideal Q in $\mathrm{M}_n(A)$, let

$$Q_* := \{x \in A : x \in \mathrm{Im}(\pi_j B) \text{ for some } B \in Q \text{ and } j, 1 \leq j \leq n\}$$

Lemma 3.1[6]

(1) L^* is an $r-$ideal of $\mathrm{M}_n(A)$
(2) Q_* is an $r-$ideal of A.

(3) $a \in Q_*$ iff $f_{11}{}^a \in Q$ iff $f_{ij}{}^a \in Q$ for all $1 \le i,j \le n$.
(4)$(L^*)_* = L$.
(5)$(Q_*)^* = Q$.

Remark 3.2

For any abstract affine near-ring A, any matrix B in $M_n(A)$ can be represented as $B = \sum_{i,j=1}^{n} f_{ij}^{b_{ij}}$,$b_{ij} \in A$ [6] .If $b_{ij} = r_{ij} + c_{ij}$ is the standard decomposition of b_{ij} into a a zero-symmetric part r_{ij} and a constant part c_{ij} ,then

$B = E + C$, where $E = \sum_{i,j=1}^{n} f_{ij}^{r_{ij}}$ and $C = \sum_{i,j=1}^{n} f_{ij}^{c_{ij}}$. One can easily see that $C = \sum_{i=1}^{n} f_{ii}{}^{c_i}$, where $c_i = \sum_{k=1}^{n} c_{ik}$ (i.e. C is a diagonal matrix). By Lemma 2.1 of [7], $M_n(A)$ is isomorphic to $M_n(R) *^n M$,for some ring R and an $R-$module M.Thus the matrix B can be written as $B = ((r_{ij}),(c_i))$ where (r_{ij}) is an $n \times n$ matrix over the ring R , $1 \le i,j \le n$.If $B = D + T$ is another representation of B, where $D = \sum_{i,j=1}^{n} f_{ij}^{d_{ij}}$ is a zero-symmetric matrix and $T = \sum_{i=1}^{n} f_{ii}{}^{t_i}$ is a digonal constant matrix, then $r_{ij} = d_{ij}$,for all $1 \le i,j \le n$, and $c_i = t_i$, for all $1 \le i \le n$.

Lemma 3.3

Let A be an abstract affine near-ring and H an $r-$ideal of A. Any matrix $B \in H^*$can be written as $B = \sum_{i,j=1}^{n} f_{ij}^{b_{ij}}$, where $b_{ij} \in H$, for all $1 \le i,j \le n$.

Proof

We use induction on $\omega(B)$.
For $\omega(B) = 1$, $B = f_{ij}^{b}$, for some $1 \le i,j \le n$, $b \in A$. Let ϵ_j be the element of A^n whose j th component is equal to 1 and all other components are equal to 0. Then $B\ \epsilon_j = f_{ij}^{b}\ \epsilon_j \in H^n$, and so$b \in H$. Assume the result holds for all matrices with weight less than m, $m \ge 2$. If $\omega(B) = m$, then $B = C + D$ or $B = CD$ where $\omega(C), \omega(D) < m$. By the induction hypothesis $C = \sum_{i,j=1}^{n} f_{ij}^{c_{ij}}$ and $D = \sum_{i,j=1}^{n} f_{ij}^{d_{ij}}$ where c_{ij}, $d_{ij} \in H$ for all $1 \le i,j \le n$. Since A is abelian, $B = C + D = \sum_{i,j=1}^{n} f_{ij}^{c_{ij}+d_{ij}}$, $c_{ij} + d_{ij} \in H$ for all $1 \le i,j \le n$. In the seconed

case $B = CD = \sum_{i,j=1}^{n} f_{ij}^{c_{ij}} D$. Note that each c_{ij} may be written as $c_{ij} = r_{ij} + t_{ij}$, where r_{ij} is distributive and t_{ij} is constant. So it follows from lemma3.1 of [4] that

$$\begin{aligned} B &= \sum_{i,j=1}^{n} f_{ij}^{r_{ij}} D + \sum_{i,j=1}^{n} f_{ij}^{t_{ij}} \\ &= \sum_{i,j=1}^{n} [\sum_{k=1}^{n} f_{ij}^{r_{ik} d_{kj}} + \sum_{i,j=1}^{n} f_{ij}^{t_{ij}}]. \end{aligned}$$

This means that B can be represented as $B = \sum_{i,j=1}^{n} f_{ij}^{u_{ij}}$ where $u_{ij} = \sum_{k=1}^{n} r_{ik} d_{kj} + t_{ij}$ .Observe that $d_{kj} \in H$, and $r_{ik} d_{kj} \in AH \subseteq H$, for all $1 \leq i, k, j \leq n$. Also, $t_{ij} = t_{ij} 0 \in AH \subseteq H$. It follows that $u_{ij} = \sum_{k=1}^{n} r_{ik} d_{kj} + t_{ij} \in H$, for all $1 \leq i, j \leq n$.■

Let $B = (B, +)$ be an abelian group. Recall that a subgroup C of B is called pure in B (Notation $C \leq_p B$) if every equation $nx = c, c \in C$, is solvable in C whenever it solvable in B, where $nx = x + x + +... + x$ (n times).
B is called torsion-free if for every $b \in B$, $mb = 0$ implies $m = 0$ or $b = 0$.In other words, each non-zero element of B has an infinite order.

Definition 3.4

Let H be an $r-$ideal of A. We say that H is pure in A if the subgroup $(H, +)$ is pure in the abelian group $(A, +)$.

Lemma 3.5

Let A be an abstract affine near ring. Then $(A, +)$ is torsion free iff $(M_n(A), +)$ is torsion free.

Proof

Suppose $(A, +)$ is torsion free. Let $mC = O$,$C \in M_n(A)$, $m \neq 0$. Then for any $u \in A^n$,$mCu = Ou = 0$. Suppose$Cu = (a_1, ..., a_n)$.It follows that $(ma_1, ..., ma_n) = 0$, and so $ma_i = 0$ for all $1 \leq i \leq n$. Since $(A, +)$ is torsion free,$a_i = 0$ for all $1 \leq i \leq n$. Hence, $Cu = O$ for all $u \in A^n$, and so $C = O$.

Conversely, if $(M_n(A), +)$ is torsion free and $ma = 0, a \in A$.Consider the matrix $f_{11}^{a} \in M_n(A)$. Note that $mf_{11}^{a} = f_{11}^{ma} = f_{11}^{0} = O$, and so $m = 0$.Hence $(A, +)$ is torsion free.■

Lemma 3.6

Let A be an abstract affine near ring. Suppose $(A, +)$ is torsion free. Let H

be an $r-$ideal of A and K an $r-$ideal of $\mathrm{M}_n(A)$. Then
(1) If $H \leq_p A$ then $H^* \leq_p M_n\ (A)$
(2) If $K \leq_p \mathrm{M}_n(A)$ then $K_* \leq_p A$

Proof

(1) Suppose that $tX = B, B \in H^*$ is solvable in $\mathrm{M}_n(A)$.Then there exists $C \in \mathrm{M}_n(A)$ such that $tC = B$, it follows that $tC\alpha = B\alpha$, for all $\alpha \in A^n$. If $B\alpha = (a_1, a_2, ..., a_n) \in H^n$ and $C\alpha = (b_1, b_2, ..., b_n)$, then ,$(tb_1, tb_2, ..., tb_n) = (a_1, a_2, ..., a_n)$ and so , $tb_1 = a_1, ..., tb_n = a_n$. Since $H \leq_p A$, then there are $c_1, c_2, ..., c_n \in H$ such that $tc_1 = a_1, ..., tc_n = a_n$. By lemma3.5, $c_i = b_i$, for all i, $1 \leq i \leq n$. Hence , $C\alpha = (c_1, c_2, ..., c_n) \in H^n$. This means that $C \in H^*$and thus, $H^* \leq_p \mathrm{M}_n(A)$.

(2) Suppose that $tx = a, a \in K_*$ is solvable in A.Then there exists $b \in A$ such that $tb = a$. Note that $f_{11}^a = f_{11}^{t\ b} = tf_{11}^b$ and by Lemma 3.1(3), $f_{11}^a \in K$. Now the equation $tX = f_{11}^a, f_{11}^a \in K$, is solvable in $\mathrm{M}_n(A)$. Since K is pure in $\mathrm{M}_n(A)$, this equation is solvable in K, and so, there is $B \in K$ such that $tB = f_{11}^a$. It follows that $tB -\ tf_{11}^b = t(\ B -\ f_{11}^b) = O$. Since $(\mathrm{M}_n(A), +)$ is torsion free, then $B = f_{11}^b \in K$ and so $b \in K_*$.

Thus $K_*$is pure in A.■

Theorem 3.7

Let A be an abstract affine near ring. Suppose $(A, +)$ is torsion free.Then there is a bijection between the set of pure $r-$ideals of A and those of $M_n(A)$.

Proof

Let $P(A)$ be the set of pure $r-$ideals of A and $P(M_n(A))$ be the set of pure $r-$ideals of $\mathrm{M}_n(A)$.

Let $\varphi : P(A) \to P(M_n(A))$, $L \to L^*$ and its inverse $\psi : P(M_n(A)) \to P(A), Q \to Q_*$. Apply Lemma 3.1(4)-(5) and Lemma 3.6.■

In the sequel, we study the relation between the strongly pure $r-$ideals of A and those of $M_n(A)$.

Definition 3.8 [2]

Let $(B, +)$ be an abelian group. A subgroup C of B is called strongly pure in B (Notation $(L, +) \leq_{sp} (B, +)$) if for every $c \in C$ there is a group homorophism $\phi_c : (B, +) \to (C, +)$ that leaves c fixed.

It is well known from [3] that if$C$ is strongly pure in B and S is a finite subset of C, then there exists a homomorphism$\phi_S : (B, +) \to (C, +)$ that leaves the elements of S fixed.

Definition 3.9

let H be an $r-$ideal of A. We say that H is strongly pure in A if the subgroup $(H, +)$ is strongly pure in the abelian group $(A, +)$.

Lemma 3.10

Let A be an abstract affine near ring. Suppose $(A, +)$ is torsion free. For

$r-$ideals K in $\mathrm{M}_n(A)$ and H in A , we have
(1) If $(H,+) \leq_{sp} (A,+)$ then $(H^*,+) \leq_{sp} (\mathrm{M}_n(A),+)$
(2) If $(K,+) \leq_{sp} (M_n(A),+)$ then $(K_*,+) \leq_{sp} (A,+)$.

Proof

(1) Any $B \in H^*$can be written as $B = \sum_{i,j=1}^{n} f_{ij}^{b_{ij}}$, $b_{ij} \in H$, $1 \leq i,j \leq n$.Let $S = \{ \ b_{ij} : 1 \leq i,j \leq n \ \} \subseteq H$. Since $(H,+) \leq_{sp} (A,+)$, then there exists a homorophism $\phi = \phi_S : A \to H$ that leaves the elements of S fixed.

Define $\Phi_B : \mathrm{M}_n(A) \to H^*$ by $\Phi_B(M) = \Phi_B \ (\sum_{i,j=1}^{n} f_{ij}^{m_{ij}}) = \sum_{i,j=1}^{n} f_{ij}^{\ \phi(m_{ij})}$. Observe that Φ_B is a group homorophism and that $\Phi_B(B) = B$.■

(2) Let $b \in K_*$. By lemma3.1(3), $B = f_{11}^b \in K$. Since $(K,+) \leq_{sp} (M_n(A,+)$, there exist a group homorophism $\Phi_B : (M_n(A),+) \to (K,+)$, that leaves B fixed. For any $a \in A$, $\Phi_B(f_{11}^a)$ is a matrix in $K \ = (K_*)^*$.Using Remark3.2, $\Phi_B(f_{11}^a) = \sum_{i,j=1}^{n} f_{ij}^{r_{ij}} + \sum_{i\ =1}^{n} f_{ii}^{\ c_i}$ .It follows from Lemma 3.3, that $r_{ij} \in K_*$ and $c_i.\in K_*$,for all $1 \leq i,j \leq n$. Now we can define $\Psi_b : (A,+) \to (K_*,+)$, by $\Psi_b(a) = (r_{11} + c_1) \in K_*$. Observe that Ψ_b is well defined as it follows from Remark 3.2. Also,Ψ_b can be expressed as $\Psi_b(a) = \pi_1 \Phi_B(f_{11}^a)\epsilon_1$ where $\pi_1$is the first projection.HenceΨ_b a group homorophism. Since $\Phi_B(f_{11}^b) = f_{11}^b$, then $\Psi_b(b) = \pi_1 \Phi_B(f_{11}^b)\epsilon_1 = \pi_1 \ f_{11}^b \ \epsilon_1 = b$, and so $(K_*,+) \leq_{sp} (A,+)$.■

Theorem 3.11

Let A be an abstract affine near ring. Suppose $(A,+)$ is torsion free. Then there is a bijection between the set of strongly pure $r-$ideals I of A and those of $M_n(A)$.

Proof

Let $SP(A)$ be the set of strongly pure $r-$ideals I of A , and let $SP(\mathrm{M}_n(A))$ be the set of strongly pure $r-$ideals J of $\mathrm{M}_n(A)$
Define $\Phi : SP(A) \to SP(M_n(A))$ by $L \to L^*$ and its inverse by $Q \to Q_*$., so by [6] (Theorem5.8) and lemma3.10 Φ is one to one and onto.■

References

[1] H. Gonshor; On abstract affine near-rings, Pacific J.Math.14(1964),1237-1240.
[2] P. Hill and C.Megibben, Pure subgroups of torsion-free groups, Transactions of the American Mathematical Society. vol.303, no.2 , 1987, 765 - 778.

[3] S.Janakiraman and K.Rangaswamy, strongly pure subgroups of abelian groups, Lectures Notes in Math. Springer-Verlg, New York, 1977, 75-64.

[4] J.D.P. Meldrum and A.P.J Van der Walt, Matrix near ring, Arch. Math. 47, 312-319(1986)

[5] G. Pilz, Near-Rings. Amasterdam: North-Holland (1977).

[6] Maher Zayed, On matrix near ring, Arch.Math.77 (2001) pp. 163-169

[7] Maher Zayed and Nawal Nour ElDean, On a Question of J.h.Meyer, Communication in Algebra, Vol.31, No10, pp.5039-5046, 2003

Received: November, 2011