Section 7-5
Estimating a Population Variance

This section covers the estimation of a population variance σ^2 and standard deviation σ.

Estimator of σ^2

The **sample variance s^2** is the best point estimate of the **population variance σ^2**.

Estimator of σ

The **sample standard deviation** s is a commonly used **point estimate of** σ.

Construction of confidence intervals for σ^2

We use the chi-square distribution, denoted by Greek character χ^2 (pronounced *chi-square*).

Chi-Square Distribution

In a normally distributed population with variance σ^2 assume that we randomly select independent samples of size n and, for each sample, compute the sample variance s^2 (which is the square of the sample standard deviation s). The sample statistic χ^2 (pronounced chi-square) has a sampling distribution called the **chi-square distribution**.

Chi-Square Distribution

$$\chi^2 = \frac{(n - 1)\, s^2}{\sigma^2}$$

where

n = sample size

s^2 = sample variance

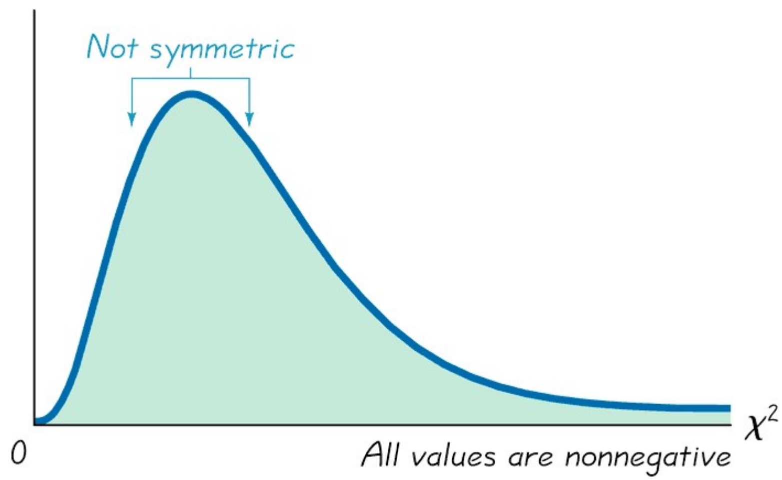
σ^2 = population variance

degrees of freedom = $n - 1$

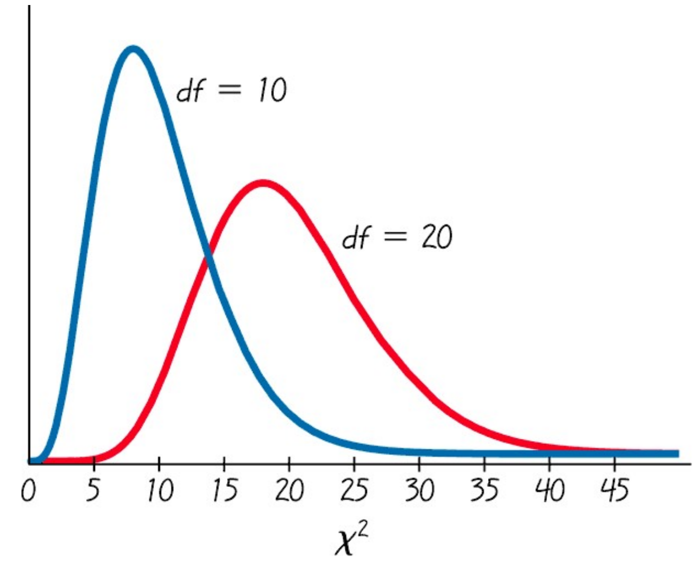
Properties of the Chi-Square Distribution

1. The chi-square distribution is **not symmetric**, unlike the normal and Student *t* distributions.

degrees of freedom = *n* – 1



Chi-Square Distribution



Chi-Square Distribution for df = 10 and df = 20

Properties of the Chi-Square Distribution

2. The values of chi-square can be zero or positive, but they cannot be negative.

3. The chi-square distribution is different for each number of degrees of freedom, which is df = $n - 1$.

   In Table A-4, each critical value of χ^2 corresponds to an area given in the top row of the table, and that area represents the cumulative area located to the right of the critical value.

Example

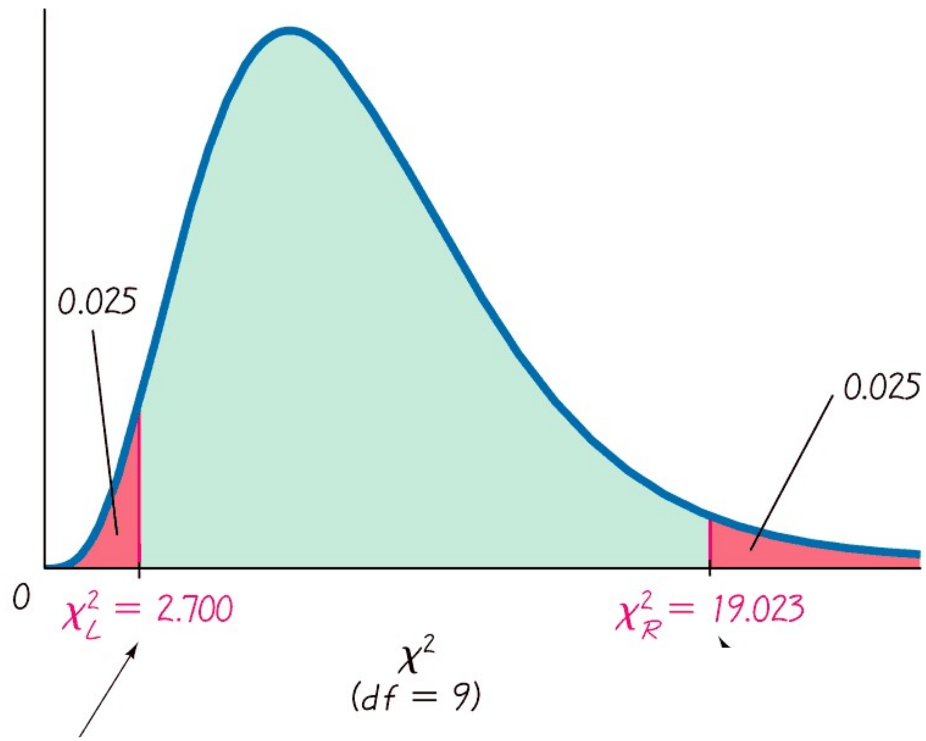
A sample of ten voltage levels is obtained.

Construction of a confidence interval for the population standard deviation σ requires the left and right critical values of χ^2 corresponding to a confidence level of 95% and a sample size of $n = 10$.

Find the critical value of χ^2 separating an area of 0.025 in the left tail, and find the critical value of χ^2 separating an area of 0.025 in the right tail.

Example

Critical Values of the Chi-Square Distribution



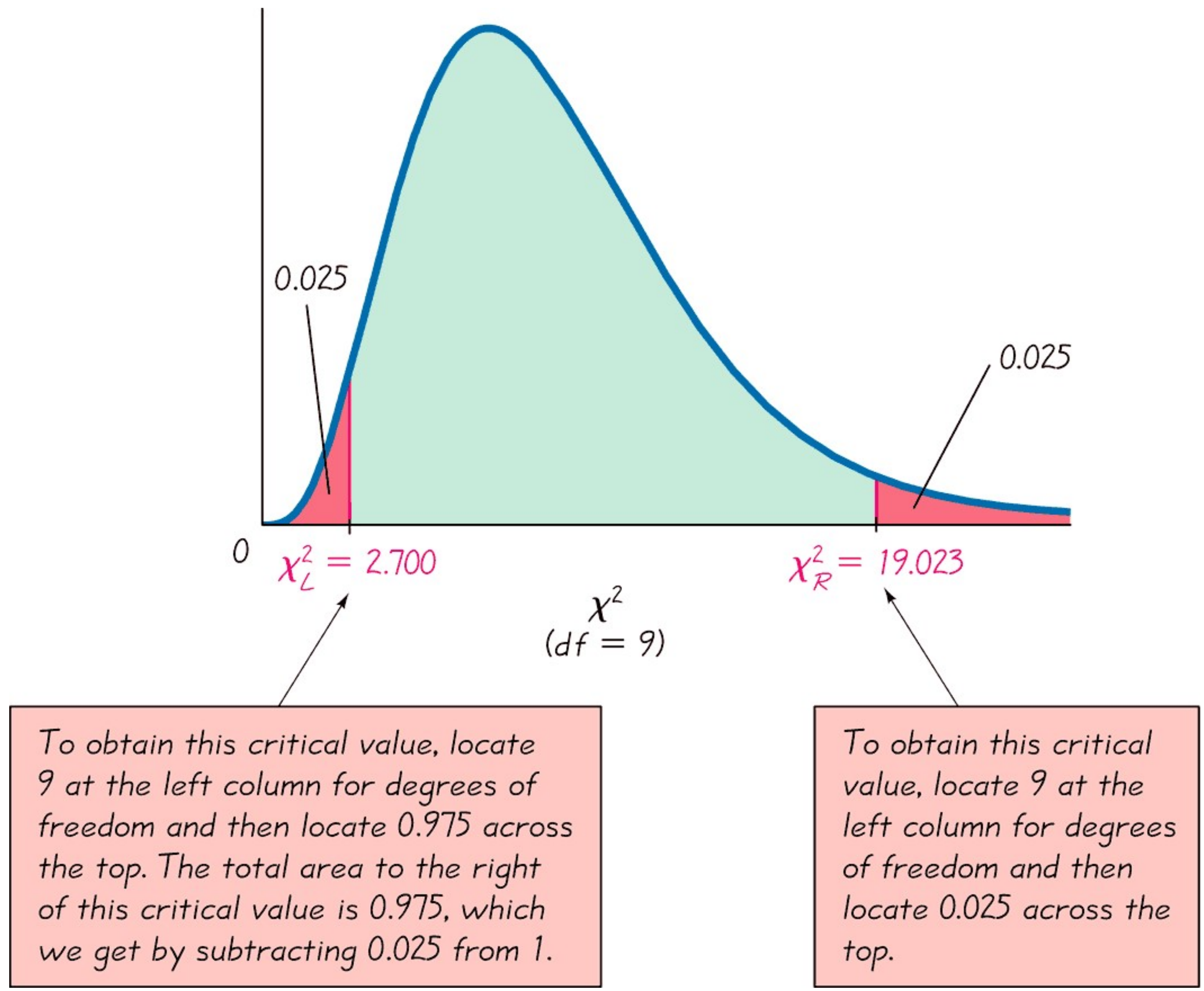
Chi-Square Distribution

TABLE A-4 Chi-Square (χ^2) Distribution

Degrees of Freedom	Area to the Right of the Critical Value									
	0.995	0.99	0.975	0.95	0.90	0.10	0.05	0.025	0.01	0.005
1	—	—	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	15.987	18.307	20.483	23.209	25.188

Example

Critical Values of the Chi-Square Distribution



Confidence Interval for Estimating a Population Standard Deviation or Variance

σ = population standard deviation

s = sample standard deviation

n = number of sample values

χ^2_L = left-tailed critical value of χ^2

σ^2 = population variance

s^2 = sample variance

E = margin of error

χ^2_R = right-tailed critical value of χ^2

Confidence Interval for Estimating a Population Standard Deviation or Variance

Requirements:

1. **The sample is a simple random sample.**
2. **The population must have normally distributed values (even if the sample is large).**

Confidence Interval for Estimating a Population Variance

$$\chi^2 = \frac{(n-1)\,s^2}{\sigma^2}$$

$$\frac{(n-1)s^2}{\chi_R^2} < \sigma^2 < \frac{(n-1)s^2}{\chi_L^2}$$

Confidence Interval for Estimating a Population Standard Deviation

$$\sqrt{\frac{(n-1)s^2}{\chi_R^2}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi_L^2}}$$

Procedure for Constructing a Confidence Interval for σ or σ^2 - cont

4. If a confidence interval estimate of σ is desired, take the square root of the upper and lower confidence interval limits and change σ^2 to σ.

5. Round the resulting confidence level limits. If using the original set of data to construct a confidence interval, round the confidence interval limits to one more decimal place than is used for the original set of data. If using the sample standard deviation or variance, round the confidence interval limits to the same number of decimals places.

Example:

The proper operation of typical home appliances requires voltage levels that do not vary much. Listed below are ten voltage levels (in volts) recorded in the author's home on ten different days. These ten values have a standard deviation of $s = 0.15$ volt. Use the sample data to construct a 95% confidence interval estimate of the standard deviation of all voltage levels.

123.3 123.5 123.7 123.4 123.6 123.5 123.5 123.4 123.6 123.8

Example:

Requirements are satisfied: simple random sample and normality

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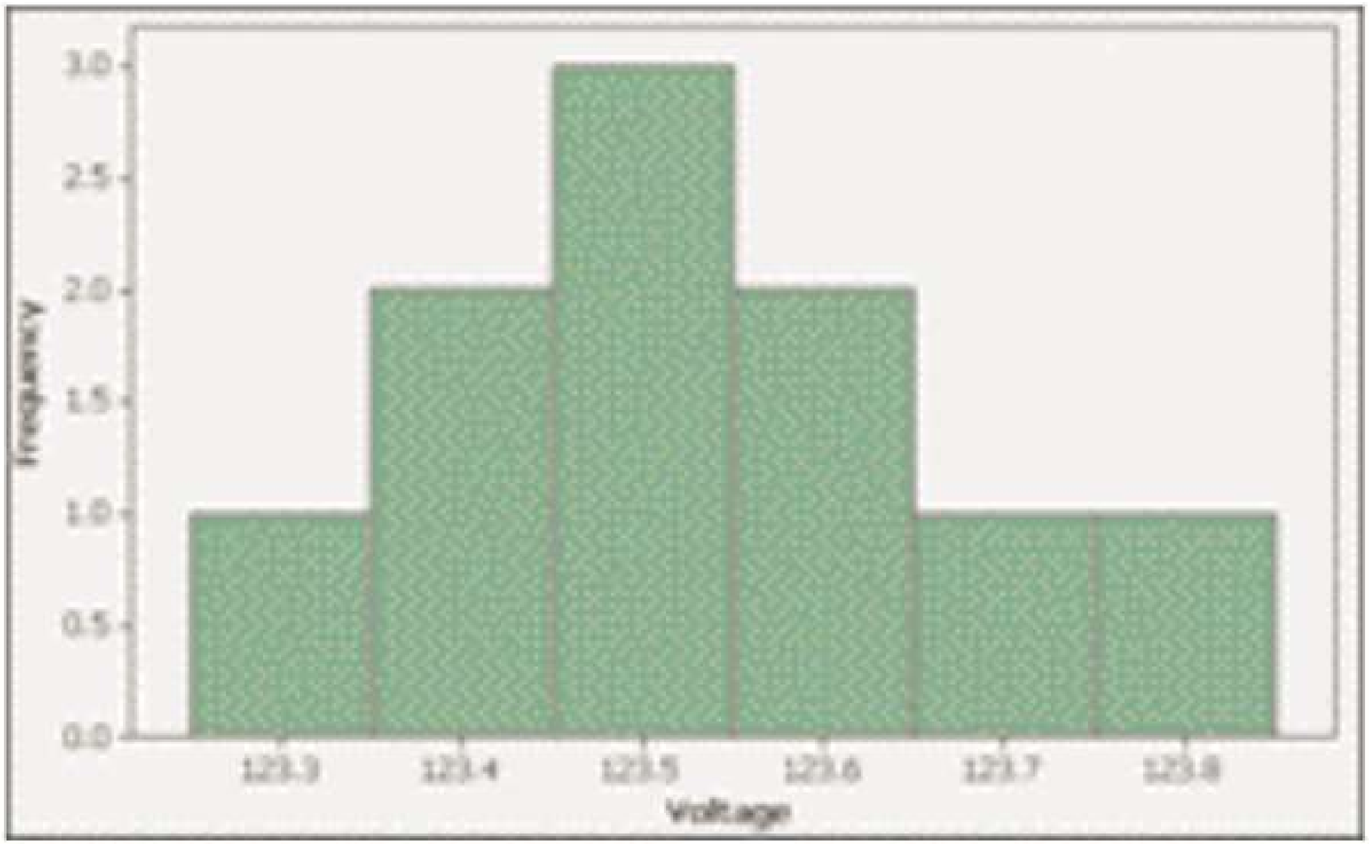

Example:

n = 10 so df = 10 – 1 = 9

Use table A-4 to find:

$$\chi_L^2 = 2.700 \text{ and } \chi_R^2 = 19.023$$

Construct the confidence interval: n = 10, s = 0.15

$$\frac{(n-1)s^2}{\chi_R^2} < \sigma^2 < \frac{(n-1)s^2}{\chi_L^2}$$

$$\frac{(10-1)(0.15)^2}{19.023} < \sigma^2 < \frac{(10-1)(0.15)^2}{2.700}$$

Example:

Evaluation the preceding expression yields:

$$0.010645 < \sigma^2 < 0.075000$$

Finding the square root of each part (before rounding), then rounding to two decimal places, yields this 95% confidence interval estimate of the population standard deviation:

$$0.10 \text{ volt} < \sigma < 0.27 \text{ volt.}$$

Based on this result, we have 95% confidence that the limits of 0.10 volt and 0.27 volt contain the true value of σ.

Recap

In this section we have discussed:

- The chi-square distribution.
- Using Table A-4.
- Confidence intervals for the population variance and standard deviation.
- Determining sample sizes. (Not covered)