# FUZZY INFERENCE

# Fuzzy Inference

- **A fuzzy inference** system consists of linguistic variables, fuzzy rules and a fuzzy inference mechanism.
- **Linguistic variables** allow us to interpret linguistic expressions in terms of fuzzy mathematical quantities.
- **Fuzzy Rules** are a set of rules that make association between typical input and output data.
- **A fuzzy inference mechanism** is able to model the process of approximate reasoning.

# Linguistic Variables

**Definition 6.1.** *(Zadeh [156]) A linguistic variable is a quintuple*

$$(X, T, U, G, M)$$

*where*

$X$ *is the name of the variable*

$T$ *is the set of linguistic terms which can be values of the variable*

$U$ *is the universe of discourse*

$G$ *is a collection of syntax rules, grammar, that produces correct expressions in* $T$.

$M$ *is a set of semantic rules that map* $T$ *into fuzzy sets in* $U$.

**Example 6.2.** *We consider an example of a linguistic variable* $(Age, T, U, G, M)$ *where*

$X = Age.$

$T = \{young,\ very\ young,\ very\ very\ young, ...\}$

$U = [0, 100]$ *is the universe of discourse for age.*

$G$ : *The syntax rules can be expressed as follows:* $young \in G$. *If* $x \in G$ *then* $very\ x \in G$.

$M : T \to \mathcal{F}(X)$, $M(young) = u$, *where* $u = (0, 0, 18, 40)$.

$M(very^n\ young) = u^n(x)$.

**Example 6.3.** *We consider an example linked to room temperature control* $(Temperature, T, U, G, M)$ *where*

$X = Temperature.$

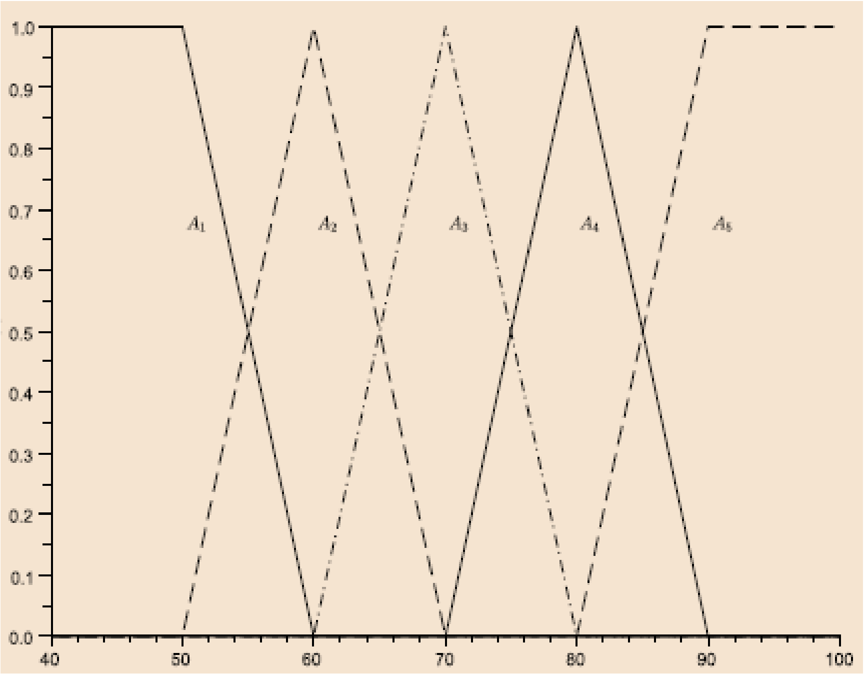
$T = \{cold,\ very\ cold, ...cool,\ very\ cool,...,....,hot,\ very\ hot,...\}.$

$U = [40, 100]$ *is the universe of discourse for temperature.*

$G$ : *The syntax rules can be expressed as follows: cold, cool, just right, warm, hot* $\in G$. *If* $x \in G$ *then very* $x \in G$. *Let us observe that very warm*$\neq$*hot, they have different membership functions. Also, the expression very just right is correctly obtained by applying the hedge very to just right. It is a correct expression in our syntax however the linguistic term for it is not used in natural language.*

$M : T \to \mathcal{F}(X)$, $M(cold) = u_1$, *where* $u_1 = (40, 40, 50, 60)$, $M(cool) = u_2$, *where* $u_2 = (50, 60, 70)$, ..., $M(hot) = u_5$, *where* $u_5 = (80, 90, 100, 100)$.

$M(very^n\ cold) = u_1^n(x)$,..., $M(very^n\ hot) = u_5^n(x)$. *See Figures* 6.1, 6.2.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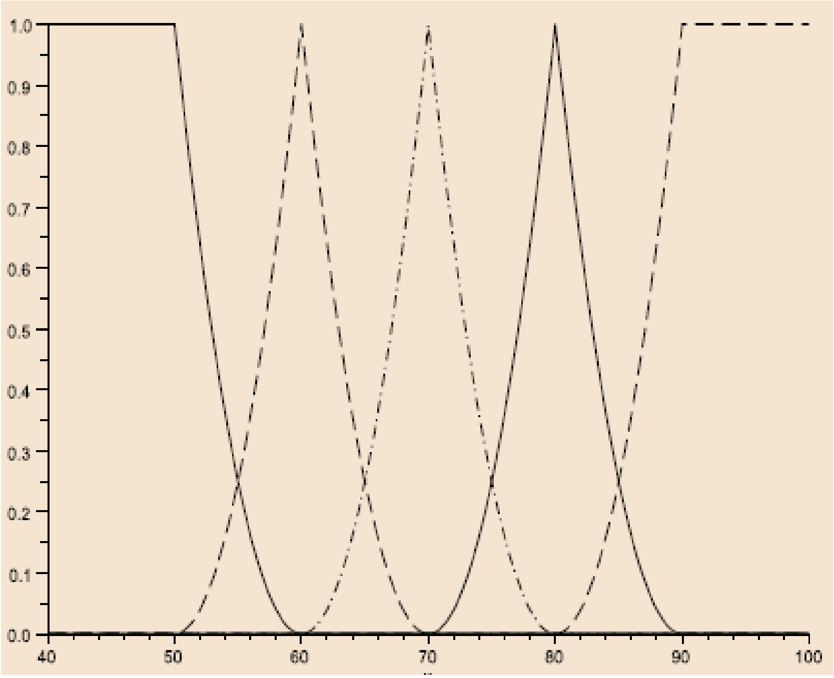
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# Fuzzy Rules

A fuzzy rule is a triplet $(A, B, R)$ that consists of an antecedent $A \in \mathcal{F}(X)$, a consequence $B \in \mathcal{F}(X)$ that are linguistic variables, linked through a fuzzy relation $R \in \mathcal{F}(X \times Y)$.

Using fuzzy sets a fuzzy rule is written as follows:

$$\text{If } x \text{ is } A \text{ then } y \text{ is } B$$

**Example 6.4.** *An example of a fuzzy rule that naturally can be considered in the room temperature control problem is the following:*

*If temperature is cold then heat is high*

**Definition 6.5.** *(Mamdani-Assilian [107]) We define the* ***fuzzy rule***

*If x is A then y is B.*

*as a fuzzy relation as follows*

*(i) Mamdani rule:*

$$R_M(x, y) = A(x) \wedge B(y);$$

*(ii) Larsen rule:*

$$R_L(x, y) = A(x) \cdot B(y);$$

*(iii) t-norm rule:*

$$R_T(x, y) = A(x) T B(y),$$

*with T being an arbitrary t-norm.*

*(iv) Gödel rule:*

$$R_G(x, y) = A(x) \to B(y),$$

*with* $\to$ *being Gödel implication;*

*(v) Gödel residual rule:*

$$R_R(x, y) = A(x) \to_T B(y)$$

*with* $\to_T$ *being a residual implication with a given t-norm.*

**Example 6.6.** *An example of a fuzzy rule interpreted as a fuzzy relation is represented in Fig. 6.3. Two Gaussian fuzzy sets* $A : [0,1] \to [0,1]$ *and* $B : [0,1] \to [0,1]$ *are considered with* $\bar{x}_1 = 0.6$ *and* $\sigma_1 = 0.1$ *being the mean and spread of* $A$*, while* $\bar{x}_2 = 0.4$ *and* $\sigma_2 = 0.1$ *are the mean and the spread of* $B$*.*

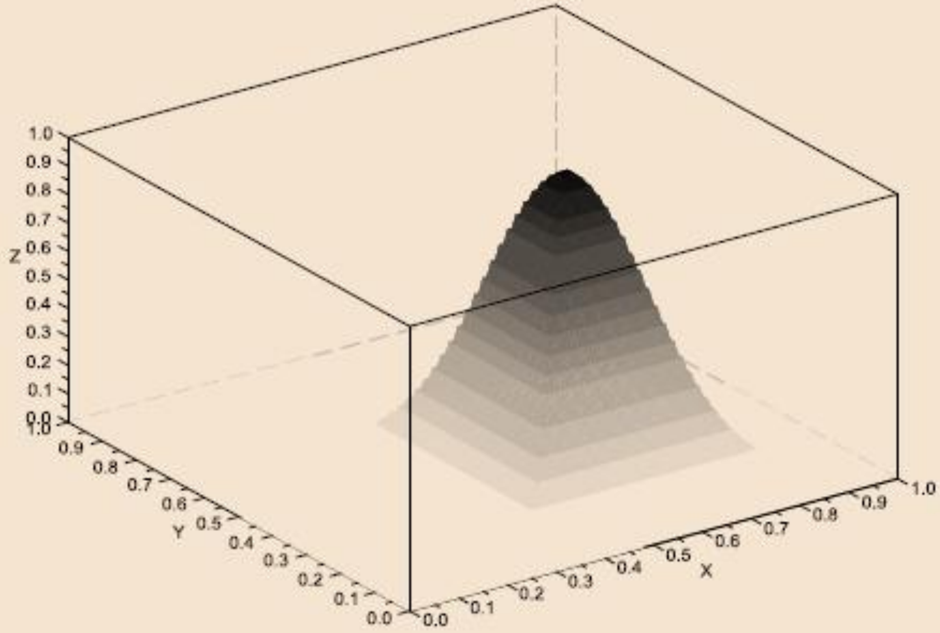


**Fig. 6.3** A fuzzy rule interpreted as a fuzzy relation

**Remark 6.7.** *In many applications a fuzzy rule will have several antecedents that are used in conjunction to build our fuzzy rule. For example a more complex fuzzy rule can be considered*

$$\textit{If } x \textit{ is } A \textit{ and } y \textit{ is } B \textit{ then } z \textit{ is } C.$$

*In this case the antecedents are naturally combined into a fuzzy relation*

$$D(x, y) = A(x) \wedge B(y)$$

*For example, in this case the Mamdani rule will be*

$$R_M(x, y, z) = A(x) \wedge B(y) \wedge C(z).$$

*the Gödel rule as*

$$R_G(x, y, z) = A(x) \wedge B(y) \rightarrow C(z).$$

# Fuzzy Rule Base

**Definition 6.9.** *(Mamdani-Assilian [107]) We define the* ***fuzzy rule base***

$$\text{If } x \text{ is } A_i \text{ then } y \text{ is } B_i, \quad i = 1, ..., n.$$

*as a fuzzy relation as follows:*

*(i) Mamdani rule base:*

$$R_M(x,y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y)$$

*(ii) Larsen rule base:*

$$R_L(x,y) = \bigvee_{i=1}^{n} A_i(x) \cdot B_i(y)$$

*(iii) max-t-norm rule base:*

$$R_T(x,y) = \bigvee_{i=1}^{n} A_i(x) T B_i(y),$$

*with $T$ being an arbitrary t-norm*

*(iv) Gödel rule base:*

$$R_G(x,y) = \bigwedge_{i=1}^{n} A_i(x) \to B_i(y),$$

*with* $\to$ *being Gödel implication*

*(v) Gödel residual rule base:*

$$R_R(x,y) = \bigwedge_{i=1}^{n} A_i(x) \to_T B_i(y),$$

*with* $\to_T$ *being a residual implication with a given t-norm* $T$.

**Remark 6.10.** *As individual rules may have more antecedents linked through conjunctions we can have the same situation for a fuzzy rule base*

$$\textit{If } x \textit{ is } A_i \textit{ and } y \textit{ is } B_i \textit{ then } z \textit{ is } C_i, i = 1, ..., n.$$

*In this case the antecedents are combined into a fuzzy relation* $A_i(x) \wedge B_i(y)$. *Then the Mamdani rule will be*

$$R_M(x, y, z) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y) \wedge C_i(z).$$

*Between the antecedents we do not have an implication, instead we will have conjunction or eventually a t-norm. So, when a fuzzy rule base with more antecedents is translated into a fuzzy relation we obtain the Gödel rule base as*

$$R_G(x, y, z) = \bigwedge_{i=1}^{n} A_i(x) \wedge B_i(y) \to C_i(z).$$

**Example 6.11.** *Considering the temperature control example described above, the output can be scaled on the* $[-2, 2]$ *interval (*$-2$ *being the strongest cooling and* $2$ *stands for the strongest heating. The consequences* $B_1, ..., B_5$ *are triangular fuzzy numbers* $B_1 = (-2, -2, -1)$, $B_2 = (-2, -1, 0)$, $B_3 = (-1, 0, 1)$, $B_4 = (0, 1, 2)$, $B_5 = (1, 2, 2)$. *The Mamdani rule base can be represented as in Fig. 6.4. The Gödel rule base of the same example is shown in Fig. 6.5.*

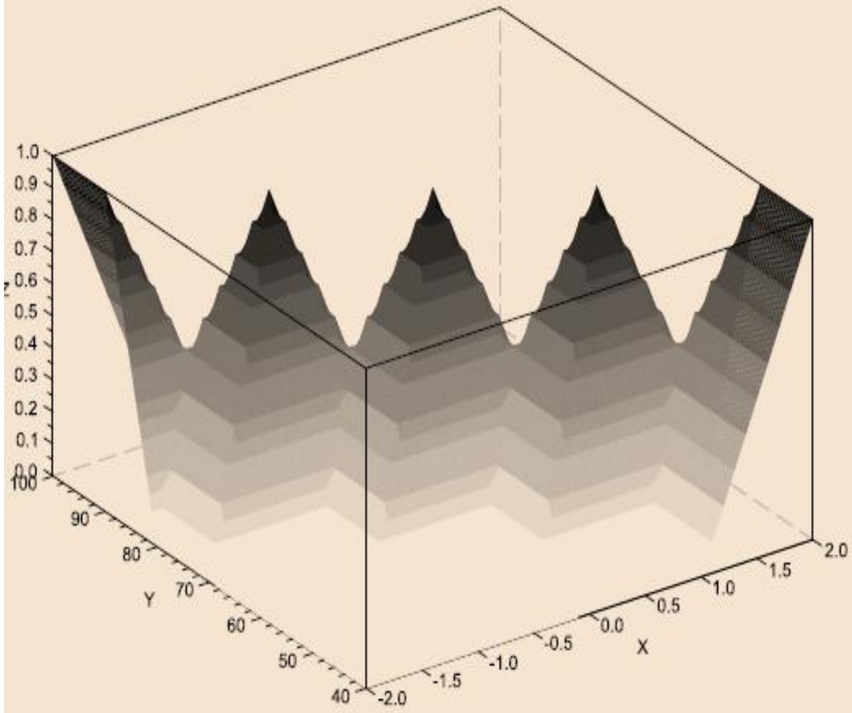


Fig. 6.4 The Mamdani rule base

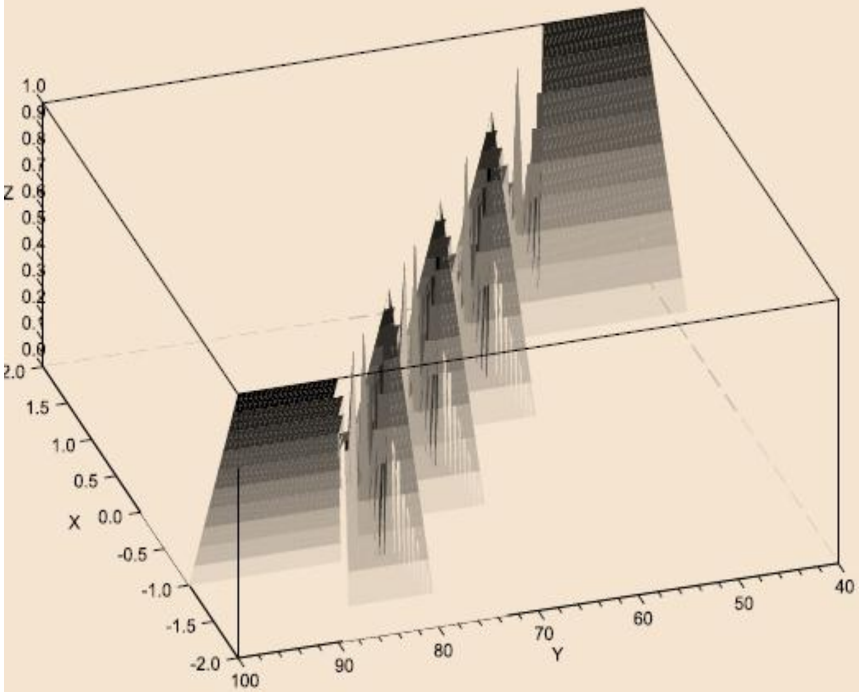


Fig. 6.5 The Gödel rule base

# Fuzzy Inference

Fuzzy inference is the process of obtaining a conclusion for a given input that was possibly never encountered before. The basic rule (law) for a fuzzy inference system is the **compositional rule of inference** (Zadeh [155]) It is based on the classical rule of Modus Ponens. Let us recall first the classical Modus Ponens of Boolean logic:

$$premise: \text{ if } p \text{ then } q$$

$$fact: \; p$$

$$conclusion: \; q$$

Given a fuzzy rule or a fuzzy rule base $R \in \mathcal{F}(X \times Y)$, the compositional rule of inference is a function $F : \mathcal{F}(X) \to \mathcal{F}(Y)$ determined through a composition $B' = F(A') = A' * R$, with $* : \mathcal{F}(x) \times \mathcal{F}(X \times Y) \to \mathcal{F}(Y)$ being a composition of fuzzy relations.

The compositional rule of inference consists of a

$$premise: \text{ if } x \text{ is } A_i \text{ then } y \text{ is } B_i, i = 1, ..., n$$

$$fact: \; x \text{ is } A'$$

$$conclusion: \; y \text{ is } B'$$

**Definition 6.13.** *(Mamdani-Assilian [107], see also e.g. Fullér [64], Fullér [63]) We define a* ***fuzzy inference*** *based on a composition law as follows:*

*(i) Mamdani Inference:*

$$B'(y) = A' \circ R(x, y) = \bigvee_{x \in X} A'(x) \wedge R(x, y)$$

*(ii) Larsen inference:*

$$B'(y) = A' \circ_L R(x, y) = \bigvee_{x \in X} A'(x) \cdot R(x, y)$$

*(iii) Generalized modus ponens or t-norm-based inference:*

$$B'(y) = A' \circ_T R(x, y) = \bigvee_{x \in X} A'(x) T R(x, y)$$

*with $T$ being an arbitrary t-norm.*

*(iv) Gödel Inference*

$$B'(y) = A' \triangleleft R(x, y) = \bigwedge_{x \in X} A'(x) \rightarrow R(x, y)$$

*with* $\rightarrow$ *being Gödel implication.*

*(v) Gödel residual inference*

$$B'(y) = A' \triangleleft_T R(x, y) = \bigwedge_{x \in X} A'(x) \rightarrow_T R(x, y)$$

*with* $\rightarrow_T$ *being a residual implication with a given t-norm.*

*Throughout the definition,* $R(x, y)$ *is a fuzzy relation that is used for interpreting the fuzzy rule base in the premise.*

**Example 8.1**

Let sets of values of variables $\mathcal{X}$ and $\mathcal{Y}$ be $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively. Assume that a proposition "if $\mathcal{X}$ is $A$, then $\mathcal{Y}$ is $B$" is given, where $A = .5/x_1 + 1/x_2 + .6/x_3$ and $B = 1/y_1 + .4/y_2$. Then, given a fact expressed by the proposition "$x$ is $A'$," where $A' = .6/x_1 + .9/x_2 + .7/x_3$, we want to use the generalized modus ponens (8.41) to derive a conclusion in the form "$\mathcal{Y}$ is $B'$."

Using, for example, the Lukasiewicz implication (8.14), we obtain

$$R = 1/x_1, y_1 + .9/x_1, y_2 + 1/x_2, y_1 + .4/x_2, y_2 + 1/x_3, y_1, +.8/x_3, y_2$$

by (8.40). Then, by the compositional rule of inference (8.39), we obtain

$$\begin{aligned} B'(y_1) &= \sup_{x \in X} \min[A'(x), R(x, y_1)] \\ &= \max[\min(.6, 1), \min(.9, 1), \min(.7, 1)] \\ &= .9 \\ B'(y_2) &= \sup_{x \in X} \min[A'(x), R(x, y_2)] \\ &= \max[\min(.6, .9), \min(.9, .4), \min(.7, .8)] \\ &= .7 \end{aligned}$$

Thus, we may conclude that $\mathcal{Y}$ is $B'$, where $B' = .9/y_1 + .7/y_2$.

**Remark 6.15.** *As previously discussed we may have a fuzzy rule base with more antecedents, so we also need a fuzzy inference system to be able to deal with more antecedents. This is possible to be done by using the same strategy to combine the premises as the one used for the antecedents in the fuzzy rule base.*

*For example if we have a fuzzy inference system of the following form*

$$premise: \ \textit{If } x \textit{ is } A_i \textit{ and } y \textit{ is } B_i \textit{ then } z \textit{ is } C_i$$

$$fact: \ x \textit{ is } A' \textit{ and } y \textit{ is } B'$$

$$conclusion \ z \textit{ is } C'$$

*where $C'$ is determined based on a composition as for example*

$$C'(z) = (A' \wedge B') \circ R(x, y, z) = \bigvee_{x \in X, y \in Y} A'(x) \wedge B'(y) \wedge R(x, y, z)$$

*(for a Mamdani inference) and*

$$C'(z) = (A' \wedge B') \triangleleft R(x, y, z)$$
$$= \bigwedge_{x \in X, y \in Y} A'(x) \wedge B'(y) \to R(x, y, z)$$

*(for Gödel inference).*

*The fuzzy relation $R(x, y, z)$ is being used here to interpret the fuzzy rule base in the premise.*

**Example 6.16.** *To build a fuzzy inference system that solves the room temperature control problem discussed in the previous sections we can use e.g. a Mamdani or Gödel inference together with a Mamdani or a Gödel rule base. In Fig. 6.6 the input of the system is represented.*

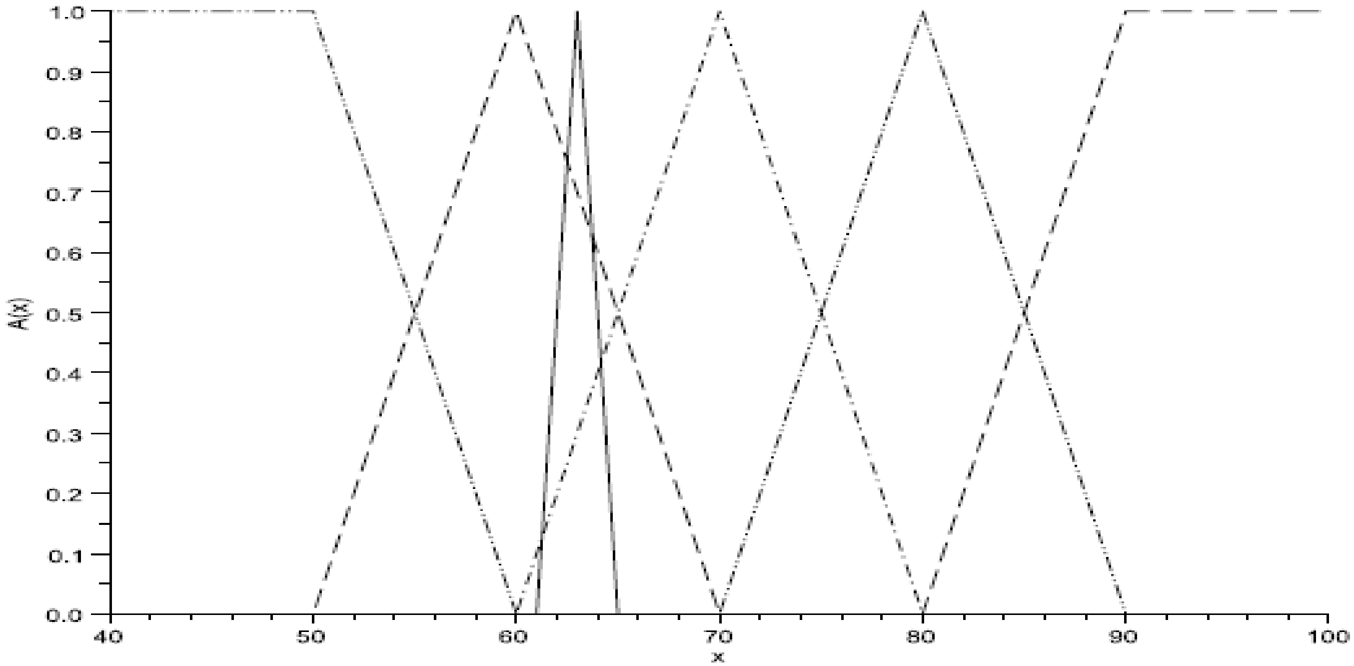


**Fig. 6.6** The input of a fuzzy inference system

$$\begin{array}{ll} \text{Rule 1:} & \text{If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Y} \text{ is } B \\ \text{Rule 2:} & \text{If } \mathcal{Y} \text{ is } B, \text{ then } \mathcal{Z} \text{ is } C \\ \hline \text{Conclusion:} & \text{If } \mathcal{X} \text{ is } A, \text{ then } \mathcal{Z} \text{ is } C \end{array} \tag{8.43}$$

In this case, $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are variables taking values in sets $X, Y, Z$, respectively, and $A, B, C$ are fuzzy sets on sets $X, Y, Z$, respectively.

For each conditional fuzzy proposition in (8.43), there is a fuzzy relation determined by (8.40). These relations are determined for each $x \in X$, $y \in Y$, and $z \in Z$ by the equations

$$R_1(x, y) = \mathcal{J}[A(x), B(y)],$$

$$R_2(y, z) = \mathcal{J}[B(y), C(z)],$$

$$R_3(x, z) = \mathcal{J}[A(x), C(z)].$$

Given $R_1, R_2, R_3$, obtained by these equations, we say that the generalized hypothetical syllogism holds if

$$R_3(x, z) = \sup_{y \in Y} \min[R_1(x, y), R_2(y, z)], \tag{8.44}$$

which again expresses the compositional rule of inference. This equation may also be written in the matrix form

$$\mathbf{R}_3 = \mathbf{R}_1 \circ \mathbf{R}_2. \tag{8.45}$$

**Example 8.3**

Let $X, Y$ be the same as in Example 8.1, and let $Z = \{z_1, z_2\}$. Moreover, let $A = .5/x_1 + 1/x_2 + .6/x_3$, $B = 1/y_1 + .4/y_2$, $C = .2/z_1 + 1/z_2$, and

$$\mathcal{J}(a, b) = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b. \end{cases}$$

Then, clearly,

$$\mathrm{R}_1 = \begin{bmatrix} 1 & .4 \\ 1 & .4 \\ 1 & .4 \end{bmatrix}, \quad \mathrm{R}_2 = \begin{bmatrix} .2 & 1 \\ .2 & 1 \end{bmatrix}, \quad \mathrm{R}_3 = \begin{bmatrix} .2 & 1 \\ .2 & 1 \\ .2 & 1 \end{bmatrix}$$

The generalized hypothetical syllogism holds in this case since $\mathrm{R}_1 \circ \mathrm{R}_2 = \mathrm{R}_3$.

# The Interpolation Property of a Fuzzy Inference System

- A natural property to be required for a fuzzy inference system is the following **interpolation property.**

Let us start by considering the Mamdani inference with Mamdani rule. The system is described by

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \;\; R(x,y) = A(x) \wedge B(y)$$

The interpolation property says that when the input of the system coincides with the antecedent ($A' = A$) then the output has to be coincident with the consequence ($B' = B$), i.e., our fuzzy inference is a generalization of the Modus Ponens of classical logic.

**Proposition 6.17.** *The Mamdani inference with Mamdani rule*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y),\ \ R(x,y) = A(x) \wedge B(y),$$

*such that there exists an $x_0 \in X$ with $A(x_0) = 1$, satisfies the interpolation property, i.e., if $A' = A$, then $B' = B$.*

**Proof.** We have

$$B'(y) = \bigvee_{x \in X} A(x) \wedge R(x,y) = \bigvee_{x \in X} A(x) \wedge A(x) \wedge B(y)$$

$$= \bigvee_{x \in X} (A(x) \wedge B(y)) = \left( \bigvee_{x \in X} A(x) \right) \wedge B(y).$$

Since $A$ is normalized we have $\bigvee_{x \in X} A(x) = 1$ and then $B'(y) = 1 \wedge B(y) = B(y)$. ■

**Proposition 6.19.** *The Mamdani inference with Larsen rule*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \; R(x,y) = A(x) \cdot B(y)$$

*such that there exists an* $x_0 \in X$ *with* $A(x_0) = 1$ *satisfies the interpolation property, i.e., if* $A' = A$ *then* $B' = B$.

**Proposition 6.20.** *The Mamdani inference with t-norm rule*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \;\; R(x,y) = A(x)TB(y)$$

*such that there exists an* $x_0 \in X$ *with* $A(x_0) = 1$ *satisfies the interpolation property, i.e., if* $A' = A$ *then* $B' = B$.

**Proof.** From the properties of a t-norm we immediately get $A(x)TB(y) \leq A(x)$. We have

$$B'(y) = \bigvee_{x \in X} A(x) \wedge R(x,y)$$

$$= \bigvee_{x \in X} A(x) \wedge (A(x)TB(y)) = \bigvee_{x \in X} A(x)TB(y)$$

$$= \left( \bigvee_{x \in X} A(x) \right) T B(y) = 1 T B(y) = B(y).$$

Let us consider the Mamdani inference with Gödel rule

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x, y), \;\; R(x, y) = A(x) \to B(y)$$

**Remark 6.21.** *We observe here that we can interpret the interpolation property of a Mamdani inference with Gödel rule as a fuzzy relational equation. Indeed, the interpolation property can be written as* $A \circ R = B$. *This equation, by Sanchez Theorem* 3.24 *has a solution if and only if* $R(x,y) = A^{-1} \triangleleft B$ *is a solution, and in this case it is the greatest solution* ($A(x) = A(1,x)$, $B(y) = B(1,y)$ *are interpreted here as row vectors then*

$$R(x,y) = A^{-1} \triangleleft B(x,y)$$

$$= \bigwedge_{i=1} A(x,1) \to B(1,y) = A(x) \to B(y)$$

*becomes the Gödel implication. As a conclusion, the fuzzy relational equation is solvable if and only if the inference system with Mamdani inference and Gödel rule has the interpolation property.*

**Proposition 6.22.** *The Mamdani inference with Gödel rule*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x, y), \;\; R(x, y) = A(x) \rightarrow B(y)$$

*such that* $A$ *is continuous and there exists an* $x_0 \in X$ *such that* $A(x_0) = 1$ *fulfills the interpolation property, i.e., if* $A' = A$ *then* $B' = B$.

**Remark 6.23.** *We observe here that we can interpret the interpolation property of a Gödel inference with Mamdani rule as a fuzzy relational equation* $A \triangleleft R = B$. *This equation, by Miyakoshi-Shimbo Theorem 3.27 has a solution if and only if* $R(x,y) = A^{-1} \circ B$ *is a solution, and in this case it is the least solution and then*

$$R(x,y) = A^{-1} \circ B(x,y) = \bigvee_{i=1} A(x,1) \wedge B(1,y) = A(x) \wedge B(y)$$

*becomes the Mamdani rule. As a conclusion, the equation is solvable if and only if the inference system with Mamdani inference and Gödel rule has the interpolation property.*

**Proposition 6.24.** *The Gödel inference with Mamdani rule*

$$B'(y) = \bigwedge_{x \in X} A'(x) \to R(x,y), \;\; R(x,y) = A(x) \wedge B(y)$$

*with* $A \in \mathcal{F}$ *such that there exists an* $x_0 \in X$ *such that* $A(x_0) = 1$ *possesses the interpolation property, i.e., if* $A' = A$ *then* $B' = B$.

**Proposition 6.25.** *Let us consider the Mamdani inference with Mamdani rule base*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \;\; R(x,y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y)$$

*such that*

*(i) for every $i = 1, ..., n$ there exists an $x_i \in X$ such that $A_i(x_i) = 1$;*

*(ii) any two antecedents $A_i, A_j$, $i \neq j$ have disjoint supports*

*Then the interpolation property is verified, i.e., if $A' = A_j$ then $B' = B_j$.*

**Proposition 6.26.** *Let us consider the Mamdani inference with Larsen rule base*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \;\; R(x,y) = \bigvee_{i=1}^{n} A_i(x) \cdot B_i(y)$$

*such that*

*(i) for every* $i = 1, ..., n$ *there exists an* $x_i \in X$ *such that* $A_i(x_i) = 1$*;*

*(ii) any two antecedents* $A_i, A_j$, $i \neq j$ *have disjoint supports*

*Then the interpolation property is verified, i.e., if* $A' = A_j$ *then* $B' = B_j$.

**Proposition 6.28.** *Let us consider the Mamdani inference with Mamdani rule base*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y),\ \ R(x,y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y)$$

*such that for every* $i = 1, ..., n$ *there exists an* $x_i \in X$ *such that* $A_i(x_i) = 1$. *Under this condition, if* $A' = A_j$ *then* $B' \geq B_j$.

**Remark 6.29.** *We observe here that we can interpret the interpolation property of a Mamdani inference with Gödel rule base as a fuzzy relational equation. Indeed, the interpolation property can be written as* $A_i \circ R = B_i$, $i = 1, ..., n$. *If* $A_i(x) = A(i, x)$, $B_i(y) = B(i, y)$, $i = 1, ..., n$ *are interpreted as fuzzy relations we can rewrite our equation as* $A \circ R = B$. *This equation has a solution if and only if* $R(x, y) = A^{-1} \triangleleft B$ *is a solution, and in this case it is the greatest solution*

$$R(x, y) = A^{-1} \triangleleft B(x, y) = \bigwedge_{i=1}^{n} A(x, i) \to B(i, y)$$

$$= \bigwedge_{i=1}^{n} A_i(x) \to B_i(y).$$

*As a conclusion, the equation is solvable if and only if the inference system with Mamdani inference and Gödel rule base has the interpolation property.*

**Proposition 6.30.** *Let us consider a Mamdani inference with Gödel rule base*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \;\; R(x,y) = \bigwedge_{i=1}^{n} A_i(x) \to B_i(y).$$

*(i) If $A_i$ are normal and continuous and*

*(ii) if the core of any $A_i$ and the support of any $A_j$ are disjoint when $i \neq j$,*

*then the Mamdani inference with Gödel rule base satisfies the interpolation property: if $A' = A_j$ then $B' = B_j$.*

**Proposition 6.31.** *Let us consider a Mamdani inference with Gödel rule base*

$$B'(y) = \bigvee_{x \in X} A'(x) \wedge R(x,y), \;\; R(x,y) = \bigwedge_{i=1}^{n} A_i(x) \to B_i(y).$$

*If $A_i$ are normal and if $A' = A_j$ then $B' \leq B_j$.*

**Remark 6.32.** *We observe here that we can interpret the interpolation property of a Gödel inference with Mamdani rule base as a fuzzy relational equation* $A_i \triangleleft R = B_i, i = 1, ..., n$ *or considering* $A(i, x) = A_i(x)$ *and* $B(i, x) = B_i(x)$ *we can write* $A \triangleleft R = B$. *This equation, by Miyakoshi-Shimbo theorem has a solution if and only if* $R(x, y) = A^{-1} \circ B$ *is a solution, and in this case it is the least solution and we also have*

$$R(x, y) = A^{-1} \circ B(x, y) = \bigvee_{i=1} A(x, i) \wedge B(i, y)$$

$$= \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y).$$

*becomes the Gödel implication. As a conclusion, the equation is solvable if and only if the inference system with Mamdani inference and Gödel rule base has the interpolation property.*

**Proposition 6.33.** *Let us consider the Gödel inference with Mamdani rule base*

$$B'(y) = \bigwedge_{x \in X} A'(x) \to R(x, y), \;\; R(x, y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y).$$

*(i) If $A_i$ are normal and continuous*

*(ii) and if the core of any $A_i$ and the support of any $A_j$ are disjoint when $i \neq j$,*

*then the interpolation property holds i.e., if $A' = A_j$ then $B' = B_j$.*

**Proposition 6.34.** *Let us consider the Gödel inference with Mamdani rule base*

$$B'(y) = \bigwedge_{x \in X} A'(x) \to R(x,y), \;\; R(x,y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y).$$

*If $A_i$ are normal and if $A' = A_j$ then $B' \geq B_j$.*

**Proposition 6.35.** *Let us consider the Gödel residual inference with Mamdani rule base*

$$B'(y) = \bigwedge_{x \in X} A'(x) \to_T R(x,y), \;\; R(x,y) = \bigvee_{i=1}^{n} A_i(x) \wedge B_i(y).$$

*(i) If $A_i$ are normal and continuous*

*(ii) and if the core of any $A_i$ and the support of any $A_j$ are disjoint when $i \neq j$,*

*then the interpolation property holds i.e., if $A' = A_j$ then $B' = B_j$.*

# Example of a Fuzzy Inference System

**Example 6.36.** *"Mr. John Smith has been shot dead in his house. He was found by his friend, Mr. Carry. Lt. Columbo suspects Mr. Carry to be the murderer.*

*Mr. Carry's testimony is the following: I have started from my home at about 6:30, arrived to John's house at about 7, found John dead and went immediately to the phone box to call police. They told me to wait and came immediately.*

*Lt. Columbo has found the following evidence about dead Mr. Smith: He had high quality suit with broken wristwatch stopped at 5:45. No evidence of strong strike on his body. Lt. Columbo touched engine of Mr. Carry's car and found it to be more or less cold."*

*To be able to analyze the problem we need to understand and implement (following the ideas in Dvorak-Novak [54]) commonsense knowledge. We will use fuzzy if then rules for this task. Let us start with describing how engine temperature depends on drive duration.*

1. *If drive duration is big and time stopped is small then engine is hot.*

2. *If drive duration is small then engine is cold.*

3. *If time stopped is big then engine is cold.*

*Another set of fuzzy rules concerns the wristwatch quality.*

1. *If suit quality is high then wristwatch quality is high.*

2. *If suit quality is low then wristwatch quality is low.*

*The last set of fuzzy rules concerns how likely the wristwatch is broken.*

1. *If wristwatch quality is high and strike is unlikely then broken is unlikely.*
2. *If wristwatch quality is low and strike is likely then broken is likely.*
3. *If strike is likely then broken is more or less likely.*

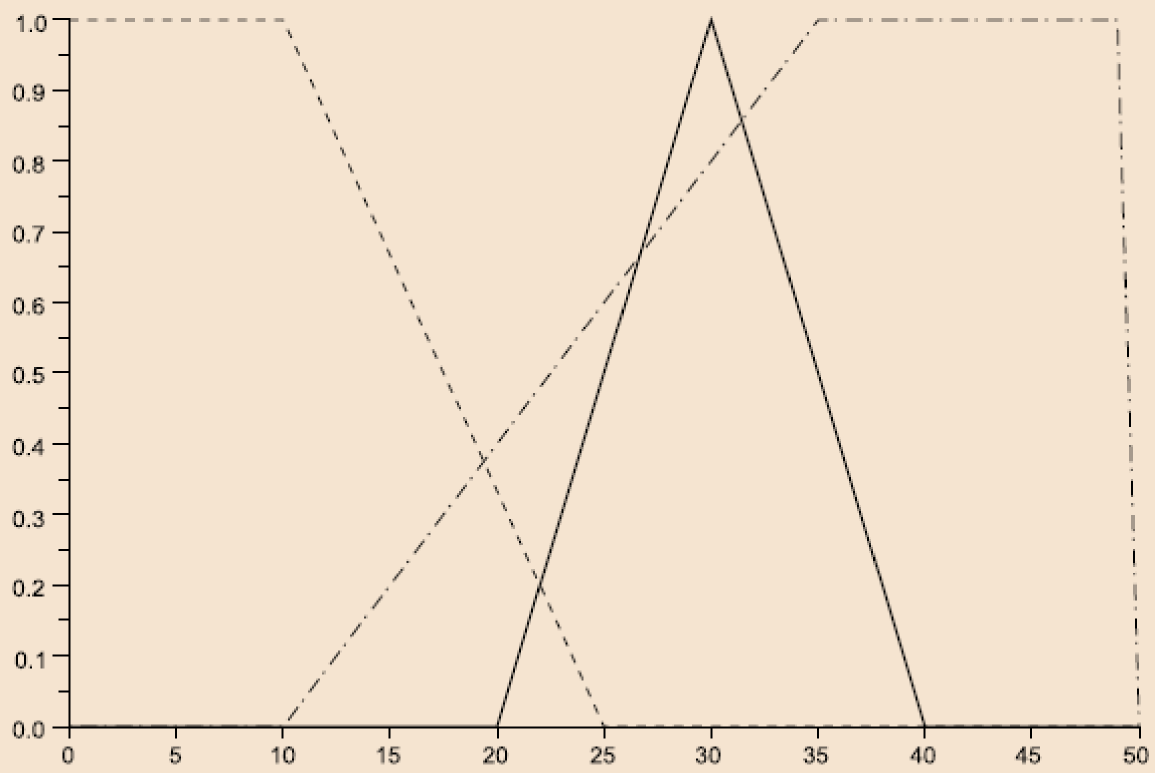


**Fig. 6.11** Drive duration small (dash), big (dash-dot) and about 30' from Mr. Carry's testimony (solid line)

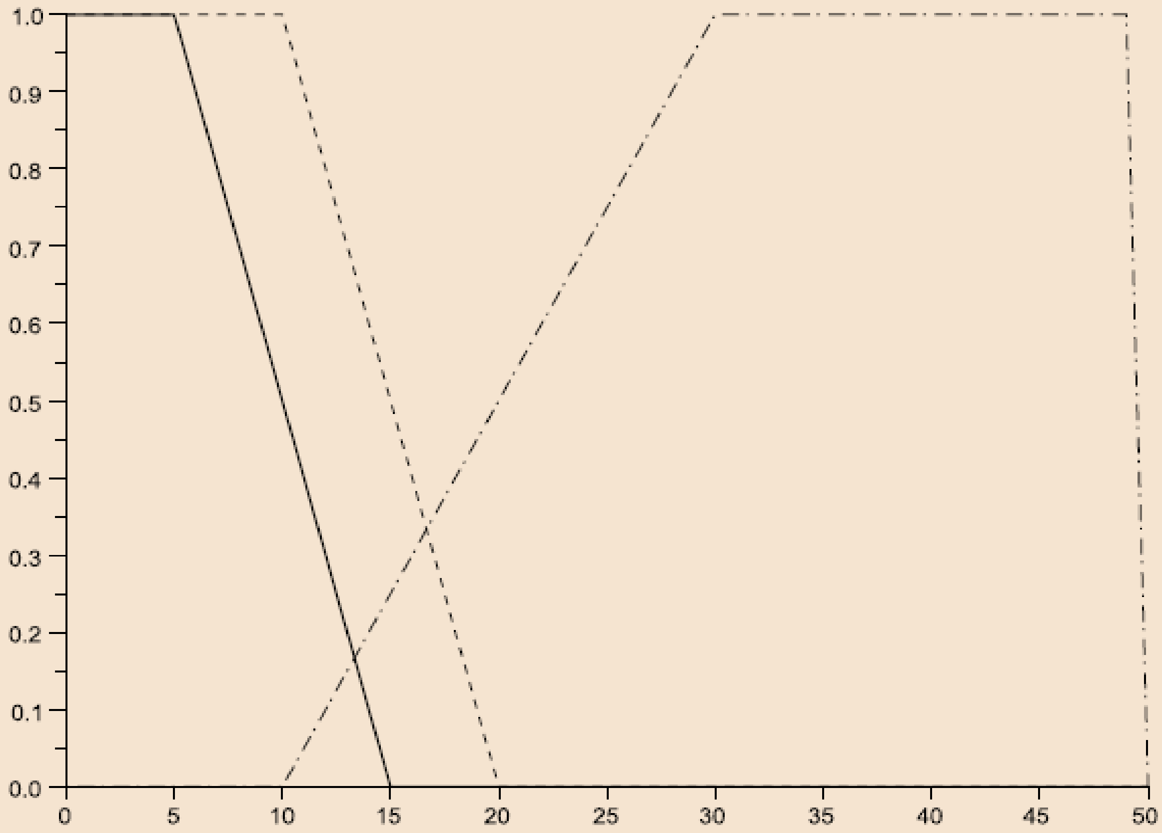


**Fig. 6.12** Time stooped small (dash), big (dash-dot) and the fuzzy set representing the fact that police came immediately (solid line)