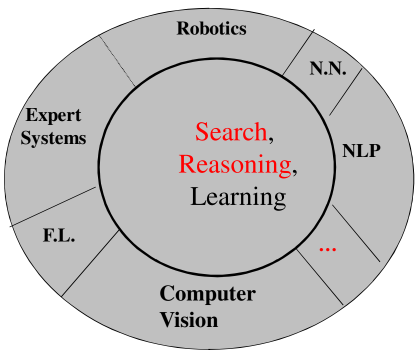
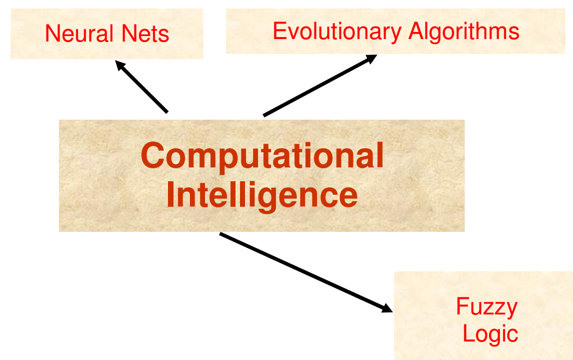
A.I.



"Soft Computing"



Soft Computing

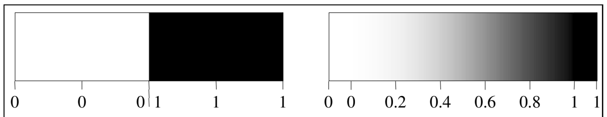
- Soft computing differs from conventional (hard) computing in that:" Its tolerance to imprecision, uncertainty, partial truth, and approximation.
- The model for soft computing is the human mind.
- The guiding principle of soft computing is: Exploit the tolerance for **imprecision**, **uncertainty**, **partial truth**, and **approximation** to achieve tractability, robustness and low solution cost.
- Soft computing is used as an umbrella term for sub-disciplines of computing, including fuzzy logic and fuzzy control, neural networks based computing and machine learning, and genetic algorithms, together with chaos theory in physics.

Introduction to Fuzzy Logic

Fuzzy logic is based on **degrees of membership.**

Unlike two-valued Boolean logic, fuzzy logic is **multi-valued**. It deals with **degrees of membership** and **degrees of truth**. Fuzzy logic uses the continuum of logical values between 0 (completely false) and 1 (completely true). Instead of just black and white, it employs the spectrum of colours, accepting that things can be partly true and partly false at the same time.

Range of logical values in Boolean and fuzzy logic



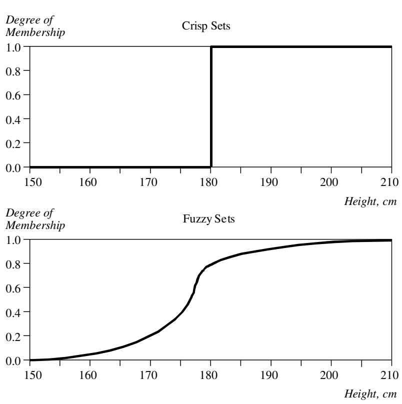
(*a*) Boolean Logic. (*b*) Multi-valued Logic.

Fuzzy sets

- The classical example in fuzzy sets is ***tall men***. The elements of the fuzzy set "tall men" are all men, but their degrees of membership depend on their height.

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

Crisp and fuzzy sets of "***tall men***"



Fuzzy Logic

"Because **Fuzzy Logic** is similar to the way we talk and think, it is easier for us to adjust" Cynthia Taylor

- Fuzzy Logic is particularly good at handling uncertainty, vagueness and imprecision.
- This is especially useful where a problem can be described linguistically (using words) or, as with neural networks, where there is data and you are looking for relationships or patterns within that data.
- Fuzzy Logic uses imprecision to provide robust, tractable solutions to problems.
- Fuzzy logic relies on the concept of a ***fuzzy set***.

Fuzzy sets, logic, inference, control

Explanation of some terms. The following have become widely accepted:

Fuzzy logic system

anything that uses fuzzy set theory

Fuzzy control

any control system that employs fuzzy logic

Fuzzy associative memory

any system that evaluates a set of fuzzy *if-then* rules uses fuzzy inference. Also known as **fuzzy rule base** or **fuzzy expert system**

Fuzzy inference control

a system that uses fuzzy control and fuzzy inference

Fuzzy sets, logic, inference, control

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Fuzzy inference control

a system that uses fuzzy control and fuzzy inference

Remark: different authors and researchers use the same term either for the same thing or for different things.

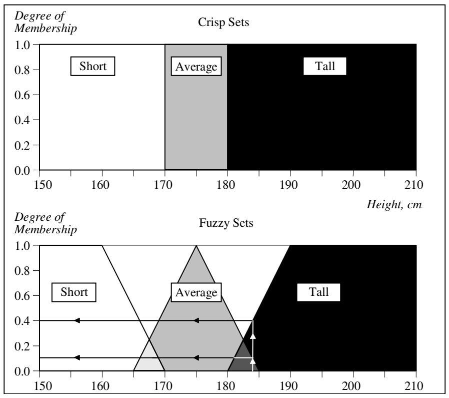
Fuzzy Logic

- It is hard to characterize the truth of "John is old" as unambiguously true or false if John is 60 years old.
- In some respects he is old, being eligible for senior citizen benefits at many establishments,
- But in other respects he is not old since he still can work.
- ***The truth value of the statement could be tv(John is old) = 0.70***
- ***The negation of the statement may be*** $tv(\neg p) = 1 - tv(p)$
- ***We need also to find*** $tv(p \wedge q)$ ***and*** $tv(p \wedge q)$

How to represent a fuzzy set in a computer?

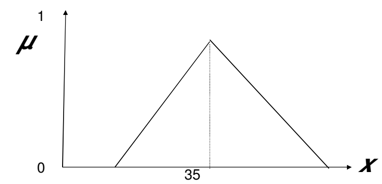
- First, we determine the membership functions. In our "***tall men***" example, we can obtain fuzzy sets of ***tall***, ***short*** and ***average*** men.
- The universe of discourse – the men's heights – consists of three sets: ***short***, ***average*** and ***tall men***. As you will see, a man who is 184 cm tall is a member of the ***average men*** set with a degree of membership of 0.1, and at the same time, he is also a member of the ***tall men*** set with a degree of 0.4.

Crisp and fuzzy sets of short, average and tall men



Imprecision / Vagueness / Uncertainty

In many physical systems, measurements are never precise
Fuzzy numbers are one way of capturing this imprecision by having a fuzzy set representing a real number where the numbers in an interval near to the number are in the fuzzy set to some degree.



The fuzzy number 'About 35'

Fuzzy sets

- The *x*-axis represents the **universe of discourse** – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.
- The *y*-axis represents the **membership value of the fuzzy set**. In our case, the fuzzy set of "*tall men*" maps height values into corresponding membership values.

Fuzzy sets

A fuzzy set is a set with fuzzy boundaries

- Let X be the universe of discourse and its elements be denoted as x. In the classical set theory, crisp set A of X is defined as function $\psi_A(x)$ called the **characteristic function of** A

$$\psi_A(x): X \to \{0,1\}$$, where $$\psi_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

This set maps universe X to a set of two elements. For any element x of universe X, characteristic function is $\psi_A(x)$ equal to 1 if x is an element of set A, and is equal to 0 if x is not an element of A.

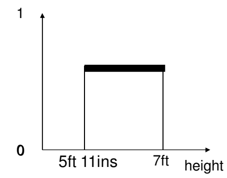
Fuzzy sets

For any fuzzy set, (let’s say) A, the function μ_A represents the membership function for which $\mu_A(x)$ indicates the degree of membership that x, of the universal “crisp” set X, belongs to set A and is, usually, expressed as a number between 0 and 1

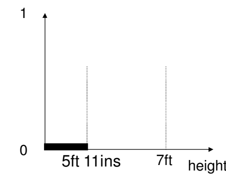
$$\mu_A(x) : X \rightarrow [0,1]$$

Fuzzy sets can be either discrete or continuous

Let’s consider the first example (How tall/short we are?)

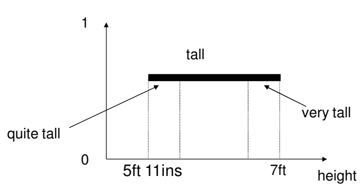


A crisp way of modelling tallness



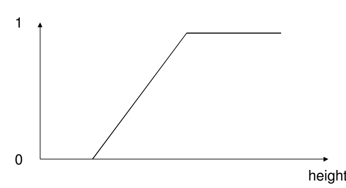
A crisp version of short

Let’s consider the first example (How tall/short we are?)

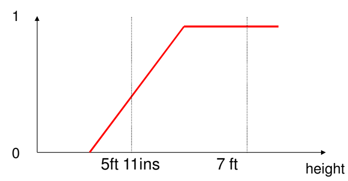


crisp definitions for tallness

Definition in a Fuzzy Set (How tall/short we are?)

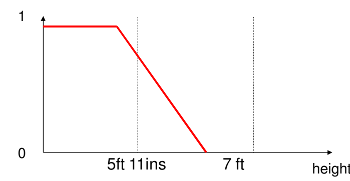


Definition in a Fuzzy Set
(How tall/short we are?)



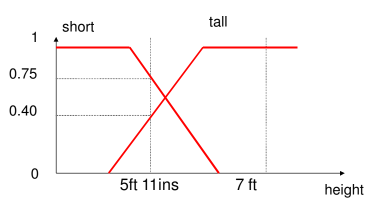
A possible fuzzy set tall

Definition in a Fuzzy Set
(How tall/short we are?)



A possible fuzzy set short

Definition in a Fuzzy Set
(How tall/short we are?)



Membership functions that represent tallness and short

Membership function

Membership functions can

- either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
- Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)

There are different shapes of membership functions; <u>triangular</u>, <u>trapezoidal</u>, piecewise-linear, Gaussian, bell-shaped, etc.

Discrete Fuzzy sets

The notation for fuzzy sets: for the member, *x*, of a discrete set with membership *μ*, x is a member of the set to degree *μ*.

Discrete sets are defined as:

$$A = \frac{\mu_1}{x_1} + \frac{\mu_2}{x_2} + \ldots + \frac{\mu_n}{x_n}$$

or (in a more compact form)

$$A = \sum_{i=1}^{n} \frac{\mu_i}{x_i}$$

$x_1, x_2, \ldots x_n$:members of the set *A*

$\mu_1, \mu_2, \ldots \mu_n : x_1, x_2, \ldots x_n$'s degree of membership.

Example Discrete case

Suppose we have the following (discrete) fuzzy sets:

A = 0.4/1+0.6/2+0.7/3+0.8/4

B = 0.3/1+0.65/2+0.4/3+0.1/4

The union of the fuzzy sets A and B

=0.4/1+0.65/2+0.7/3+0.8/4

The intersection of the fuzzy sets A and B

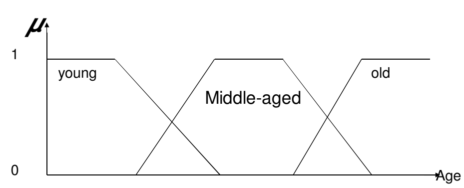
=0.3/1+0.6/2+0.4/3+0.1/4

The complement of the fuzzy set A

= 0.6/1+0.4/2+0.3/3+0.2/4

Continuous Fuzzy sets

Example: describing people as "young", "middle-aged", and "old"



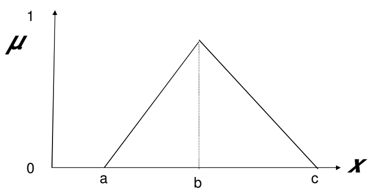
Fuzzy Logic allows modelling of linguistic terms using linguistic variables and linguistic values. The fuzzy sets "young", "middle-aged", and "old" are fully defined by their membership functions. The linguistic variable "Age" can then take linguistic values.

Most Common Membership functions

- TRIANGULAR: $\text{tri}(x; a, b, c) = \max\left\{\min\left\{\frac{x-a}{b-a}, \frac{c-x}{c-b}\right\}, 0\right\}$
- TRAPEZOIDAL: $\text{trap}(x; a, b, c, d) = \max\left\{\min\left\{\frac{x-a}{b-a}, \frac{d-x}{d-c}, 1\right\}, 0\right\}$
- GAUSSIAN: $\text{gauss}(x; c, \sigma) = \exp\left[-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2\right]$
- GENERALISED BELL: $\text{gbell}(x; a, b) = \dfrac{1}{1 + \left|\frac{x-b}{a}\right|^{2a}}$

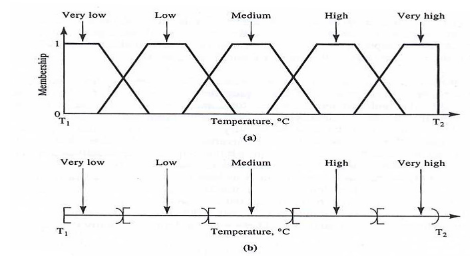
Triangular Fuzzy set

$$\mathrm{tri}(x;a,b,c) = \max\left\{\min\left\{\frac{x-a}{b-a}, \frac{c-x}{c-b}\right\}, 0\right\}$$



Trapezoidal Fuzzy Sets

$$\mathrm{trap}(x;a,b,c,d) = \max\left\{\min\left\{\frac{x-a}{b-a}, \frac{d-x}{d-c}, 1\right\}, 0\right\}$$



Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

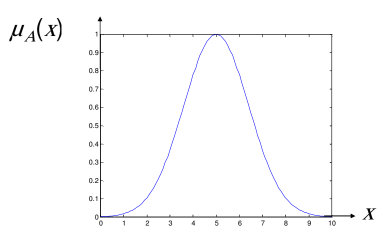
figure from G. Klir "Fuzzy Sets and Fuzzy Logic, theory and applications

Gaussian Fuzzy membership fn

Gaussian membership function

- *c:* centre
- *s:* width
- *m:* fuzzification factor (e.g., *m*=2)

$$\mu_A(x,c,s,m) = \exp\left[-\frac{1}{2}\left|\frac{x-c}{s}\right|^m\right]$$

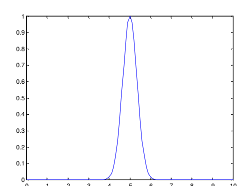


c=5

s=2

m=2

Gaussian Fuzzy membership fn

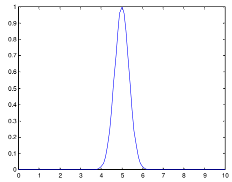


c=5

s=0.5

m=2

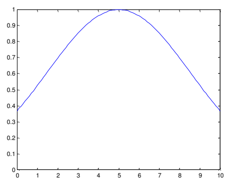
Gaussian Fuzzy membership fn



c=5

s=0.5

m=2

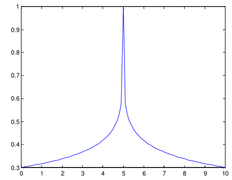


c=5

s=5

m=2

Gaussian Fuzzy membership fn

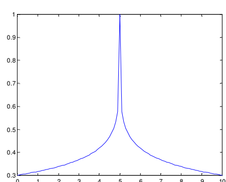


c=5

s=2

m=0.2

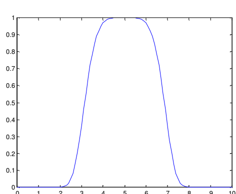
Gaussian Fuzzy membership fn



c=5

s=2

m=0.2



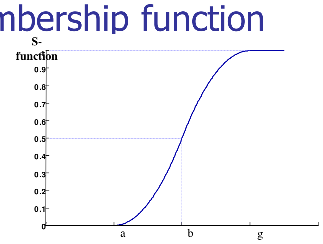
c=5

s=5

m=5

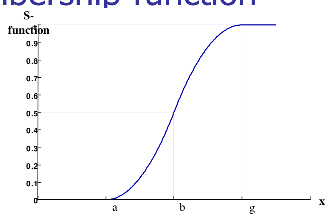
Sigmoid Membership function

$$S(x;a,b,g) = \begin{cases} 0\, for\, x \le \alpha \\ 2\left(\dfrac{x-a}{g-a}\right)^2 for\, a \langle x \le b \\ 1-2\left(\dfrac{x-a}{g-a}\right)^2 for\, b \langle x \le g \\ 1\, for\, x \rangle \gamma \end{cases}$$

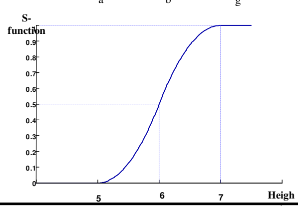


Sigmoid Membership function

$$S(x;a,b,g)=\begin{cases}0\,for\,x\le\alpha\\ 2\left(\dfrac{x-a}{g-a}\right)^2 for\,a\langle x\le b\\ 1-2\left(\dfrac{x-a}{g-a}\right)^2 for\,b\langle x\le g\\ 1\,for\,x\rangle\gamma\end{cases}$$



$$S(x;5,6,7)=\begin{cases}0\,for\,x\le 5\\ \left(\dfrac{x-5}{2}\right)^2 for\,5\langle x\le 6\\ 1-\left(\dfrac{x-7}{2}\right)^2 for\,6\langle x\le 7\\ 1\,for\,x\rangle 7\end{cases}$$

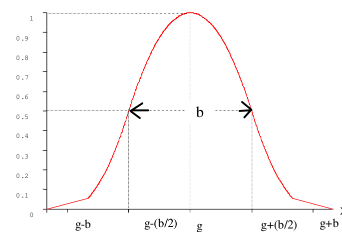


P function

$$\Pi(x;b,g)=\begin{cases}S(x;g-b,g-\dfrac{b}{2},g)\text{ for x}\le g\\ 1-S(x;g,g+\dfrac{b}{2},g+b)\text{ for x}\ge g\end{cases}$$

The P-function goes to zero at γ < β, and the 0.5 point is at γ = (β/2). Notice that the β parameter represents the bandwidth of the 0.5 points.

Pfunction



Fuzzy Operations

Fuzzy logic begins by borrowing notions from crisp logic, same as fuzzy set theory borrows from crisp set theory. As in the extension of crisp set theory to fuzzy set theory, the extension of crisp logic to fuzzy logic is made by replacing membership functions of crisp logic with fuzzy membership functions

In Fuzzy Logic, intersection, union and complement are defined in terms of their membership functions

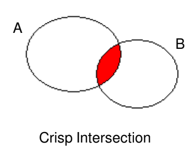
Fuzzy intersection and union correspond to 'AND' and 'OR', respectively, in classic/crisp/Boolean logic

Classic/Crisp/Boolean Logic

Logical AND (∩)

Truth Table

A	B	A ∩ B
0	0	0
0	1	0
1	0	0
1	1	1



Crisp Intersection

- Logical OR (**U**)

Truth Table

A	B	A **U** B
0	0	0
0	1	1
1	0	1
1	1	1

Crisp Union

Fuzzy Union

The union (OR) is calculated using t-conorms

t-conorm operator is a function s

Satisfying:

i. s(1,1) = 1, s(a,0) = s(0,a) = a (boundary)
ii. s(a,b) ≤ s(c,d) if a ≤ c and b ≤ d (monotonicity)
iii. s(a,b) = s(b,a) (commutativity)
iv. s(a,s(b,c)) = s(s(a,b),c) (associativity)

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iii. s(a,b) = s(b,a) (commutativity)
iv. s(a,s(b,c)) = s(s(a,b),c) (associativity)

The most commonly used method for fuzzy union is to take the maximum. That is, given two fuzzy sets *A* and *B* with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{AUB}(x) = \max(\mu_A(x), \mu_B(x))$$

Fuzzy Union

Additional requirements may be added to any t-conorm to satisfy. The most famous are:

i. u is continuous function
ii. u(a, a) > a (subidempotency)
iii. If $a<b$ and $c<d$ then $u(a,c)<u(b,d)$ (strict monotonicity)

T-conorms frequently used$_{\text{Klir}}$

Standard union: $u(a,b) = \max(a,b)$.

Algebraic sum: $u(a,b) = a + b - ab$.

Bounded sum: $u(a,b) = \min(1, a+b)$.

Drastic union: $u(a,b) = \begin{cases} a & \text{when } b = 0 \\ b & \text{when } a = 0 \\ 1 & \text{otherwise.} \end{cases}$

T-conorms frequently used$_{Klir}$

TABLE 3.3 SOME CLASSES OF FUZZY UNIONS (t-CONORMS)

Reference	Formula $u(a, b)$	Increasing generator $g(a)$	Parameter range	As parameter converges to 0	As parameter converges to 1 or −1	As parameter converges to ∞ or −∞
Dombi [1982]	$\left\{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{-\frac{1}{\lambda}}\right\}^{-1}$	$\left(\frac{1}{a}-1\right)^{-\lambda}$	$\lambda > 0$	$u_{max}(a, b)$	$\frac{a+b-2ab}{1-ab}$ when $\lambda = 1$	$\max(a, b)$
Frank [1979]	$1-\log_s\left[1+\frac{(s^{1-a}-1)(s^{1-b}-1)}{s-1}\right]$	$-\ln\left(\frac{s^{1-a}-1}{s-1}\right)$	$s > 0, s \neq 1$	$\max(a, b)$	$a+b-ab$ as $s \to 1$	$\min(1, a+b)$
Hamacher [1978]	$\frac{a+b+(r-2)ab}{r+(r-1)ab}$	$-\ln\left(\frac{1-a}{r+(1-r)(1-a)}\right)$	$r > 0$	$\frac{a+b-2ab}{1-ab}$	$a+b-ab$ when $r = 1$	$u_{max}(a, b)$
Schweizer & Sklar 1 [1963]	$1-\{\max(0, (1-a)^p+(1-b)^p-1)\}^{\frac{1}{p}}$	$1-(1-a)^p$	$p \neq 0$	$a+b-ab$	$\min(1, a+b)$, when $p = 1$; $\frac{a+b-2ab}{1-ab}$, when $p = -1$.	$u_{max}(a, b)$ as $p \to \infty$; $\min(a, b)$ as $p \to -\infty$
Schweizer & Sklar 2	$[a^p+b^p-a^pb^p]^{\frac{1}{p}}$	$\ln[1-a^p]^{\frac{1}{p}}$	$p > 0$	$u_{max}(a, b)$	$a+b-ab$ when $p = 1$	$\max(a, b)$
Schweizer & Sklar 3	$1-\exp(-(\lvert\ln(1-a)\rvert^p+\lvert\ln(1-b)\rvert^p)^{\frac{1}{p}})$	$\lvert\ln(1-a)\rvert^p$	$p > 0$	$u_{max}(a, b)$	$a+b-ab$ when $p = 1$	$\max(a, b)$
Schweizer & Sklar 4	$1-\frac{(1-a)(1-b)}{[(1-a)^p+(1-b)^p-(1-a)^p(1-b)^p]^{\frac{1}{p}}}$	$(1-a)^{-p}-1$	$p > 0$	$a+b-ab$	$\min\left(1, \frac{a+b}{1-ab}\right)$ when $p = 1$	$\max(a, b)$
Yager [1980f]	$\min\left[1, (a^w+b^w)^{\frac{1}{w}}\right]$	a^w	$w > 0$	$u_{max}(a, b)$	$\min(1, a+b)$ when $w = 1$	$\max(a, b)$
Dubois & Prade [1980]	$1-\frac{(1-a)(1-b)}{\max((1-a), (1-b), \alpha)}$		$\alpha \in [0, 1]$	$\max(a, b)$	$a+b-ab$ when $\alpha = 1$	
Weber [1983]	$\min\left(1, a+b-\frac{\lambda}{1-\lambda}ab\right)$	$\frac{1}{\lambda}\ln\frac{1+\lambda}{1+\lambda(1-a)}$	$\lambda > -1$	$\min(1, a+b)$	$u_{max}(a, b)$ as $\lambda \to -1$; $\min(1, (a+b-ab/2)$ when $\lambda = 1$.	$a+b-ab$
Yu [1985]	$\min(1, a+b+\lambda ab)$	$\frac{1}{\lambda}\ln(1+\lambda a)$	$\lambda > -1$	$\min(1, a+b)$	$a+b-ab$ as $\lambda \to -1$; $\min(1, a+b+ab)$	$u_{max}(a, b)$

Fuzzy Intersection

The intersection (AND) is calculated using t-norms.

t-norm operator is a function i

Satisfying:

i. i(0,0) = 0, t(a,1) = i(1,a) = a (boundary)
ii. i(a,b) ≤ i(c,d) if a ≤ c and b ≤ d (monotonicity)
iii. i(a,b) = i(b,a) (commutativity)
iv. i(a, i(b,c)) = i(i(a,b),c) (associativity)

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iii. i(a,b) = i(b,a) (commutativity)
iv. i(a, i(b,c)) = i(i(a,b),c) (associativity)

The most commonly adopted t-norm is the minimum. That is, given two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Fuzzy Intersection

Additional requirements may be added to any t-norm to satisfy. The most famous are:

i. i is continuous function
ii. i(a, a) < a (subidempotency)
iii. If a<b and c<d then i(a,c)<i(b,d) (strict monotonicity)

T-norms frequently used$_{\text{Klir}}$

Standard intersection : $i(a, b) = \min(a, b)$.

Algebraic product : $i(a, b) = ab$.

Bounded difference : $i(a, b) = \max(0, a + b - 1)$.

Drastic intersection : $i(a, b) = \begin{cases} a & \text{when } b = 1 \\ b & \text{when } a = 1 \\ 0 & \text{otherwise.} \end{cases}$

T-norms frequently used$_{\text{Klir}}$

TABLE 3.2 SOME CLASSES OF FUZZY INTERSECTIONS (t-NORMS)

Reference	Formula $i(a, b)$	Decreasing generator $f(a)$	Parameter range	As parameter converges to 0	As parameter converges to 1 or −1	As parameter converges to ∞ or −∞
Dombi [1982]	$\left\{1 + \left[\left(\frac{1}{a} - 1\right)^{\lambda} + \left(\frac{1}{b} - 1\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\}^{-1}$	$\left(\frac{1}{a} - 1\right)^{\lambda}$	$\lambda > 0$	$i_{\min}(a, b)$	$\frac{ab}{a + b - ab}$ when $\lambda = 1$	$\min(a, b)$
Frank [1979]	$\log_s \left[1 + \frac{(s^a - 1)(s^b - 1)}{s - 1}\right]$	$-\ln\left(\frac{s^a - 1}{s - 1}\right)$	$s > 0, s \neq 1$	$\min(a, b)$	ab as $s \to 1$	$\max(0, a + b - 1)$
Hamacher [1978]	$\frac{ab}{r + (1 - r)(a + b - ab)}$	$-\ln\left(\frac{a}{r + (1 - r)a}\right)$	$r > 0$	$\frac{ab}{a + b - ab}$	ab when $r = 1$	$i_{\min}(a, b)$
Schweizer & Sklar 1 [1963]	$\{\max(0, a^p + b^p - 1)\}^{\frac{1}{p}}$	$1 - a^p$	$p \neq 0$	ab	$\max(0, a + b - 1)$, when $p = 1$; $\frac{ab}{a + b - ab}$, when $p = -1$.	$i_{\min}(a, b)$ as $p \to \infty$; $\min(a, b)$ as $p \to -\infty$.
Schweizer & Sklar 2	$1 - [(1 - a)^p + (1 - b)^p - (1 - a)^p(1 - b)^p]^{\frac{1}{p}}$	$\ln[1 - (1 - a)^p]^{\frac{1}{p}}$	$p > 0$	$i_{\min}(a, b)$	ab when $p = 1$	$\min(a, b)$
Schweizer & Sklar 3	$\exp(-(\lvert\ln a\rvert^p + \lvert\ln b\rvert^p)^{\frac{1}{p}})$	$\lvert\ln a\rvert^p$	$p > 0$	$i_{\min}(a, b)$	ab when $p = 1$	$\min(a, b)$
Schweizer & Sklar 4	$\frac{ab}{[a^p + b^p - a^p b^p]^{\frac{1}{p}}}$	$a^{-p} - 1$	$p > 0$	ab	$\frac{ab}{a + b - ab}$, when $p = 1$	$\min(a, b)$
Yager [1980f]	$1 - \min\left\{1, [(1 - a)^w + (1 - b)^w]^{\frac{1}{w}}\right\}$	$(1 - a)^w$	$w > 0$	$i_{\min}(a, b)$	$\max(0, a + b - 1)$ when $w = 1$	$\min(a, b)$
Dubois & Prade [1980]	$\frac{ab}{\max(a, b, \alpha)}$		$\alpha \in [0, 1]$	$\min(a, b)$	ab when $\alpha = 1$	
Weber [1983]	$\max\left(0, \frac{a + b + \lambda ab - 1}{1 + \lambda}\right)$	$\frac{1}{\lambda}\ln[1 + \lambda(1 - a)]$	$\lambda > -1$	$\max(0, a + b - 1)$	$i_{\min}(a, b)$ as $\lambda \to -1$; $\max[0, (a + b + ab - 1)/2]$ when $\lambda = 1$.	ab
Yu [1985]	$\max[0, (1 + \lambda)(a + b - 1) - \lambda ab]$	$\frac{1}{\lambda}\ln\frac{1 + \lambda}{1 + \lambda a}$	$\lambda > -1$	$\max(0, a + b - 1)$	ab as $\lambda \to -1$; $\max[0, 2(a + b - ab/2 - 1)]$ when $\lambda = 1$.	$i_{\min}(a, b)$

Fuzzy Complement

To be able to develop fuzzy systems we also have to deal with NOT or complement.

This is the same in fuzzy logic as for Boolean logic

For a fuzzy set A,A^c denotes the fuzzy complement of A

Membership function for fuzzy complement is

$$\mu_{A^c}(x) = 1 - \mu_A(x)$$

Example Discrete case

Suppose we have the following (discrete) fuzzy sets:

$A =$ 0.4/1+0.6/2+0.7/3+0.8/4

$B =$ 0.3/1+0.65/2+0.4/3+0.1/4

The union of the fuzzy sets A and B

= 0.4/1+0.65/2+0.7/3+0.8/4

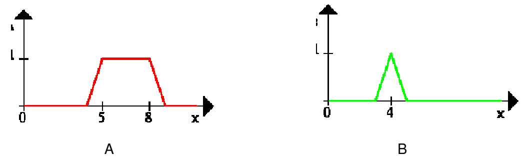
The intersection of the fuzzy sets A and B

= 0.3/1+0.6/2+0.4/3+0.1/4

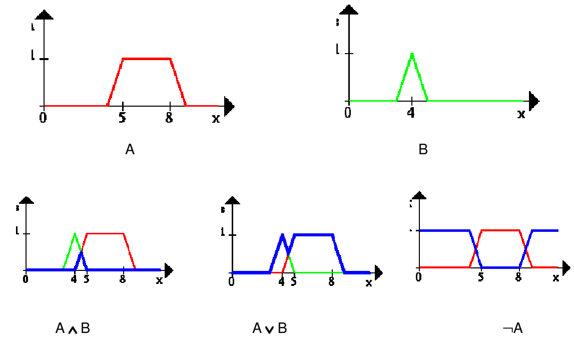
The complement of the fuzzy set A

= 0.6/1+0.4/2+0.3/3+0.2/4

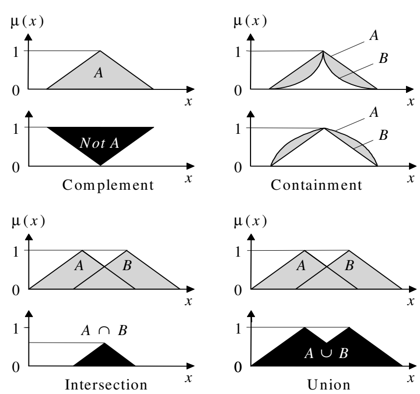
Example Continuous case



Example Continuous case



Fuzzy Operations



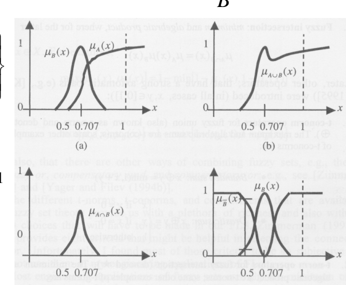
Example

Figure (a): $\mu_A(x)$, $\mu_B(x)$ *Figure (b):* $\mu_{A \cup B}(x)$

Figure (c): $\mu_{A \cap B}(x)$ *Figure (d):* $\mu_B(x)$, $\mu_{B^c}(x)$

$$\mu_A(x) = \begin{cases} 0 & if\ 0 \le x \le 0.5 \\ \dfrac{1}{1+(x-0.5)^{-2}} & if\ 0.5 < x \le 1 \end{cases}$$

$$\mu_B(x) = \frac{1}{1+(x-0.707)^4} \qquad 0 \le x \le 1$$



Example

This example demonstrates that for fuzzy sets, the Law of Excluded Middle and Concentration are broken, i.e., *for fuzzy sets A and B*:

$$A \cup A^c \neq X \text{ and } A \cap A^c \neq \emptyset$$

In fact, one of the ways to describe the difference between crisp set theory and fuzzy set theory is to explain that these two laws do not hold in fuzzy set theory

Definitions

- A is EMPTY iff for all x, $\mu_A(x) = 0.0$.
- A = B iff for all x: $\mu_A(x) = \mu_B(x)$
- A is CONTAINED in B or $A \subseteq B$ iff $\mu_A A \leq \mu_B B$.

for all $x \in X$

Proper Subset:

$A \subset B$ if $\mu_A(x) \leq \mu_B(x)$ and $\mu_A(x) < \mu_B(x)$ for at least one $x \in X$

Basic Properties of set operations

Involution : $(A^c)^c = A$. (3.3)

Commutativity : $A \cup B = B \cup A, A \cap B = B \cap A$. (3.4)

Associativity : $(A \cup B) \cup C = A \cup (B \cup C)$, (3.5)
$(A \cap B) \cap C = A \cap (B \cap C)$. (3.6)

Distributivity : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, (3.7)
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (3.8)

Idempotency : $A \cap A = A, A \cup A = A$. (3.9)

Law of Contradiction : $A \cap A^c = \phi$. (3.10)

Law of Excluded Middle : $A \cup A^c = X$. (3.11)

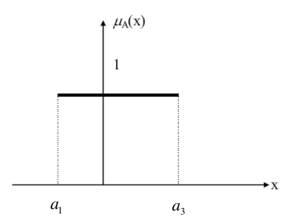
De Morgan : $(A \cup B)^c = A^c \cap B^c$, (3.12)
$(A \cap B)^c = A^c \cup B^c$. (3.13)

Identity : $A \cup \phi = A, A \cap \phi = \phi$, (3.14)
$A \cup X = X, A \cap X = A$. (3.15)

Interval

$[a_1, a_3]$

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ 1, & a_1 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}$$



α-cut

$A_\alpha = \{x \in X \,|\, \mu_A(x) \geq \alpha\}$

$A_{\alpha^+} = \{x \in X \,|\, \mu_A(x) > \alpha\}$

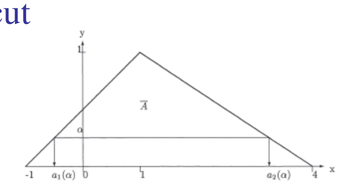


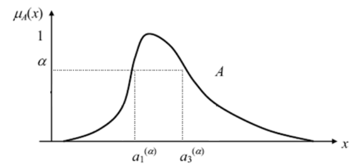
Figure 3.9: Continuous Fuzzy Set

Definitions

A_{0^+} is called the support of A

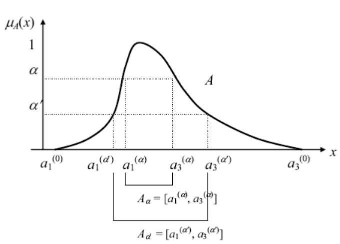
A_1 is called the core of A

α-cut



$A_\alpha = \left[a_1^{(\alpha)}, a_3^{(\alpha)}\right]$

α-cut



α-cut

In case of discrete fuzzy set

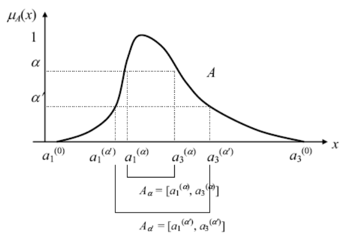
Find α-cuts of

$$\overline{A} = \{\frac{0}{x_1}, \frac{.7}{x_2}, \frac{1}{x_3}, \frac{.5}{x_4}, \frac{.2}{x_5}\},$$

where $X = \{x_1, \ldots, x_5\}$. Then

$$\overline{A}[\alpha] = \{x_2, x_3, x_4, x_5\}, 0 < \alpha \leq 0.2,$$
$$\overline{A}[\alpha] = \{x_2, x_3, x_4\}, 0.2 < \alpha \leq 0.5,$$
$$\overline{A}[\alpha] = \{x_2, x_3\}, 0.5 < \alpha \leq 0.7,$$
$$\overline{A}[\alpha] = \{x_3\}, 0.7 < \alpha \leq 1.$$

Properties of α-cut



$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$

$(\alpha' < \alpha) \Rightarrow (a_1^{(\alpha')} \le a_1^{(\alpha)}, a_3^{(\alpha')} \ge a_3^{(\alpha)}) \quad \Rightarrow (A_\alpha \subset A_{\alpha'})$

Interval Arithmetic

A* = [*a

$A = [a_1, a_2]$, $B = [b_1, b_2]$

Interval Operations

Addition

$[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$

Subtractio n

$[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$

Multiplica tion

$[a_1, a_2] * [b_1, b_2] = [\min\{ a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2 \}, \max \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}]$

Division

$[a_1, a_2] / [b_1, b_2] = [a_1, a_2] * [1/b_2, 1/b_1]$

$0 \notin [b_1, b_2]$

Examples

Addition

[2,5]+[1,3]=[3,8]   [0,1]+[-6,5]=[-6,6]

Subtraction

[2,5]-[1,3]=[-1,4]   [0,1]-[-6,5]=[-5,7]

Multiplication

[-1,1]*[-2,-0.5]=[-2,2]   [3,4]*[2,2]=[6,8]

Division

[-1,1]/[-2,-0.5]=[-2,2]   [4,10]*[1,2]=[2,10]

Properties of Interval Operations

Commutative

$A + B = B + A \qquad A \cdot B = B \cdot A$

Assocoative

$(A + B) + C = A + (B + C) \quad (A \cdot B) \cdot C = A \cdot (B \cdot C)$

Identity $\quad 0 = [0,0] \quad 1 = [1,1]$

$A = A + 0 = 0 + A \qquad A = A \cdot 1 = 1 \cdot A$

Subdistributive

$A \cdot (B + C) \subseteq A \cdot B + A \cdot C$

Inverse

$0 \in A\text{-}A \quad 1 \in A/A$

Monotonicity for any operations *

If $A \subseteq E$ and $B \subseteq F$ then $A*B \subseteq E*F$

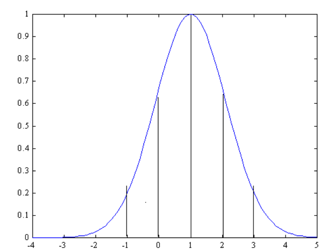
Fuzzy Number

N is called a fuzzy number if:

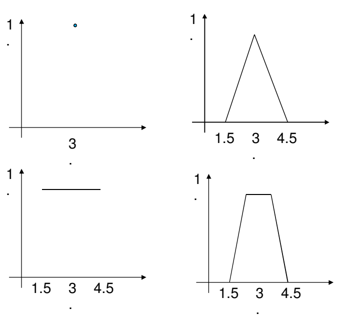
1. The core of *N* is not empty
2. All α-cuts of *N* are closed, bounded intervals
3. The support of N is bounded

Ex. Fuzzy number 1

Example: Discrete and Continuous fuzzy sets to represent the fuzzy number 1



Examples of fuzzy number 3



Examples

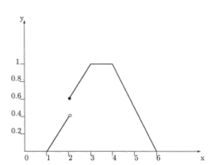
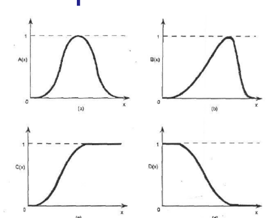
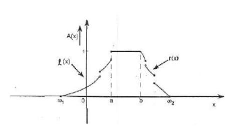
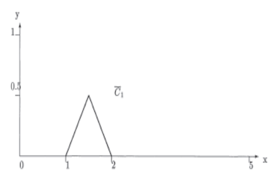


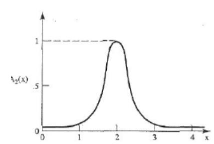
Figure 4.3: A General Fuzzy Number





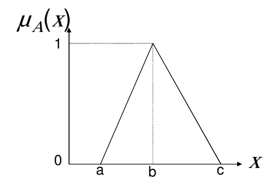
Examples: Not Fuzzy Numbers





Triangular Fuzzy Number

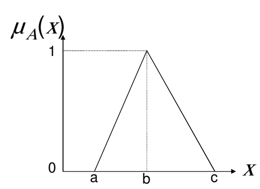
a, b and c represent the *x* coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)



Triangular Fuzzy Number

a, b and c represent the *x* coordinates of the three vertices of $\mu_A(x)$ in a fuzzy set A (a: lower boundary and c: upper boundary where membership degree is zero, b: the centre where membership degree is 1)

$$\mu_A(x) = \begin{cases} 0 & if\ x \le a \\ \dfrac{x-a}{b-a} & if\ a \le x \le b \\ \dfrac{c-x}{c-b} & if\ b \le x \le c \\ 0 & if\ x \ge c \end{cases}$$



Triangular/trapezoidal Shaped Buckley

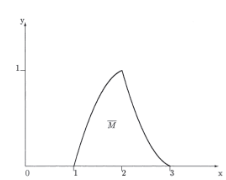


Figure 4.4: Triangular Shaped Fuzzy Number

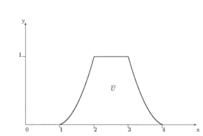


Figure 4.5: Trapezoidal Shaped Fuzzy Number

Operations of Fuzzy Numbers

Operations on fuzzy numbers can be done through two ways:

- Interval Arithmetic
- Extension Principle

Operations of Fuzzy Numbers

α-cut $A(+)B = \bigcup_{\alpha\in[0,1]} \alpha(A_\alpha(+)B_\alpha)$

Addition:

$A = [a_1, a_3] \quad a_1, a_3 \in \Re$

$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}], \forall\alpha \in [0, 1], a_1^{(\alpha)}, a_3^{(\alpha)} \in \Re$

$B = [b_1, b_3], \quad b_1, b_3, \in \Re$

$B_\alpha = [b_1^{(\alpha)}, b_3^{(\alpha)}], \forall\alpha \in [0, 1], b_1^{(\alpha)}, b_3^{(\alpha)} \in \Re$

operations between A_α and B_α can be described as follows :

$[a_1^{(\alpha)}, a_3^{(\alpha)}] (+) [b_1^{(\alpha)}, b_3^{(\alpha)}] = [a_1^{(\alpha)} + b_1^{(\alpha)}, a_3^{(\alpha)} + b_3^{(\alpha)}]$

Operations of Fuzzy Number

α-cut $A(-)B = \bigcup_{\alpha\in[0,1]} \alpha(A_\alpha(-)B_\alpha)$

Subtraction:

$A(\cdot)B = \bigcup_{\alpha\in[0,1]} \alpha(A_\alpha(\cdot)B_\alpha)$

Multiplication:

$A(/)B = \bigcup_{\alpha\in[0,1]} \alpha(A_\alpha(/)B_\alpha)$

Division:

$A(\wedge)B = \bigcup_{\alpha\in[0,1]} \alpha(A_\alpha(\wedge)B_\alpha)$

Minimum:

$A(\vee)B = \bigcup_{\alpha\in[0,1]} \alpha(A_\alpha(\vee)B_\alpha)$

Maximum:

Operations of Fuzzy Number

α-cut

α = 0.5

$A_{0.5} = [2,3],\ B_{0.5} = [3,4]$

$A_{0.5}(+)B_{0.5} = [5,7]$

α = 1.0

$A_{1.0} = 2,\ B_{1.0} = 3$

$A_{1.0}(+)B_{1.0} = 5$

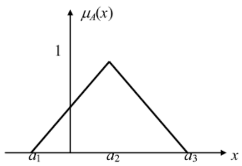
Triangular fuzzy number

Definition: It is a number represented with three points as follows :

$A=(a_1, a_2, a_3)$

With the representation function:

$$\mu_{(A)}(x)=\begin{cases}0, & x<a_1\\ \dfrac{x-a_1}{a_2-a_1}, & a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2}, & a_2\le x\le a_3\\ 0, & x>a_3\end{cases}$$



Triangular fuzzy number

Now if you get crisp interval by α-cut operation, interval A_α shall be obtained as follows $\forall\alpha \in [0, 1]$
from

$$\frac{a_1^{(\alpha)}-a_1}{a_2-a_1}=\alpha, \quad \frac{a_3-a_3^{(\alpha)}}{a_3-a_2}=\alpha$$

we get

$$a_1^{(\alpha)}=(a_2-a_1)\alpha+a_1$$
$$a_3^{(\alpha)}=-(a_3-a_2)\alpha+a_3$$

thus

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$
$$=[(a_2-a_1)\alpha+a_1, -(a_3-a_2)\alpha+a_3]$$

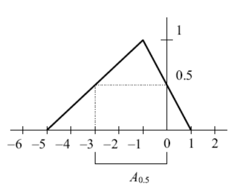
Triangular fuzzy number

Example: Consider ***A=(-5,-1,1),***
Therefore its membership fn. is :

$A=(a_1, a_2, a_3)$

With the representation function:

$$\mu_{(A)}(x)=\begin{cases}0, & x<-5\\ \dfrac{x+5}{4}, & -5\le x\le -1\\ \dfrac{1-x}{2}, & -1\le x\le 1\\ 0, & x>1\end{cases}$$



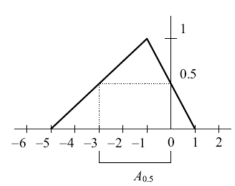
Triangular fuzzy number

α-cut interval from this fuzzy number is

$$\frac{x+5}{4}=\alpha \quad \Rightarrow \quad x=4\alpha-5$$
$$\frac{1-x}{2}=\alpha \quad \Rightarrow \quad x=-2\alpha+1$$
$$A_\alpha=[a_1^{(\alpha)}, a_3^{(\alpha)}]=[4\alpha-5, -2\alpha+1]$$

if $\alpha=0.5$, substituting 0.5 for α, we get $A_{0.5}$

$$A_{0.5}=[a_1^{(0.5)}, a_3^{(0.5)}]=[-3, 0] \quad \square$$



Operations of triangular fuzzy number

Same important properties of operations on triangular fuzzy number are summarized

(1) The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.

(2) The results from multiplication or division are not triangular fuzzy numbers.

but we often assume that the operational results of multiplication or division to be TFNs as approximation values.

Operations of triangular fuzzy number

1) Operation of triangular fuzzy number

first, consider addition and subtraction. Here we need not use membership function. Suppose triangular fuzzy numbers A and B are defined as,

$$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3)$$

i) Addition

$$\begin{aligned} A(+)B &= (a_1, a_2, a_3)(+)(b_1, b_2, b_3) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3) \end{aligned}$$: triangular fuzzy number

ii) Subtraction

$$\begin{aligned} A(-)B &= (a_1, a_2, a_3)(-)(b_1, b_2, b_3) \\ &= (a_1 - b_3, a_2 - b_2, a_3 - b_1) \end{aligned}$$: triangular fuzzy number

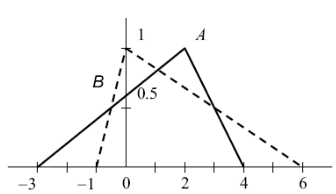
iii) Symmetric image

$-(A) = (-a_3, -a_2, -a_1)$: triangular fuzzy number

Operations of triangular fuzzy number

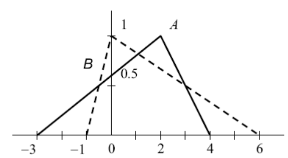
Example: Consider ***A=(-3,2,4), B=(-1,0,6)***

A+B=(-4,2,10)
A-B=(-9,2,5)

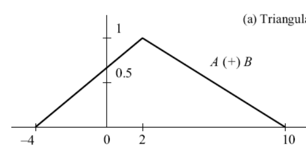


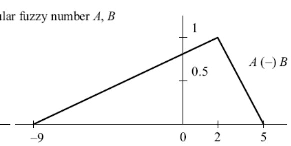
(a) Triangular fuzzy number A, B

Operations of triangular fuzzy number



(a) Triangular fuzzy number A, B





(c) A (–) B triangular fuzzy numbers

(b) A (+) B of triangular fuzzy numbers

Operations of triangular fuzzy number

2) Operations with α-cut

The operations may be done through alpha cut opertions

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] = [5\alpha - 3, -2\alpha + 4]$$

$$B_\alpha = [b_1^{(\alpha)}, b_3^{(\alpha)}] = [(b_2 - b_1)\alpha + b_1, -(b_3 - b_2)\alpha + b_3] = [\alpha - 1, -6\alpha + 6]$$

performing the addition of two α-cut intervals A_α and B_α,

$$A_\alpha (+) B_\alpha = [6\alpha - 4, -8\alpha + 10]$$

especially for $\alpha = 0$ and $\alpha = 1$,

$$A_0 (+) B_0 = [-4, 10]$$
$$A_1 (+) B_1 = [2, 2] = 2$$

Operations of triangular fuzzy number

three points from this procedure coincide with the three points of triangular fuzzy number (-4, 2, 10) from the result $A(+)B$ given in the previous example.

Likewise, after obtaining $A_\alpha(-)B_\alpha$, let's think of the case when $\alpha = 0$ and $\alpha = 1$

$$A_\alpha (-) B_\alpha = [11\alpha - 9, -3\alpha + 5]$$

substituting $\alpha = 0$ and $\alpha = 1$ for this equation,

$$A_0 (-) B_0 = [-9, 5]$$
$$A_1 (-) B_1 = [2, 2] = 2$$

these also coincide with the three points of $A(-)B = (-9, 2, 5)$. □

Consequently, we know that we can perform operations between fuzzy number using α-cut interval.

Arithmetic Operation using the Extension Principle

$\forall x, y, z \in \Re$

Addition: $A(+)B$ $\quad \mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}$

Subtraction: $A(-)B$ $\quad \mu_{A(-)B}(z) = \underset{z=x-y}{sum} \min\{\mu_A(x), \mu_B(y)\}$

Multiplication: $A(\bullet)B$ $\quad \mu_{A(\bullet)B}(z) = \sup_{z=x\bullet y} \min\{\mu_A(x), \mu_B(y)\}$

Division: $A(/)B$ $\quad \mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}$

Minimum: $A(\wedge)B$ $\quad \mu_{A(\wedge)B}(z) = \sup_{z=x\wedge y} \min\{\mu_A(x), \mu_B(y)\}$

Maximum: $A(\vee)B$ $\quad \mu_{A(\vee)B}(z) = \sup_{z=x\vee y} \min\{\mu_A(x), \mu_B(y)\}$

Operations of triangular fuzzy number

Example: Consider the flowing A and B defined by their membership fn. Using extension principle (approx. calculations!)

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -3 \\ \frac{x+3}{2+3}, & -3 \le x \le 2 \\ \frac{4-x}{4-2}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases} \qquad \mu_{(B)}(y) = \begin{cases} 0, & y < -1 \\ \frac{y+1}{0+1}, & -1 \le y \le 0 \\ \frac{6-y}{6-0}, & 0 \le y \le 6 \\ 0, & y > 6 \end{cases}$$

for the two fuzzy number $x \in A$ and $y \in B$, $z \in A$ (+) B shall be obtained by their membership functions.

Operations of triangular fuzzy number

Let's think when $z = 8$. Addition to make $z = 8$ is possible for following cases :

$$2 + 6,\ 3 + 5,\ 3.5 + 4.5,\ \ldots$$

so

$$\begin{aligned} \mu_{A(+)B} &= \bigvee_{8=x+y} [\mu_A(2) \wedge \mu_B(6),\ \mu_A(3) \wedge \mu_B(5),\ \mu_A(3.5) \wedge \mu_B(4.5), \cdots] \\ &= \vee\ [1 \wedge 0,\ 0.5 \wedge 1/6,\ 0.25 \wedge 0.25, \cdots] \\ &= \vee\ [0, 1/6,\ 0.25, \cdots] \end{aligned}$$

Operations of triangular fuzzy number

If we go on these kinds of operations for all $z \in A\ (+)\ B$, we come to the following membership functions, and these are identical to the three point expression for triangular fuzzy number $A = (-4, 2, 10)$.

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < -4 \\ \dfrac{z+4}{6}, & -4 \le z \le 2 \\ \dfrac{10-z}{8}, & 2 \le z \le 10 \\ 0, & z > 10 \end{cases}$$

□

Operations of triangular fuzzy number

Example 5.11 Multiplication $A\ (\bullet)\ B$

Let triangular fuzzy numbers A and B be

$$A = (1, 2, 4),\ B = (2, 4, 6)$$

$$\mu_{(A)}(x) = \begin{cases} 0, & x < 1 \\ x - 1, & 1 \le x < 2 \\ -\dfrac{1}{2}x + 2, & 2 \le x < 4 \\ 0, & x \ge 4 \end{cases} \qquad \mu_{(B)}(y) = \begin{cases} 0, & y < 2 \\ \dfrac{1}{2}y - 1, & 2 \le y < 4 \\ -\dfrac{1}{2}y + 3, & 4 \le y < 6 \\ 0, & y \ge 6 \end{cases}$$

Operations of triangular fuzzy number

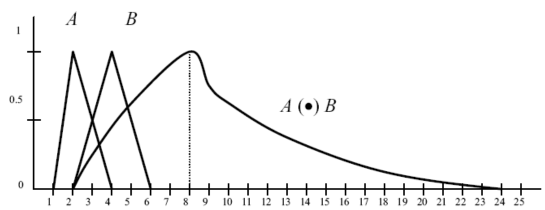
calculating multiplication $A\ (\bullet)\ B$ of A and B, $z = x \bullet y = 8$ is possible when $z = 2 \bullet 4$ or $z = 4 \bullet 2$

$$\begin{aligned} \mu_{A(\bullet)B} &= \bigvee_{x \bullet y = 8} [\mu_A(2) \wedge \mu_B(4),\ \mu_A(4) \wedge \mu_B(2), \cdots] \\ &= \vee [1 \wedge 1,\ 0 \wedge 0, \cdots] \\ &= 1 \end{aligned}$$

also when $z = x \bullet y = 12$, $3 \bullet 4$, $4 \bullet 3$, $2.5 \bullet 4.8$, … are possible.

$$\begin{aligned} \mu_{A(\bullet)B} &= \bigvee_{x \bullet y = 12} [\mu_A(3) \wedge \mu_B(4),\ \mu_A(4) \wedge \mu_B(3),\ \mu_A(2.5) \wedge \mu_B(4.8), \cdots] \\ &= \vee [0.5 \wedge 1,\ 0 \wedge 0.5,\ 0.75 \wedge 0.6,\ \cdots] \\ &= \vee [0.5,\ 0,\ 0.6,\ \cdots] \\ &= 0.6 \end{aligned}$$

Operations of triangular fuzzy number



Trapezoidal fuzzy number

Definition (Trapezoidal fuzzy number) We can define trapezoidal fuzzy number A as

$$A = (a_1, a_2, a_3, a_4)$$

the membership function of this fuzzy number will be interpreted as follows(Fig 5.10).

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4}, & a_3 \le x \le a_4 \\ 0, & x > a_4 \end{cases}$$

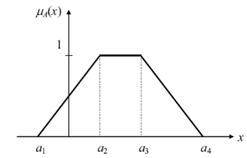


Fig. 5.10. Trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$

α-cut interval for this shape is written below.

$\forall \alpha \in [0, 1]$

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, \quad -(a_4 - a_3)\alpha + a_4]$$

when $a_2 = a_3$, the trapezoidal fuzzy number coincides with triangular one.

Operations of Trapezoidal Fuzzy Number

Let's talk about the operations of trapezoidal fuzzy number as in the triangular fuzzy number,

(1) Addition and subtraction between fuzzy numbers become trapezoidal fuzzy number.

(2) Multiplication, division, and inverse need not be trapezoidal fuzzy number.

(1) Addition

$$\begin{aligned} A(+)B &= (a_1, a_2, a_3, a_4)(+)(b_1, b_2, b_3, b_4) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4) \end{aligned}$$

(2) Subtraction

$$A(-)B = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$

Operations of Trapezoidal Fuzzy Number

Example 5.14 Multiplication

Multiply two trapezoidal fuzzy numbers as following:

$$A = (1, 5, 6, 9)$$
$$B = (2, 3, 5, 8)$$

For exact value of the calculation, the membership functions shall be used and the result is described in (Fig. 5.11) For the approximation of operation results, we use α-cut interval

$$A_\alpha = [4\alpha + 1, -3\alpha + 9]$$
$$B_\alpha = [\alpha + 2, -3\alpha + 8]$$

since, for all $\alpha \in [0, 1]$, each element for each interval is positive, multiplication between α-cut intervals will be

$$\begin{aligned} A_\alpha(\bullet)B_\alpha &= [(4\alpha + 1)(\alpha + 2), \quad (-3\alpha + 9)(-3\alpha + 8)] \\ &= [4\alpha^2 + 9\alpha + 2, \quad 9\alpha^2 - 51\alpha + 72] \end{aligned}$$

Operations of Trapezoidal Fuzzy Number

$A_\alpha = [4\alpha + 1, -3\alpha + 9]$
$B_\alpha = [\alpha + 2, -3\alpha + 8]$

$$A_\alpha(\bullet)B_\alpha = [(4\alpha+1)(\alpha+2),\ \ (-3\alpha+9)(-3\alpha+8)] = [4\alpha^2+9\alpha+2,\ \ 9\alpha^2-51\alpha+72]$$

$$\mu_{(A(\bullet)B)}(z) = \begin{cases} 0 & z < 2 \\ \dfrac{\sqrt{z+3.0625}-2.25}{2} & 2 \le z \le 15 \\ 1 & 15 \le z \le 30 \\ \dfrac{-\sqrt{z+0.25}+8.5}{3} & 30 \le z \le 72 \\ 0 & z > 72 \end{cases}$$

$$4\alpha^2 + 9\alpha + 2 = z$$
$$(2\alpha + 2.25)^2 = z + 3.0625$$
$$\alpha = \frac{\sqrt{z+3.0625}-2.25}{2}$$

$$9\alpha^2 - 51\alpha + 72 = z$$
$$(3\alpha - 8.5)^2 = z + 0.25$$
$$\alpha = \frac{-\sqrt{z+0.25}+8.5}{3}$$

Extension Principle for discrete Fuzzy Numbers

By Extension Principle

Note:

* can be any operations including arithmetic operations.

Example *A* = { 1/2 , 0.5/3}, *B* = { 1/3, 0.8/4}

Extension Principle for discrete Fuzzy Numbers

A = { 1/2 , 0.5/3}, *B* = { 1/3, 0.8/4}

A+*B* = {1/5, 0.8/6, 0.5/7 }

$i)\ z < 5$ No such case.

$\mu_{A(+)B}(z) = 0$

$ii)\ z = 5$

$x + y = 2 + 3$

$\mu_A(2) \wedge \mu_B(3) = 1$

$iii)\ z = 6$

$x + y = 3 + 3$ or $x + y = 2 + 4$

$\mu_A(3) \wedge \mu_B(3) = 0.5$

$\mu_A(2) \wedge \mu_B(4) = 0.8$ $\quad \mu_{A(+)B}(6) = \bigvee_{\substack{6=3+3 \\ 6=2+4}} (0.5,\ 0.8) = 0.8$

$iv)\ z = 7$

$\mu_A(3) \wedge \mu_B(4) = \min(0.5,\ 0.8) = 0.5$

Example

A = { 1/2 , 0.5/3}, *B* = { 1/3, 0.8/4}

Max (A,B) = { (3 , 1) , (4 , 0.5) }

$i)\ z \le 2$ No such case.

$\mu_{A(\vee)B}(z) = 0$

$ii)\ z = 3$

$x \vee y = 2 \vee 3$ or $x \vee y = 3 \vee 3$

$\mu_A(2) \wedge \mu_B(3) = 1 \wedge 1 = 1$ $\quad \mu_A(3) \wedge \mu_B(3) = 0.5 \wedge 1 = 0.5$

$\mu_{A(\vee)B}(3) = \bigvee_{\substack{3=2\vee 3 \\ 3=3\vee 3}} (1, 0.5) = 1$

$iii)\ z = 4$

$x \vee y = 2 \vee 4$ or $x \vee y = 3 \vee 4$

$\mu_A(2) \wedge \mu_B(4) = 1 \wedge 0.5 = 0.5$ $\quad \mu_A(3) \wedge \mu_B(4) = 0.5 \wedge 0.5 = 0.5$

$\mu_{A(\vee)B}(4) = \bigvee_{\substack{4=2\vee 4 \\ 4=3\vee 4}} (0.5, 0.5) = 0.5$

$iv)\ z \ge 5$ No such case.

$\mu_{A(\vee)B}(z) = 0$