Latest to accept your presentations is May 4rd.

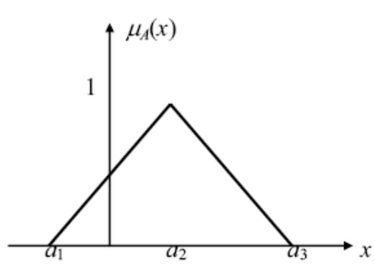
Triangular fuzzy number

Definition: It is a number represented with three points as follows :

$A=(a_1, a_2, a_3)$

With the representation function:

$$\mu_{(A)}(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & x > a_3 \end{cases}$$



Triangular fuzzy number

Now if you get crisp interval by α-cut operation, interval A_α shall be obtained as follows $\forall\alpha \in [0, 1]$
from

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha, \quad \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha$$

we get

$$a_1^{(\alpha)} = (a_2 - a_1)\alpha + a_1$$
$$a_3^{(\alpha)} = -(a_3 - a_2)\alpha + a_3$$

thus

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$
$$= [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]$$

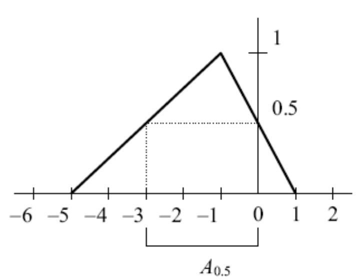
Triangular fuzzy number

Example: Consider ***A=(-5,-1,1),***
Therefore its membership fn. is :

$A=(a_1, a_2, a_3)$

With the representation function:

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -5 \\ \frac{x+5}{4}, & -5 \le x \le -1 \\ \frac{1-x}{2}, & -1 \le x \le 1 \\ 0, & x > 1 \end{cases}$$



Triangular fuzzy number

α-cut interval from this fuzzy number is

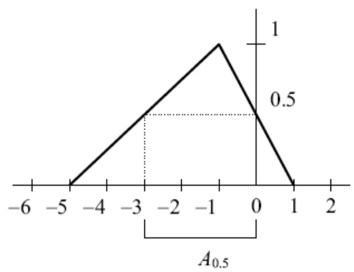
$$\frac{x+5}{4}=\alpha \quad \Rightarrow \quad x=4\alpha-5$$

$$\frac{1-x}{2}=\alpha \quad \Rightarrow \quad x=-2\alpha+1$$

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}] = [4\alpha - 5, -2\alpha + 1]$$

if $\alpha = 0.5$, substituting 0.5 for α, we get $A_{0.5}$

$$A_{0.5} = [a_1^{(0.5)}, a_3^{(0.5)}] = [-3, 0] \quad \square$$



Operations of triangular fuzzy number

Same important properties of operations on triangular fuzzy number are summarized

(1) The results from addition or subtraction between triangular fuzzy numbers result also triangular fuzzy numbers.

(2) The results from multiplication or division are not triangular fuzzy numbers.

but we often assume that the operational results of multiplication or division to be TFNs as approximation values.

Operations of triangular fuzzy number

1) Operation of triangular fuzzy number

first, consider addition and subtraction. Here we need not use membership function. Suppose triangular fuzzy numbers A and B are defined as,

$$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3)$$

i) Addition

$$A(+)B = (a_1, a_2, a_3)(+)(b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$: triangular fuzzy number

ii) Subtraction

$$A(-)B = (a_1, a_2, a_3)(-)(b_1, b_2, b_3) = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$$: triangular fuzzy number

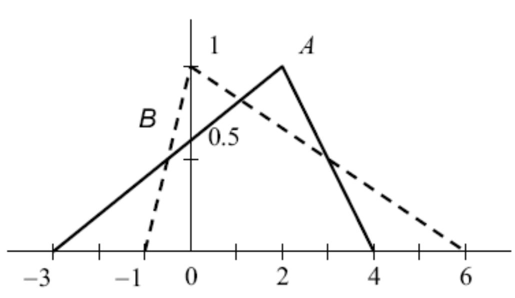
iii) Symmetric image

$$-(A) = (-a_3, -a_2, -a_1)$$: triangular fuzzy number

Operations of triangular fuzzy number

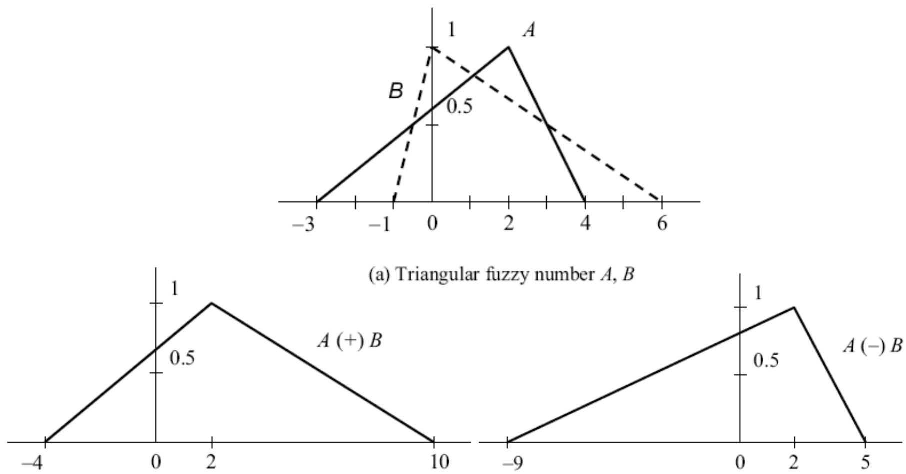
Example: Consider ***A=(-3,2,4), B=(-1,0,6)***

A+B=(-4,2,10)
A-B=(-9,2,5)



(a) Triangular fuzzy number A, B

Operations of triangular fuzzy number



(a) Triangular fuzzy number A, B

(b) A (+) B of triangular fuzzy numbers

(c) A (−) B triangular fuzzy numbers

Operations of triangular fuzzy number

2) Operations with α-cut

The operations may be done through alpha cut opertions

$$\begin{aligned} A_\alpha &= [a_1^{(\alpha)}, a_3^{(\alpha)}] = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3] \\ &= [5\alpha - 3, -2\alpha + 4] \\ B_\alpha &= [b_1^{(\alpha)}, b_3^{(\alpha)}] = [(b_2 - b_1)\alpha + b_1, -(b_3 - b_2)\alpha + b_3] \\ &= [\alpha - 1, -6\alpha + 6] \end{aligned}$$

performing the addition of two α-cut intervals A_α and B_α,

$$A_\alpha \,(+)\, B_\alpha = [6\alpha - 4, -8\alpha + 10]$$

especially for $\alpha = 0$ and $\alpha = 1$,

$$A_0 \,(+)\, B_0 = [-4, 10]$$
$$A_1 \,(+)\, B_1 = [2, 2] = 2$$

Operations of triangular fuzzy number

three points from this procedure coincide with the three points of triangular fuzzy number (-4, 2, 10) from the result $A(+)B$ given in the previous example.

Likewise, after obtaining $A_\alpha(-)B_\alpha$, let's think of the case when $\alpha = 0$ and $\alpha = 1$

$$A_\alpha\ (-)\ B_\alpha = [11\alpha - 9, -3\alpha + 5]$$

substituting $\alpha = 0$ and $\alpha = 1$ for this equation,

$$A_0\ (-)\ B_0 = [-9, 5]$$

$$A_1\ (-)\ B_1 = [2, 2] = 2$$

these also coincide with the three points of $A(-)B = (-9, 2, 5)$. □

Consequently, we know that we can perform operations between fuzzy number using α-cut interval.

Arithmetic Operation using the Extension Principle

$\forall x, y, z \in \Re$

Addition: $A(+)B$ $\quad \mu_{A(+)B}(z) = \sup_{z=x+y} \min\{\mu_A(x), \mu_B(y)\}$

Subtraction: $A(-)B$ $\quad \mu_{A(-)B}(z) = sum_{z=x-y} \min\{\mu_A(x), \mu_B(y)\}$

Multiplication: $A(\bullet)B$ $\quad \mu_{A(\bullet)B}(z) = \sup_{z=x\bullet y} \min\{\mu_A(x), \mu_B(y)\}$

Division: $A(/)B$ $\quad \mu_{A(/)B}(z) = \sup_{z=x/y} \min\{\mu_A(x), \mu_B(y)\}$

Minimum: $A(\wedge)B$ $\quad \mu_{A(\wedge)B}(z) = \sup_{z=x\wedge y} \min\{\mu_A(x), \mu_B(y)\}$

Maximum: $A(\vee)B$ $\quad \mu_{A(\vee)B}(z) = \sup_{z=x\vee y} \min\{\mu_A(x), \mu_B(y)\}$

Extension Principle$_{klir}$

Theorem 2.8. Let $f : X \to Y$ be an arbitrary crisp function. Then, for any $A_i \in \mathcal{F}(X)$ and any $B_i \in \mathcal{F}(Y)$, $i \in I$, the following properties of functions obtained by the extension principle hold:

(i) $f(A) = \varnothing$ iff $A = \varnothing$;

(ii) if $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$;

(iii) $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$;

(iv) $f(\bigcap_{i \in I} A_i) \subseteq \bigcap_{i \in I} f(A_i)$;

(v) if $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$;

(vi) $f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$;

(vii) $f^{-1}(\bigcap_{i \in I} B_i) = \bigcap_{i \in I} f^{-1}(B_i)$;

(viii) $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$;

(ix) $A \subseteq f^{-1}(f(A))$;

(x) $B \supseteq f(f^{-1}(B))$.

Alpha cuts and the Extension Principle

Theorem 2.10. Let $f : X \to Y$ be an arbitrary crisp function. Then, for any $A \in \mathcal{F}(X)$, f fuzzified by the extension principle satisfies the equation

$$f(A) = \bigcup_{\alpha \in [0,1]} f({}_{\alpha+}A). \tag{2.12}$$

The significance of Theorem 2.10 is that it provides us with an efficient procedure for calculating $f(A)$: first we calculate all images of strong α-cuts (i.e., crisp sets) under function f, convert them to the special fuzzy sets ${}_{\alpha+}A$, and then employ (2.12).

Operations of triangular fuzzy number

Example: Consider the flowing A and B defined by their membership fn. Using extension principle (approx. calculations!)

$$\mu_{(A)}(x) = \begin{cases} 0, & x < -3 \\ \dfrac{x+3}{2+3}, & -3 \le x \le 2 \\ \dfrac{4-x}{4-2}, & 2 \le x \le 4 \\ 0, & x > 4 \end{cases} \qquad \mu_{(B)}(y) = \begin{cases} 0, & y < -1 \\ \dfrac{y+1}{0+1}, & -1 \le y \le 0 \\ \dfrac{6-y}{6-0}, & 0 \le y \le 6 \\ 0, & y > 6 \end{cases}$$

for the two fuzzy number $x \in A$ and $y \in B$, $z \in A\ (+)\ B$ shall be obtained by their membership functions.

Operations of triangular fuzzy number

Let's think when $z = 8$. Addition to make $z = 8$ is possible for following cases :

$$2 + 6,\ 3 + 5,\ 3.5 + 4.5,\ \ldots$$

so

$$\begin{aligned} \mu_{A(+)B} &= \bigvee_{8=x+y} [\mu_A(2) \wedge \mu_B(6),\ \mu_A(3) \wedge \mu_B(5),\ \mu_A(3.5) \wedge \mu_B(4.5), \cdots] \\ &= \vee \quad [1 \wedge 0,\ 0.5 \wedge 1/6,\ 0.25 \wedge 0.25, \cdots] \\ &= \vee \quad [0, 1/6,\ 0.25, \cdots] \end{aligned}$$

Operations of triangular fuzzy number

If we go on these kinds of operations for all $z \in A\ (+)\ B$, we come to the following membership functions, and these are identical to the three point expression for triangular fuzzy number $A = (-4, 2, 10)$.

$$\mu_{A(+)B}(z) = \begin{cases} 0, & z < -4 \\ \dfrac{z+4}{6}, & -4 \le z \le 2 \\ \dfrac{10-z}{8}, & 2 \le z \le 10 \\ 0, & z > 10 \end{cases}$$

□

Operations of triangular fuzzy number

Example 5.11 Multiplication $A\ (\bullet)\ B$
Let triangular fuzzy numbers A and B be

$$A = (1, 2, 4),\ B = (2, 4, 6)$$

$$\mu_{(A)}(x) = \begin{cases} 0, & x < 1 \\ x-1, & 1 \le x < 2 \\ -\dfrac{1}{2}x + 2, & 2 \le x < 4 \\ 0, & x \ge 4 \end{cases} \qquad \mu_{(B)}(y) = \begin{cases} 0, & y < 2 \\ \dfrac{1}{2}y - 1, & 2 \le y < 4 \\ -\dfrac{1}{2}y + 3, & 4 \le y < 6 \\ 0, & y \ge 6 \end{cases}$$

Operations of triangular fuzzy number

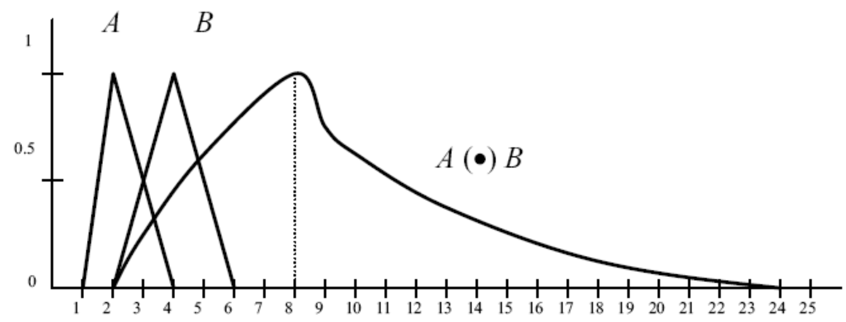
calculating multiplication $A\ (\bullet)\ B$ of A and B, $z = x \bullet y = 8$ is possible when $z = 2 \bullet 4$ or $z = 4 \bullet 2$

$$\begin{aligned}\mu_{A(\bullet)B} &= \vee_{x \bullet y=8}[\mu_A(2) \wedge \mu_B(4),\ \ \mu_A(4) \wedge \mu_B(2), \cdots] \\ &= \vee[1 \wedge 1,\ \ 0 \wedge 0, \cdots] \\ &= 1\end{aligned}$$

also when $z = x \bullet y = 12$, $3 \bullet 4$, $4 \bullet 3$, $2.5 \bullet 4.8$, … are possible.

$$\begin{aligned}\mu_{A(\bullet)B} &= \vee_{x \bullet y=12}[\mu_A(3) \wedge \mu_B(4),\quad \mu_A(4) \wedge \mu_B(3),\quad \mu_A(2.5) \wedge \mu_B(4.8), \cdots] \\ &= \vee[0.5 \wedge 1,\quad 0 \wedge 0.5,\quad 0.75 \wedge 0.6,\quad \cdots] \\ &= \vee[0.5,\quad 0,\quad 0.6,\quad \cdots] \\ &= 0.6\end{aligned}$$

Operations of triangular fuzzy number



Trapezoidal fuzzy number

Definition (Trapezoidal fuzzy number) We can define trapezoidal fuzzy number A as

$$A = (a_1, a_2, a_3, a_4)$$

the membership function of this fuzzy number will be interpreted as follows(Fig 5.10).

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4}, & a_3 \le x \le a_4 \\ 0, & x > a_4 \end{cases}$$

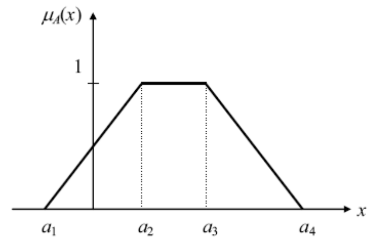


Fig. 5.10. Trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4)$

α-cut interval for this shape is written below.

$\forall \alpha \in [0, 1]$

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, \quad -(a_4 - a_3)\alpha + a_4]$$

when $a_2 = a_3$, the trapezoidal fuzzy number coincides with triangular one.

Operations of Trapezoidal Fuzzy Number

Let's talk about the operations of trapezoidal fuzzy number as in the triangular fuzzy number,

(1) Addition and subtraction between fuzzy numbers become trapezoidal fuzzy number.

(2) Multiplication, division, and inverse need not be trapezoidal fuzzy number.

(1) Addition

$$\begin{aligned} A(+)B &= (a_1, a_2, a_3, a_4)(+)(b_1, b_2, b_3, b_4) \\ &= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4) \end{aligned}$$

(2) Subtraction

$$A(-)B = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1)$$

Operations of Trapezoidal Fuzzy Number

Example 5.14 Multiplication
Multiply two trapezoidal fuzzy numbers as following:

$$A = (1, 5, 6, 9)$$
$$B = (2, 3, 5, 8)$$

For exact value of the calculation, the membership functions shall be used and the result is described in (Fig. 5.11) For the approximation of operation results, we use α-cut interval

$$A_\alpha = [4\alpha + 1, -3\alpha + 9]$$
$$B_\alpha = [\alpha + 2, -3\alpha + 8]$$

since, for all $\alpha \in [0, 1]$, each element for each interval is positive, multiplication between α-cut intervals will be

$$\begin{aligned} A_\alpha(\bullet)B_\alpha &= [(4\alpha+1)(\alpha+2), \quad (-3\alpha+9)(-3\alpha+8)] \\ &= [4\alpha^2+9\alpha+2, \quad 9\alpha^2-51\alpha+72] \end{aligned}$$

Operations of Trapezoidal Fuzzy Number

$$A_\alpha = [4\alpha + 1, -3\alpha + 9]$$
$$B_\alpha = [\alpha + 2, -3\alpha + 8]$$

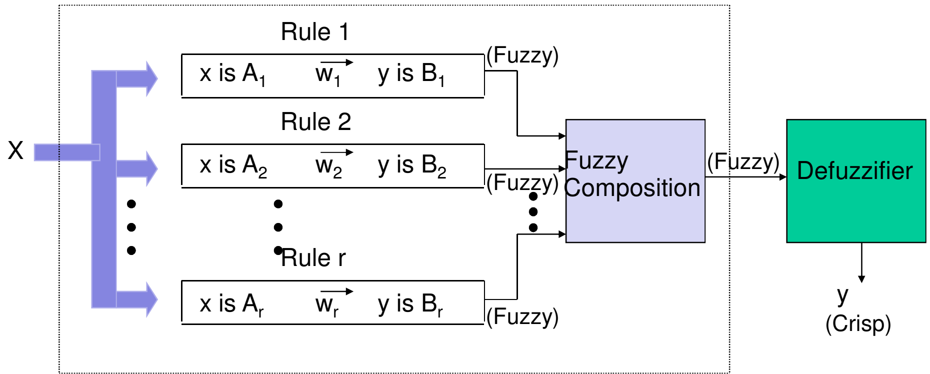
$$\begin{aligned} A_\alpha(\bullet)B_\alpha &= [(4\alpha+1)(\alpha+2), \quad (-3\alpha+9)(-3\alpha+8)] \\ &= [4\alpha^2+9\alpha+2, \quad 9\alpha^2-51\alpha+72] \end{aligned}$$

$$\mu_{(A(\cdot)B)}(z) = \begin{cases} 0 & z < 2 \\ \dfrac{\sqrt{z+3.0625}-2.25}{2} & 2 \le z \le 15 \\ 1 & 15 \le z \le 30 \\ \dfrac{-\sqrt{z+0.25}+8.5}{3} & 30 \le z \le 72 \\ 0 & z > 72 \end{cases}$$

$$4\alpha^2 + 9\alpha + 2 = z$$
$$(2\alpha + 2.25)^2 = z + 3.0625$$
$$\alpha = \frac{\sqrt{z+3.0625}-2.25}{2}$$

$$9\alpha^2 - 51\alpha + 72 = z$$
$$(3\alpha - 8.5)^2 = z + 0.25$$
$$\alpha = \frac{-\sqrt{z+0.25}+8.5}{3}$$

Fuzzy System Design

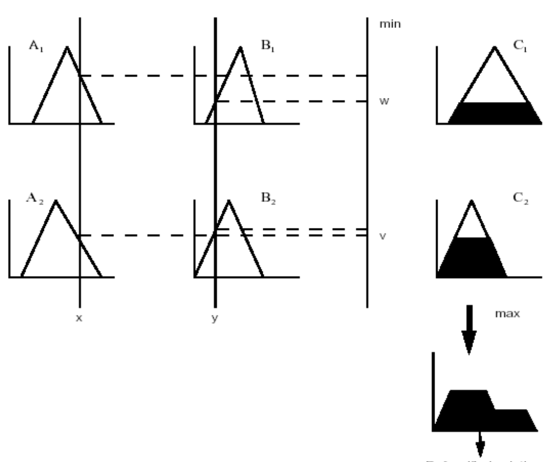


A fuzzy system is a computer system that uses fuzzy sets in either the antecedent and/or the consequent of fuzzy if-then rules. It consists of the following components: (i) The 'base' fuzzy sets that describe the problem, (ii) The if-then rules, (iii) Rule composition, and (iv) Defuzzification

The Mamdani Model for two rules

IF x is A_1 and y is B_1 THEN z is C_1

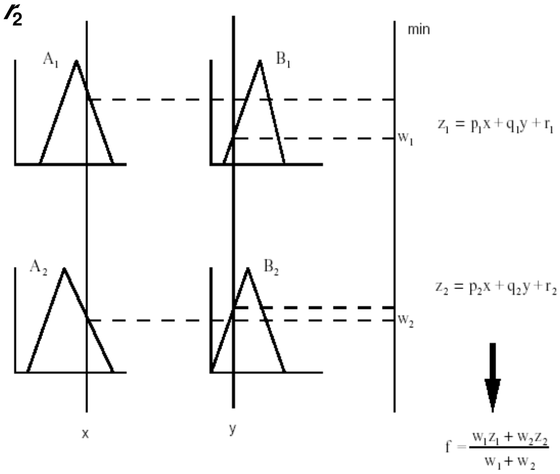
IF x is A_2 and y is B_2 THEN z is C_2



The Takagi-Sugeno Model for two rules

IF x is A_1 *and y is* B_1 *THEN* $z_1 = p_1x + q_1y + r_1$

IF x is A_2 *and y is* B_2 *THEN* $z_2 = p_2x + q_2y + r_2$



Defuzzification

Without the defuzzification phase, the final output of the FIS is a fuzzy set

Defuzzification is used to obtain a crisp output from the FIS

Methods for Defuzzification

- The Centre of Area (COA)
- The Mean of Maximum (MOM)
- Bisector of Area (BOA)
- Smallest of Maximum (SOM)
- Largest of Maximum (LOM)