Fuzzy Operations

Fuzzy logic begins by borrowing notions from crisp logic, same as fuzzy set theory borrows from crisp set theory. As in the extension of crisp set theory to fuzzy set theory, the extension of crisp logic to fuzzy logic is made by replacing membership functions of crisp logic with fuzzy membership functions

In Fuzzy Logic, intersection, union and complement are defined in terms of their membership functions

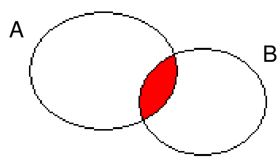
Fuzzy intersection and union correspond to 'AND' and 'OR', respectively, in classic/crisp/Boolean logic

Classic/Crisp/Boolean Logic

Logical AND (∩)

Truth Table

A	B	A ∩ B
0	0	0
0	1	0
1	0	0
1	1	1

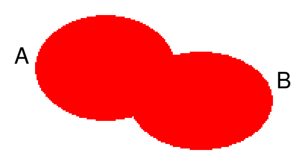


Crisp Intersection

- Logical OR (**U**)

Truth Table

A	B	A **U** B
0	0	0
0	1	1
1	0	1
1	1	1



Crisp Union

Fuzzy Union

The union (OR) is calculated using t-conorms

t-conorm operator is a function s

Satisfying:

i. s(1,1) = 1, s(a,0) = s(0,a) = a (boundary)
ii. s(a,b) ≤ s(c,d) if a ≤ c and b ≤ d (monotonicity)
iii. s(a,b) = s(b,a) (commutativity)
iv. s(a,s(b,c)) = s(s(a,b),c) (associativity)

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iv. s(a,s(b,c)) = s(s(a,b),c) (associativity)

The most commonly used method for fuzzy union is to take the maximum. That is, given two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{AUB}(x) = \max(\mu_A(x), \mu_B(x))$$

Fuzzy Union

Additional requirements may be added to any t-conorm to satisfy. The most famous are:

i. u is continuous function

ii. u(a, a) > a (subidempotency)

iii. If $a<b$ and $c<d$ then $u(a,c)<u(b,d)$ (strict monotonicity)

T-conorms frequently used$_{\text{Klir}}$

Standard union: $u(a, b) = \max(a, b)$.

Algebraic sum: $u(a, b) = a + b - ab$.

Bounded sum: $u(a, b) = \min(1, a + b)$.

Drastic union: $u(a, b) = \begin{cases} a & \text{when } b = 0 \\ b & \text{when } a = 0 \\ 1 & \text{otherwise.} \end{cases}$

T-conorms frequently used$_{\text{Klir}}$

TABLE 3.3 SOME CLASSES OF FUZZY UNIONS (t-CONORMS)

Reference	Formula $u(a,b)$	Increasing generator $g(a)$	Parameter range	As parameter converges to 0	As parameter converges to 1 or −1	As parameter converges to ∞ or −∞
Dombi [1982]	$\left\{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{-\frac{1}{\lambda}}\right\}^{-1}$	$\left(\frac{1}{a}-1\right)^{-\lambda}$	$\lambda > 0$	$u_{max}(a,b)$	$\frac{a+b-2ab}{1-ab}$ when $\lambda = 1$	$\max(a,b)$
Frank [1979]	$1-\log_s\left[1+\frac{(s^{1-a}-1)(s^{1-b}-1)}{s-1}\right]$	$-\ln\left(\frac{s^{1-a}-1}{s-1}\right)$	$s>0, s\neq 1$	$\max(a,b)$	$a+b-ab$ as $s\to 1$	$\min(1,a+b)$
Hamacher [1978]	$\frac{a+b+(r-2)ab}{r+(r-1)ab}$	$-\ln\left(\frac{1-a}{r+(1-r)(1-a)}\right)$	$r>0$	$\frac{a+b-2ab}{1-ab}$	$a+b-ab$ when $r=1$	$u_{max}(a,b)$
Schweizer & Sklar 1 [1963]	$1-\{\max(0,(1-a)^p+(1-b)^p-1)\}^{\frac{1}{p}}$	$1-(1-a)^p$	$p\neq 0$	$a+b-ab$	$\min(1,a+b)$, when $p=1$; $\frac{a+b-2ab}{1-ab}$, when $p=-1$.	$u_{max}(a,b)$ as $p\to\infty$; $\min(a,b)$ as $p\to -\infty$
Schweizer & Sklar 2	$[a^p+b^p-a^pb^p]^{\frac{1}{p}}$	$\ln[1-a^p]^{\frac{1}{p}}$	$p>0$	$u_{max}(a,b)$	$a+b-ab$ when $p=1$	$\max(a,b)$
Schweizer & Sklar 3	$1-\exp(-(\lvert\ln(1-a)\rvert^p+\lvert\ln(1-b)\rvert^p)^{\frac{1}{p}})$	$\lvert\ln(1-a)\rvert^p$	$p>0$	$u_{max}(a,b)$	$a+b-ab$ when $p=1$	$\max(a,b)$
Schweizer & Sklar 4	$1-\frac{(1-a)(1-b)}{[(1-a)^p+(1-b)^p-(1-a)^p(1-b)^p]^{\frac{1}{p}}}$	$(1-a)^{-p}-1$	$p>0$	$a+b-ab$	$\min\left(1,\frac{a+b}{1-ab}\right)$ when $p=1$	$\max(a,b)$
Yager [1980f]	$\min\left[1,(a^w+b^w)^{\frac{1}{w}}\right]$	a^w	$w>0$	$u_{max}(a,b)$	$\min(1,a+b)$ when $w=1$	$\max(a,b)$
Dubois & Prade [1980]	$1-\frac{(1-a)(1-b)}{\max((1-a),(1-b),\alpha)}$		$\alpha\in[0,1]$	$\max(a,b)$	$a+b-ab$ when $\alpha=1$	
Weber [1983]	$\min\left(1,a+b-\frac{\lambda}{1-\lambda}ab\right)$	$\frac{1}{\lambda}\ln\frac{1+\lambda}{1+\lambda(1-a)}$	$\lambda>-1$	$\min(1,a+b)$	$u_{max}(a,b)$ as $\lambda\to -1$; $\min(1,(a+b-ab/2)$ when $\lambda=1$.	$a+b-ab$
Yu [1985]	$\min(1,a+b+\lambda ab)$	$\frac{1}{\lambda}\ln(1+\lambda a)$	$\lambda>-1$	$\min(1,a+b)$	$a+b-ab$ as $\lambda\to -1$; $\min(1,a+b+ab)$	$u_{max}(a,b)$

Fuzzy Intersection

The intersection (AND) is calculated using t-norms.

t-norm operator is a function i

Satisfying:

i. i(0,0) = 0, t(a,1) = i(1,a) = a (boundary)
ii. i(a,b) ≤ i(c,d) if a ≤ c and b ≤ d (monotonicity)
iii. i(a,b) = i(b,a) (commutativity)
iv. i(a, i(b,c)) = i(i(a,b),c) (associativity)

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iii. i(a,b) = i(b,a) (commutativity)
iv. i(a, i(b,c)) = i(i(a,b),c) (associativity)

The most commonly adopted t-norm is the minimum. That is, given two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Fuzzy Intersection

Additional requirements may be added to any t-norm to satisfy. The most famous are:

i. i is continuous function
ii. i(a, a) < a (subidempotency)
iii. If a<b and c<d then i(a,c)<i(b,d) (strict monotonicity)

T-norms frequently used$_{\text{Klir}}$

Standard intersection : $i(a, b) = \min(a, b)$.
Algebraic product : $i(a, b) = ab$.
Bounded difference : $i(a, b) = \max(0, a + b - 1)$.
Drastic intersection : $i(a, b) = \begin{cases} a & \text{when } b = 1 \\ b & \text{when } a = 1 \\ 0 & \text{otherwise.} \end{cases}$

T-norms frequently used$_{\text{Klir}}$

TABLE 3.2 SOME CLASSES OF FUZZY INTERSECTIONS (t-NORMS)

Reference	Formula $i(a, b)$	Decreasing generator $f(a)$	Parameter range	As parameter converges to 0	As parameter converges to 1 or −1	As parameter converges to ∞ or −∞
Dombi [1982]	$\left\{1+\left[\left(\frac{1}{a}-1\right)^{\lambda}+\left(\frac{1}{b}-1\right)^{\lambda}\right]^{\frac{1}{\lambda}}\right\}^{-1}$	$\left(\frac{1}{a}-1\right)^{\lambda}$	$\lambda > 0$	$i_{\min}(a, b)$	$\frac{ab}{a+b-ab}$ when $\lambda = 1$	$\min(a, b)$
Frank [1979]	$\log_s\left[1+\frac{(s^a-1)(s^b-1)}{s-1}\right]$	$-\ln\left(\frac{s^a-1}{s-1}\right)$	$s > 0, s \neq 1$	$\min(a, b)$	ab as $s \to 1$	$\max(0, a+b-1)$
Hamacher [1978]	$\frac{ab}{r+(1-r)(a+b-ab)}$	$-\ln\left(\frac{a}{r+(1-r)a}\right)$	$r > 0$	$\frac{ab}{a+b-ab}$	ab when $r = 1$	$i_{\min}(a, b)$
Schweizer & Sklar 1 [1963]	$(\max(0, a^p + b^p - 1))^{\frac{1}{p}}$	$1 - a^p$	$p \neq 0$	ab	$\max(0, a+b-1)$, when $p = 1$; $\frac{ab}{a+b-ab}$, when $p = -1$.	$i_{\min}(a, b)$ as $p \to \infty$; $\min(a, b)$ as $p \to -\infty$.
Schweizer & Sklar 2	$1 - [(1-a)^p + (1-b)^p - (1-a)^p(1-b)^p]^{\frac{1}{p}}$	$\ln[1-(1-a)^p]^{\frac{1}{p}}$	$p > 0$	$i_{\min}(a, b)$	ab when $p = 1$	$\min(a, b)$
Schweizer & Sklar 3	$\exp(-(\lvert\ln a\rvert^p + \lvert\ln b\rvert^p)^{\frac{1}{p}})$	$\lvert\ln a\rvert^p$	$p > 0$	$i_{\min}(a, b)$	ab when $p = 1$	$\min(a, b)$
Schweizer & Sklar 4	$\frac{ab}{[a^p + b^p - a^p b^p]^{\frac{1}{p}}}$	$a^{-p} - 1$	$p > 0$	ab	$\frac{ab}{a+b-ab}$, when $p = 1$	$\min(a, b)$
Yager [1980f]	$1 - \min\left\{1, [(1-a)^w + (1-b)^w]^{\frac{1}{w}}\right\}$	$(1-a)^w$	$w > 0$	$i_{\min}(a, b)$	$\max(0, a+b-1)$ when $w = 1$	$\min(a, b)$
Dubois & Prade [1980]	$\frac{ab}{\max(a, b, \alpha)}$		$\alpha \in [0, 1]$	$\min(a, b)$	ab when $\alpha = 1$	
Weber [1983]	$\max\left(0, \frac{a+b+\lambda ab-1}{1+\lambda}\right)$	$\frac{1}{\lambda}\ln[1+\lambda(1-a)]$	$\lambda > -1$	$\max(0, a+b-1)$	$i_{\min}(a, b)$ as $\lambda \to -1$; $\max[0, (a+b+ab-1)/2]$ when $\lambda = 1$.	ab
Yu [1985]	$\max[0, (1+\lambda)(a+b-1) - \lambda ab]$	$\frac{1}{\lambda}\ln\frac{1+\lambda}{1+\lambda a}$	$\lambda > -1$	$\max(0, a+b-1)$	ab as $\lambda \to -1$; $\max[0, 2(a+b-ab/2-1)]$ when $\lambda = 1$.	$i_{\min}(a, b)$

Fuzzy Complement

To be able to develop fuzzy systems we also have to deal with NOT or complement.

This is the same in fuzzy logic as for Boolean logic

For a fuzzy set A, A^c denotes the fuzzy complement of A

Membership function for fuzzy complement is

$$\mu_{A^c}(x) = 1 - \mu_A(x)$$

Example Discrete case

Suppose we have the following (discrete) fuzzy sets:

$$A = 0.4/1 + 0.6/2 + 0.7/3 + 0.8/4$$

$$B = 0.3/1 + 0.65/2 + 0.4/3 + 0.1/4$$

The union of the fuzzy sets A and B

$= 0.4/1 + 0.65/2 + 0.7/3 + 0.8/4$

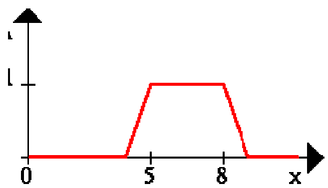
The intersection of the fuzzy sets A and B

$= 0.3/1 + 0.6/2 + 0.4/3 + 0.1/4$

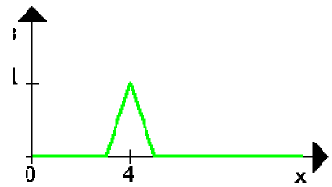
The complement of the fuzzy set A

$= 0.6/1 + 0.4/2 + 0.3/3 + 0.2/4$

Example Continuous case

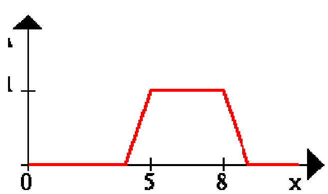


A

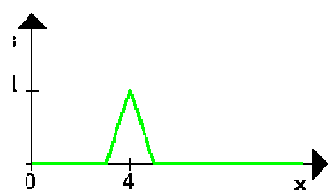


B

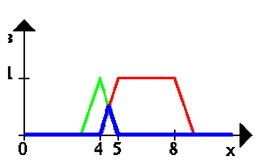
Example Continuous case



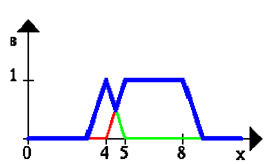
A



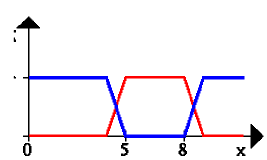
B



A ∧ B

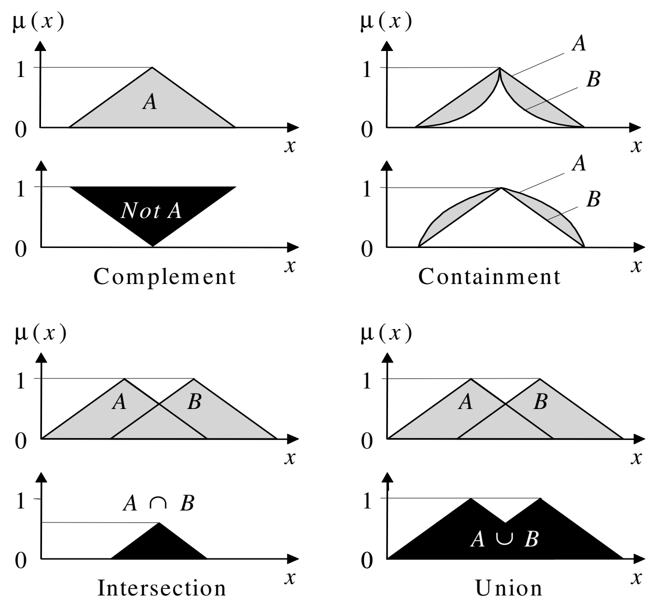


A ∨ B



¬A

Fuzzy Operations



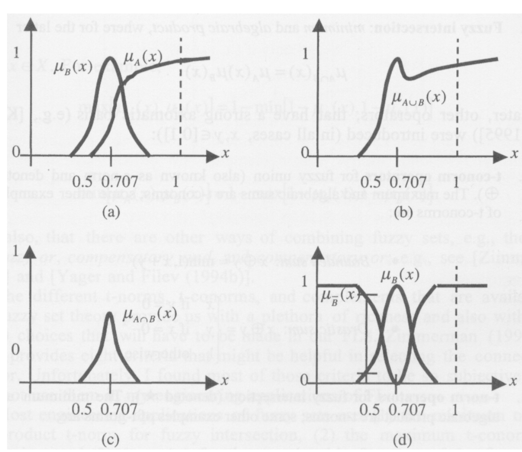
Example

Figure (a): $\mu_A(x)$, $\mu_B(x)$ *Figure (b):* $\mu_{A\cup B}(x)$

Figure (c): $\mu_{A\cap B}(x)$ *Figure (d):* $\mu_B(x)$, $\mu_{B^c}(x)$

$$\mu_A(x) = \begin{cases} 0 & if\ 0 \le x \le 0.5 \\ \dfrac{1}{1+(x-0.5)^{-2}} & if\ 0.5 < x \le 1 \end{cases}$$

$$\mu_B(x) = \frac{1}{1+(x-0.707)^4} \qquad 0 \le x \le 1$$



Example

This example demonstrates that for fuzzy sets, the Law of Excluded Middle and Concentration are broken, i.e., *for fuzzy sets A and B*:

$$A \cup A^c \neq X \text{ and } A \cap A^c \neq \emptyset$$

In fact, one of the ways to describe the difference between crisp set theory and fuzzy set theory is to explain that these two laws do not hold in fuzzy set theory

Definitions

- A is EMPTY iff for all x, $\mu_A(x) = 0.0$.
- A = B iff for all x: $\mu_A(x) = \mu_B(x)$
- A is CONTAINED in B or $A \subseteq B$ iff $\mu_A A \leq \mu_B B$.

for all $x \in X$

Proper Subset:

$A \subset B$ if $\mu_A(x) \leq \mu_B(x)$ and $\mu_A(x) < \mu_B(x)$ for at least one $x \in X$

Basic Properties of set operations

$$\text{Involution}: (A^c)^c = A. \tag{3.3}$$

$$\text{Commutativity}: A \cup B = B \cup A, A \cap B = B \cap A. \tag{3.4}$$

$$\text{Associativity}: \quad (A \cup B) \cup C = A \cup (B \cup C), \tag{3.5}$$
$$(A \cap B) \cap C = A \cap (B \cap C). \tag{3.6}$$

$$\text{Distributivity}: \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \tag{3.7}$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{3.8}$$

$$\text{Idempotency}: A \cap A = A, A \cup A = A. \tag{3.9}$$

$$\text{Law of Contradiction}: A \cap A^c = \phi. \tag{3.10}$$

$$\text{Law of Excluded Middle}: A \cup A^c = X. \tag{3.11}$$

$$\text{De Morgan}: \quad (A \cup B)^c = A^c \cap B^c, \tag{3.12}$$
$$(A \cap B)^c = A^c \cup B^c. \tag{3.13}$$

$$\text{Identity}: \quad A \cup \phi = A, A \cap \phi = \phi, \tag{3.14}$$
$$A \cup X = X, A \cap X = A. \tag{3.15}$$

Students' Topics

→ Fuzzy Controller: Find an Example of

Contains: [Fuzzifier, fuzzy inference/Aggregation of rules, defuzzifier]

→Fuzzy Expert System: Small example

→Fuzzy Neural Network

→Solving Fuzzy Inequalities