Propositional logic is a weak language

- Hard to identify "individuals." Ex. Mary, 3
- Can't directly talk about properties of individuals or relations between individuals. Ex. "Bill is tall"
- Generalizations, patterns, regularities can't easily be represented. Ex. all triangles have 3 sides
- **First-Order Logic** (abbreviated FOL or FOPC) is expressive enough to concisely represent this kind of situation.
  - FOL adds relations, variables, and quantifiers, e.g.,
    - "Every elephant is gray": $\forall$ x (elephant(x) $\rightarrow$ gray(x))
    - "There is a white elephant": $\exists$ x (elephant(x) ^ white(x))

Logical equivalence in PC

- Two sentences are logically equivalent iff true in the same models: $\alpha \equiv \beta$ iff $\alpha \models \beta$ and $\beta \models \alpha$
- EXamples:

$$
\begin{array}{rcll}
(\alpha \land \beta) & \equiv & (\beta \land \alpha) & \text{commutativity of } \land \\
(\alpha \lor \beta) & \equiv & (\beta \lor \alpha) & \text{commutativity of } \lor \\
((\alpha \land \beta) \land \gamma) & \equiv & (\alpha \land (\beta \land \gamma)) & \text{associativity of } \land \\
((\alpha \lor \beta) \lor \gamma) & \equiv & (\alpha \lor (\beta \lor \gamma)) & \text{associativity of } \lor \\
\neg(\neg\alpha) & \equiv & \alpha & \text{double-negation elimination} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg\beta \Rightarrow \neg\alpha) & \text{contraposition} \\
(\alpha \Rightarrow \beta) & \equiv & (\neg\alpha \lor \beta) & \text{implication elimination} \\
(\alpha \Leftrightarrow \beta) & \equiv & ((\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)) & \text{biconditional elimination} \\
\neg(\alpha \land \beta) & \equiv & (\neg\alpha \lor \neg\beta) & \text{de Morgan} \\
\neg(\alpha \lor \beta) & \equiv & (\neg\alpha \land \neg\beta) & \text{de Morgan} \\
(\alpha \land (\beta \lor \gamma)) & \equiv & ((\alpha \land \beta) \lor (\alpha \land \gamma)) & \text{distributivity of } \land \text{ over } \lor \\
(\alpha \lor (\beta \land \gamma)) & \equiv & ((\alpha \lor \beta) \land (\alpha \lor \gamma)) & \text{distributivity of } \lor \text{ over } \land
\end{array}
$$

First-order logic

- First-order logic (FOL) models the world in terms of
  - Objects, which are things with individual identities
  - Properties of objects that distinguish them from other objects
  - Relations that hold among sets of objects
  - Functions, which are a subset of relations where there is only one "value" for any given "input"

Ex:Objects: Students, lectures, companies, cars ...

  - Relations: Brother-of, bigger-than, outside, part-of, has-color, occurs-after, owns, visits, precedes, ...
  - Properties: blue, oval, even, large, ...
  - Functions: father-of, best-friend, second-half, one-more-than ...

FOL Syntax

- **Variable symbols**
  - E.g., x, y, John
- **Connectives**: $\neg$, $\wedge$, $\vee$, $\Rightarrow$
  - **Quantifiers**
  - Universal $\forall x$
  - Existential $\exists x$

Syntax of First-order logic

Sentence → *Atomicsentence*
/ (*Sentence Connective Sentence*)
| *Quantifier Variable,*. . . *Sentence*
/> *Sentence*
AtomicSentence → *Predicate(Term,* . . .)
| (*Term* = *Term*
Term -→ *Function(Term,* . . .)
l *Constant*
| *Variable*
Connective → ¬, ∧, ∨, ⇒
Quantifier → ∀, ∃
Constant → *A* (XI (*John* 1 . . .
Variable → *a* | *x* | s | . . .
Predicate → *Before...*
Function → *Mother* | *...*

Atomic Sentences

- Propositions are represented by a **predicate applied to a** tuple of terms. A predicate represents a property of or relation between terms that can be true or false:
- Brother(John, Fred), Left-of(Square1, Square2), GreaterThan(plus(1,1), plus(0,1))
- **Sentences in logic state facts** that are true or false.
- In FOL properties and n-ary relations do express that:

  LargerThan(2,3) is false. Brother(Mary,Pete) is false.
- Note: **Functions do not state facts** and form no sentence: Brother(Pete) refers to the object John (his brother) and is neither true nor false.
- Brother(Pete,Brother(Pete)) is True.

↑ Binary relation ↑ Function

Truth in first-order logic

- Sentences are true with respect to a model and an interpretation
- Model contains objects (domain elements) and relations among them
- Interpretation specifies referents for

constant symbols → objects
predicate symbols → relations
function symbols → functional relations

- An atomic sentence *predicate*($term_1,...,term_n$) is true iff the objects referred to by $term_1,...,term_n$ are in the relation referred to by *predicate*

Entailment

- Entailment means that one thing follows from another:

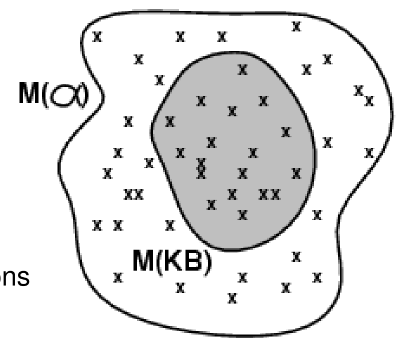
$$KB \models \alpha$$

Knowledge base ***KB*** entails sentence α if and only if α is true in all worlds where ***KB*** is true

- E.g., the KB containing "the Greens won" and "the Reds won" entails "Either the Greens or the reds won"
- E.g., x+y = 4 entails 4 = x+y
- Entailment is a relationship between sentences (i.e., syntax) that is based on semantics
- entailment: necessary truth of one sentence given another

Models

- Logicians typically think in terms of models, which are formally structured worlds with respect to which truth can be evaluated
- We say *m* is a model of a sentence α if α is true in *m*
- *M(α)* is the set of all models of α
- Then KB ╞ α iff *M(KB)* ⊆ *M(*α)
  - E.g. *KB* = Greens won and Reds won
    α = Greens won



- Think of KB and α as collections of constraints and of models m as possible states. M(KB) are the solutions to KB and M(α) the solutions to α. Then, KB ╞ α when all solutions to KB are also solutions to α.

Nested Quantifiers

- Combinations of universal and existential quantification are possible:

$$\forall x \forall y\, Father(x,y) \equiv \forall y \forall x\, Father(x,y)$$
$$\exists x \exists y\, Father(x,y) \equiv \exists y \exists x\, Father(x,y)$$
$$\forall x \exists y\, Father(x,y) \neq \exists y \forall x\, Father(x,y)$$
$$\exists x \forall y\, Father(x,y) \neq \forall y \exists x\, Father(x,y)$$
$$x,y \in \{All\ people\}$$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$
- Common mistake: using $\wedge$ as the main connective with $\forall$:
- Ex:

$\forall x$ At(x,CU) $\wedge$ Smart(x)

means "Everyone is at CU and everyone is smart"

Yet to say Everyone at CU is smart

$\forall x$ At(x,CU) $\Rightarrow$ Smart(x)

Another common mistake to avoid

- Typically, $\wedge$ is the main connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

  $\exists x$ At(x,CU) $\Rightarrow$ Smart(x)

  is true if there is anyone who is smart not at CU.

Yet to say: there exists someone in CU that is smart

$\exists x$ At(x,CU) $\wedge$ Smart(x)

Properties of quantifiers

∀x ∀y is the same as ∀y ∀x
∃x ∃y is the same as ∃y ∃x

∃x ∀y is not the same as ∀y ∃x
∃x ∀y Loves(x,y)
- "There is a person who loves everyone in the world"

∀y ∃x Loves(x,y)
- "Everyone in the world is loved by at least one person"

- Quantifier duality: each can be expressed using the other

Exp.	Negation
∀x Likes(x,IceCream)	∃x ¬Likes(x,IceCream)
∃x Likes(x,Broccoli)	∀x ¬Likes(x,Broccoli)

Equality

Equality:

term$_1$ = *term*$_2$ is true under a given interpretation if and only if *term*$_1$ and *term*$_2$ refer to the same object

FOPC can include equality as a primitive predicate or require it to be as identity relation

Equal(x,y) or x=y

Examples:

to say "that Mary is taking two courses", you need to insure that x,y are different

∃x ∃y (takes(Mary,x) ^ takes (Mary,y) ^ ~ (x=y))

To say "Everyone has exactly one father"

∀x ∃y father(y,x) ^ ∀z father(z,x) ➔ y=z

Higher Order Logic

- FOPC is called first order because it allows quantifiers to rang only over objects (terms).

  $\forall x, \forall y\ [x=y \text{ or } x>y \text{ or } y>x]$

- **Second-Order Logic** allows quantifiers to range over predicates and functions as well

  $\forall f, \forall g\ [f=g \Leftarrow\Rightarrow (\forall x\ f(x)=g(x))]$

- **Third-Order Logic** allows quantifiers to range over predicates of predicates,.. etc

Examples of FOPC

- Brothers are siblings

  $\forall x, \forall y\ \textit{Brother}(x,y) \Rightarrow \textit{Sibling}(x,y)$

- One's mother is one's female parent

  $\forall m, \forall c\ \textit{Mother}(c) = m \Leftrightarrow (\textit{Female}(m) \wedge \textit{Parent}(m,c))$

- "Sibling" is symmetric

  $\forall x, \forall y\ \textit{Sibling}(x,y) \Leftrightarrow \textit{Sibling}(y,x)$

Translating English to FOL

- Every gardener likes the sun.

  `(∀x) gardener(x) => likes(x,Sun)`

Translating English to FOL

- Every gardener likes the sun.

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- You can fool some of the people all of the time.

  `(∃x) person(x) ^ ((∀t) time(t)) => can-fool(x,t))`

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  `(∀ x) (mushroom(x) ^ purple(x)) => poisonous(x)`

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- You can fool all of the people some of the time.

  `(∀x) person(x) => ((∃t) time(t) ^ can-fool(x,t))`
- All purple mushrooms are poisonous.

  `(∀ x) (mushroom(x) ^ purple(x)) => poisonous(x)`
- No purple mushroom is poisonous.

  `~(∃x) purple(x) ^ mushroom(x) ^ poisonous(x)`
  or, equivalently,

  `(∀x) (mushroom(x) ^ purple(x)) => ~poisonous(x)`

Translating English to FOL

- There are exactly two purple mushrooms.

  ```
  (∃x)(∃ y) mushroom(x) ^ purple(x) ^ mushroom(y) ^
  purple(y) ^ ~(x=y) ^ (∀z) (mushroom(z) ^ purple(z))
  => ((x=z) v (y=z))
  ```

Inference in FOL chapter 9 in Russel

- ***KB*** $\vdash_i \alpha$ = sentence α can be derived from ***KB*** by procedure ***i***

i.e. deriving sentences from other sentences

- Soundness: ***i*** is sound if whenever ***KB*** $\vdash_i \alpha$, it is also true that ***KB*** $\models \alpha$

i.e. derivations produce only entailed sentences ***(no wrong inferences, but maybe not all inferences)***

- Completeness: ***i*** is complete if whenever ***KB*** $\models \alpha$, it is also true that ***KB*** $\vdash_i \alpha$

i.e. derivations can produce all entailed sentences (***all inferences can be made, but maybe some wrong extra ones as well)***

Validity and satisfiability

- A sentence is **valid** if it is true in **all models**,
- e.g., ***True***, $A \vee \neg A$, $A \Rightarrow A$, $(A \wedge (A \Rightarrow B)) \Rightarrow B$

Validity is connected to inference via the following:
KB $\models \alpha$ if and only if (***KB*** $\Rightarrow \alpha$) is valid

A sentence is **satisfiable** if it is true in **some model**
e.g., $A \vee B$, C

A sentence is **unsatisfiable** if it is true in **no models**
e.g., $A \wedge \neg A$

Satisfiability is connected to inference via the following:
KB $\models \alpha$ if and only if (***KB*** $\wedge \neg \alpha$) is unsatisfiable
(there is no model for which KB=true and is false)

Proof Methods in FOL

Major Families:

- GMP
- Reduction
- Resolution
- Forward chaining
- Backward chaining

Some Other inference tools:

Entailment/ Unification/

Proof Methods in FOL

- GMP: Using the generalized form of Modus Ponense
- Reduction: Reduce all FOL sentences to propositional Calculus then use inference in propositional calculus
- Resolution – Refutation
  - Negate goal
  - Convert all pieces of knowledge into clausal form (disjunction of literals)
  - See if contradiction indicated by null clause $\square$ can be derived
- Forward chaining
  - Given P, $P \rightarrow Q$, to infer Q
  - P, match $L.H.S$ of
  - Assert Q from $R.H.S$
- Backward chaining
  - Q, Match $R.H.S$ of $P \rightarrow Q$
  - assert P
  - Check if P exists