

Chapter 8

Generating Random Variate

- If a Pseudo random number Y is used to generate another variable X , then X is called random variate.
- Note that the sequence X may or may not be uniform.
- Generally, all random variates require the use of a $U(0,1)$ distribution.

1

Random Variate

- The exact technique used to generate an observation varies with the type of distribution used. However, in general we're looking for techniques that are:
 - Exact
 - Efficient in terms of time, storage, and setup
 - Low in complexity
 - Robust.

2

Generating Random Variates

- Say we have fitted an exponential distribution to inter arrival times of customers
- Every time we anticipate a new customer arrival (place an arrival even on the events list), we need to generate a realization of the arrival times
- Already know how to generate unit uniform
- Can we use this to generate exponential? (And other distributions)

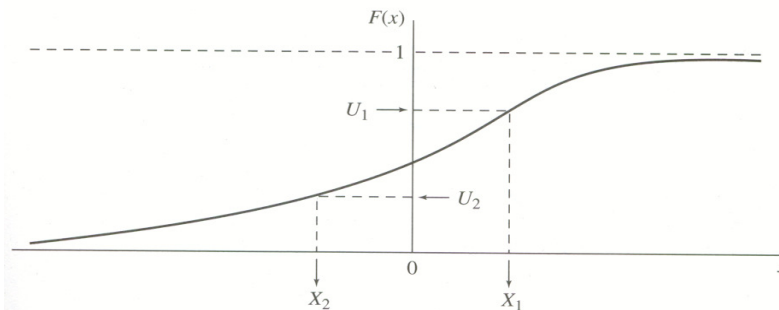
3

Two Types of Approaches

- **Direct**
 - Obtain an analytical expression
 - **Inverse transform**
 - Requires inverse of the distribution function
 - **Composition & Convolution**
 - For special forms of distribution functions
- **Indirect**
 - Acceptance-rejection

4

Inverse-Transform Method



- Place r_i on the y-axis and find the corresponding x_i

$$r = F(x) \Rightarrow x = F^{-1}(r)$$

5

Formulation

- Algorithm
 1. Generate $U \sim U(0,1)$
 2. Return $X = F^{-1}(U)$
- Proof Since $F(x)$ is a monotone function, then

$$\begin{aligned}
 P(X \leq x) &= P(F^{-1}(U) \leq x) \\
 &= P(U \leq F(x)) \\
 &= F(x)
 \end{aligned}$$

6

Example

- Let $f(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & O.W. \end{cases}$

$$F(x) = \int_0^x 4x^3 dx = x^4$$

- i.e. $r = x^4$

$$x_i = F^{-1}(r_i) = \sqrt[4]{r_i} \quad 0 < r_i \leq 1$$

- That is use a pseudo random generator to find r_i then find x_i for the new random variate ⁷

Ex. Uniform random variate

- Let $f(x) = \begin{cases} \frac{1}{b-a} & a < x \leq b \\ 0 & O.W. \end{cases}$

$$F(x) = \frac{x-a}{b-a} = r \quad 0 < r \leq 1$$

- i.e.

$$x = a + r(b-a)$$

- Use a pseudo random generator to find r_i then find x_i for the new random variate uniformly distributed on (a,b].

Exponential R.V.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \lambda > 0, x \geq 0 \\ 0 & \text{O.W.} \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$r = 1 - e^{-\lambda x} \quad 0 < r \leq 1$$

$$e^{-\lambda x} = 1 - r$$

$$x = \frac{-1}{\lambda} \ln(1 - r)$$

9

Exponential R.V.

- Algorithm:

1. Generate $U \sim U(0,1)$

2. Return $X = \frac{-1}{\lambda} \ln(1 - U)$

10

Exponential R.V.

- Algorithm:

1. Generate $U \sim U(0,1)$

2. Return $X = \frac{-1}{\lambda} \ln(1-U)$

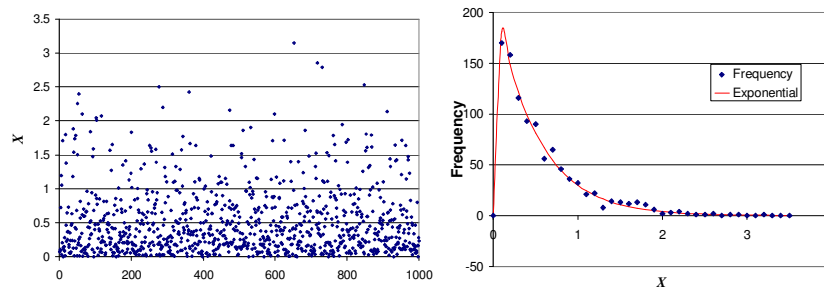
- Remark:** Since $U \sim U(0,1)$ then also $1-U \sim U(0,1)$

And hence for sake of computation we may replace $1-U$ by U in the above algorithm.

11

Exponential Distribution

1000 Exponential Random Numbers ($\lambda = 2$)



12

Exponential R.V.

- Example:
- For a sample of 5 numbers and $\lambda = 3.8$
- Use a R.N.G. to find r_i 's

r_i	0.135	0.639	0.424	0.01	0.843
$\ln r_i$	-2.002	-0.448	-0.858	-4.61	-1.171
x_i	0.526	0.118	0.226	1.213	0.045

13

Exponential distribution

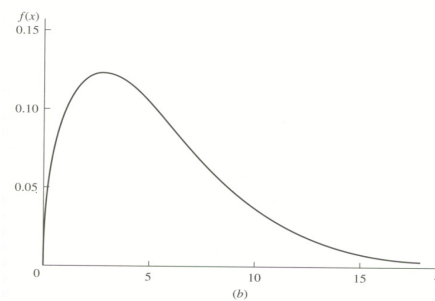
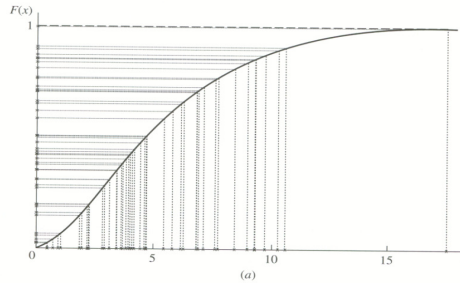
Needed for many applications

- Probability of waiting time on phone calls
- Radio active decay materials
- Time arrival of customers to some server
- Time of failure of machines/ components

Example: Weibull

- The CDF of a Weibull is

$$F(X) = 1 - e^{-\left(\frac{x}{\alpha}\right)^\beta}$$



Inverse Transformation-Weibull

$$r = 1 - e^{-\left(\frac{X}{\alpha}\right)^\beta}$$

$$(1-r) = e^{-\left(\frac{X}{\alpha}\right)^\beta}$$

$$\ln(1-r) = -\left(\frac{X}{\alpha}\right)^\beta$$

$$\left[-\ln(1-r)\right]^{\frac{1}{\beta}} = \frac{X}{\alpha}$$

$$\alpha \left[-\ln(1-r)\right]^{\frac{1}{\beta}} = X$$

- Exr. Write the algorithm for generating a Weibull random variate

Advantages of Inverse Transform Method

- Straightforward method, easy to use
- Variance reduction techniques has an advantage if inverse transform method is used

Disadvantages of the inverse method

- Must evaluate the inverse of the distribution function
 - May not exist in closed form
 - Could still use numerical methods
- May not be the fastest way

Discrete R.V.: Bernoulli

- Mass function

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$
- Algorithm
 1. Generate $U \sim U(0,1)$
 2. If $U \leq p$ return $X = 1$. Otherwise return $X = 0$

19

Geometric Distribution

$p(x) = p(1-p)^x$, where $x = 0, 1, 2, \dots$

$$F(x) = \sum_{j=0}^x p(1-p)^j = 1 - (1-p)^{x+1}$$

$$R = 1 - (1-p)^{x+1}$$

$$\Leftrightarrow (1-p)^{x+1} = 1-R$$

$$\Leftrightarrow (x+1)\ln(1-p) = \ln(1-R)$$

$$\Leftrightarrow x = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F^{-1}(R) = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F(x-1) < R \leq F(x)$$

$$X = \lfloor F^{-1}(R) \rfloor = \left\lfloor \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rfloor$$

Geometric Distribution

- Algorithm to generate random variate for

$$p(x) = p(1-p)^x, \text{ where } x = 0, 1, 2, \dots$$

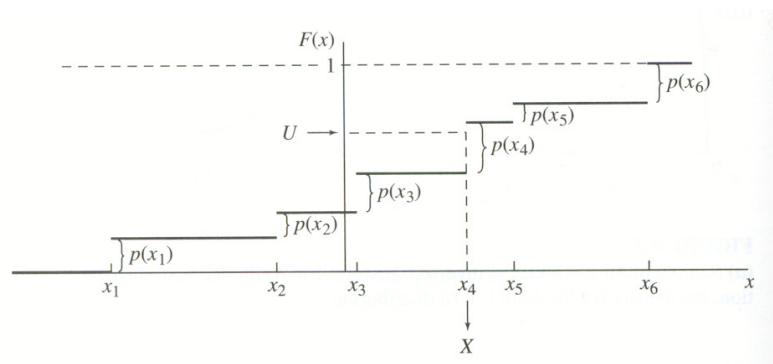
- Generate $U \sim U(0, 1)$ uniform random number

- Return $\left\lfloor \frac{\ln(1-U)}{\ln(1-p)} - 1 \right\rfloor$

- Remark**

For simplicity you may return $X = \left\lfloor \frac{\ln U}{\ln(1-p)} \right\rfloor$

Discrete Distributions



Empirical Discrete Distribution

- Find the random variate for the following discrete distribution.

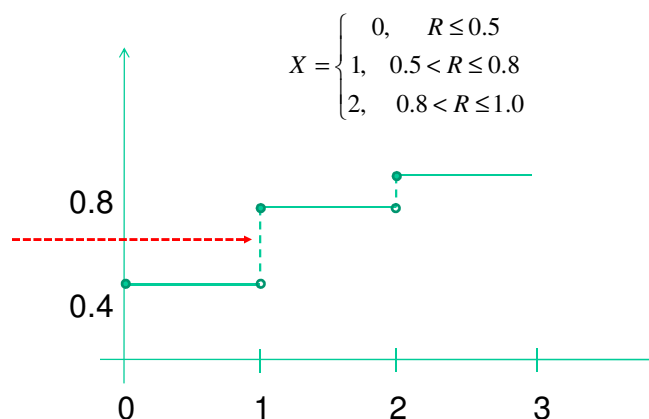
x	p(x)
0	0.50
1	0.30
2	0.20

- At first find the F

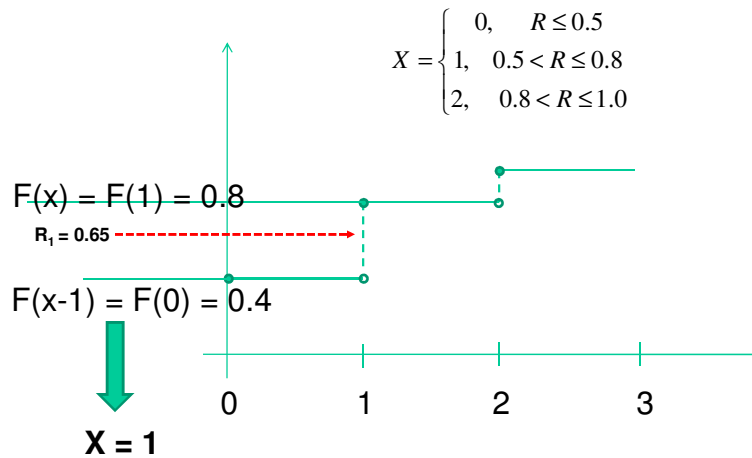
x	p(x)	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5, & 0 \leq x < 1 \\ 0.8, & 1 \leq x < 2 \\ 1, & x \geq 2 \end{cases}$$

Empirical Discrete Distribution



Empirical Discrete Distribution



Example of Empirical Distribution

- There are four pumps (1 to 4) at a petrol station. And on average 1/3 of the customers used pump 1, 1/6 of them used pump 2, another 1/3 used pump 3 and the remaining 1/6 used pump 4. Outline a procedure for selection of pumps by various customers.

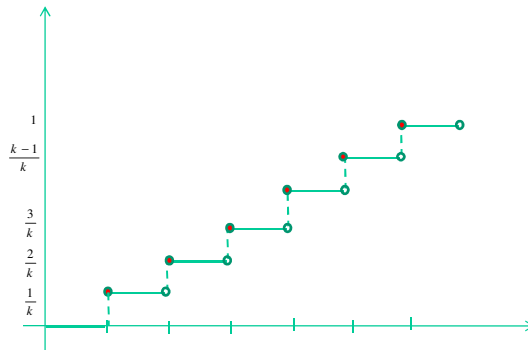
i	1	2	3	4
P(i)	1/3	1/6	1/3	1/6
F(i)	1/3	1/2	5/6	1

- The procedure:
- Generate $r \sim U(0,1)$
 - If $0 \leq r < 1/3$ select pump 1
 - If $1/3 \leq r < 1/2$ select pump 2
 - If $1/2 \leq r < 5/6$ select pump 3
 - If $5/6 \leq r < 1$ select pump 4

Discrete Uniform Distribution

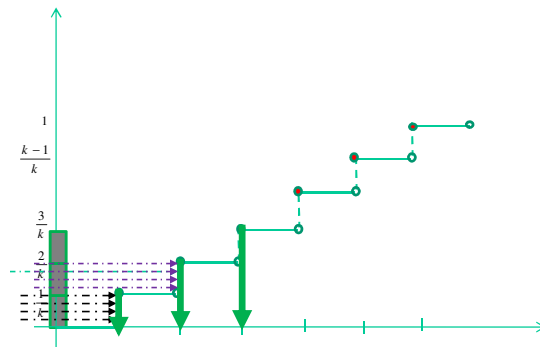
- Consider $p(x) = \frac{1}{k}, x = 1, 2, 3 \dots k$

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{k}, & 1 \leq x < 2 \\ \frac{2}{k}, & 2 \leq x < 3 \\ \vdots & \vdots \\ \frac{k-1}{k}, & k-1 \leq x < k \\ 1, & k \leq x \end{cases}$$



Discrete Uniform Distribution

- Generate $R \sim U(0,1)$



$$0 \leq R \leq \frac{1}{k}$$

$$\frac{1}{k} < R \leq \frac{2}{k}$$

$$\frac{2}{k} < R \leq \frac{3}{k}$$

$$\frac{3}{k} < R \leq \frac{4}{k}$$

$$\frac{i-1}{k} < R \leq \frac{i}{k}$$

Discrete Uniform Distribution

$$0 \leq R \leq \frac{1}{k}$$

$$\frac{1}{k} < R \leq \frac{2}{k}$$

$$\frac{2}{k} < R \leq \frac{3}{k}$$

$$\frac{3}{k} < R \leq \frac{4}{k}$$

$$\frac{i-1}{k} < R \leq \frac{i}{k}$$

$$\frac{i-1}{k} < R \leq \frac{i}{k}$$

$$\Rightarrow i-1 < Rk \leq i$$

$$\Rightarrow i-1 < Rk \leq i$$

$$\Rightarrow i < Rk + 1 \text{ and } Rk \leq i$$

$$\Rightarrow Rk \leq i < Rk + 1$$

$$\Rightarrow i = \lfloor RK \rfloor = \text{output } X$$

$$\therefore X = \lfloor RK \rfloor$$

Discrete Uniform Distribution

- Algorithm to generate random variate for $p(x)=1/k$ where $x = 1, 2, 3, \dots k$
 - Generate $R \sim U(0,1)$ uniform random number
 - Return $\lfloor RK \rfloor$