Probability

- Probability is a measure of how likely it is for an event to happen.
- We name a probability with a number from 0 to 1.
- If an event is certain to happen, then the probability of the event is 1.
- If an event is certain not to happen, then the probability of the event is 0.

Probability Vs Statistics

- In probability theory: R.V. is specified and their parameters are known.
- Goal: Compute probabilities of random values that these variables can take.
- In statistics: The values of random variables are known " from experiment" but theoretical characteristics are unknown.
- Goal: To determine the unknown theoretical characteristics of R.V.
- Probability and Statistics are complementary subjects

What is an Event?

- In probability theory, **an event** is a set of outcomes (a subset of the sample space) to which a probability is assigned.
- Typically, when the sample space is finite, any subset of the sample space is an event (i.e. all elements of the power set of the sample space are defined as events).

Fundamentals

- We measure the probability for Random Events
  - How likely an event would occur
- The set of all possible events is called Sample Space
- In each experiment, an event may occur with a certain probability (Probability Measure)
- Example:
  - Tossing a dice with 6 faces
  - The sample space is {1, 2, 3, 4, 5, 6}
  - Getting the Event « 2 » in on experiment has a probability 1/6

Examples

- A single card is pulled (out of 52 cards).
  - **Possible Events**
    - having a red card (P=1/2);
    - Having a Jack (P= 1/13);
- Two true 6-sided dice are used to consider the event where the sum of the up faces is 10.
  - P = 3 / 36 = 1/12

Probability

- The **probability** of every set of possible events is between 0 and 1, inclusive.
- The probability of the whole set of outcomes is 1.
  - Sum of all probability is equal to one
  - Example for a dice: P(1)+P(2)+P(3)+ P(4)+P(5)+P(6)=1
- If A and B are two events with no common outcomes, then the probability of their union is the sum of their probabilities.
  - Event E1={1},
  - Event E2 ={6}
  - P(E1 v E2)=P(E1)+P(E2)

Random Variables

An Experiment: is a process whose outcome is not known with certainty

Sample Space: set of outcomes S

Ex: S= {H,T} , S= {1,2,3,4,5,6}

Random Variable: also known as **stochastic variable**. is a function that assigns a real number to each point in the space

Random Variable is either discrete or continuous

A random variable: Examples.

- The waiting time of a customer in a queue
- The number of cars that enters the parking each hour
- The number of students that succeed in the exam

Probability Distribution

- The **probability distribution** of a discrete random variable is a **list of probabilities** associated with each of its possible values.
- It is also sometimes called the **probability function** or the **probability mass function** (**PMF**) for discrete random variable.

Probability Mass Function (PMF)

- The **probability distribution** or **probability mass function (PMF)** of a **discrete random variable** X is a function that gives the probability p(xi) that the random variable equals some value xi, for each value xi:
- It satisfies the following conditions:

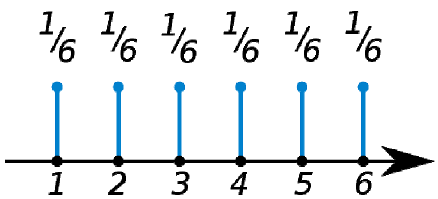
$$p(x_i) = P(X = x_i)$$

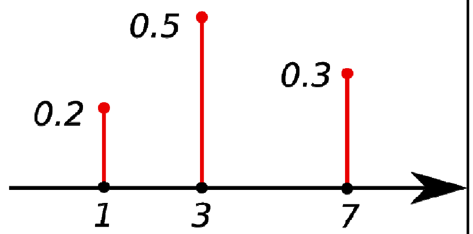
$$0 \le p(x_i) \le 1$$

$$\sum_i p(x_i) = 1$$

Probability Mass Function

PMF of a fair Dice





Continuous Random Variable

- A **continuous random variable** is one which takes an infinite number of possible values.
- Continuous random variables are usually measurements.
- Examples include height, weight, the amount of sugar in an orange, the time required to run a mile.

Distribution function *aggregates*

- For the **case of continuous variables**, we do not want to ask what the probability of "1/6" is, because the answer is always 0...
- Rather, we ask **what is the probability that the value is in the interval (a,b)**.
- So for continuous variables, we care about the derivative of the distribution function at a point (that's the derivative of an integral). This is called a **probability density function (PDF)**.
- The probability that a random variable has a value in a set A is the integral of the p.d.f. over that set A.

Probability Density Function (PDF)

- The **Probability Density Function (PDF)** of a **continuous random variable** is a function that can be integrated to obtain the probability that the random variable takes a value in a given interval.
- More formally, the probability density function, $f(x)$, of a **continuous random variable** X is the derivative of the cumulative distribution function $F(x)$:

$$f(x) = \frac{d}{dx} F(x)$$

- Since $F(x)=P(X \le x)$, it follows that:

$$F(b) - F(a) = P(a \le X \le b) = \int_a^b f(x) \cdot dx$$

Cumulative Distribution Function (CDF)

- The **Cumulative Distribution Function (CDF)** is a function giving the probability that the random variable X is less than or equal to x, for every value x.
- Formally
  - the cumulative distribution function F(x) is defined to be:

$$\forall -\infty < x < +\infty,$$

$$F(x) = P(X \le x)$$

Cumulative Distribution Function (CDF)

- For a **discrete random variable**, the cumulative distribution function is found by summing up the probabilities as in the example below.

$$\forall -\infty < x < +\infty,$$

$$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i) = \sum_{x_i \le x} p(x_i)$$

- For a **continuous random variable**, the cumulative distribution function is the integral of its **probability density function f(x)**.

$$F(a) - F(b) = P(a \le X \le b) = \int_a^b f(x) \cdot dx$$

Cumulative Distribution Function (CDF)

► **EX- Discrete case**: Suppose a random variable X has the following probability mass function p(xi):

xi	0	1	2	3	4	5
p(xi)	1/32	5/32	10/32	10/32	5/32	1/32

► The cumulative distribution function F(x) is then:

xi	0	1	2	3	4	5
F(xi)	1/32	6/32	16/32	26/32	31/32	32/32

Discrete Distribution Function

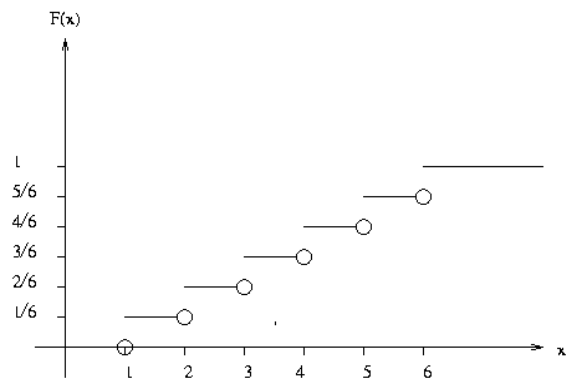


Figure 1.4: Fair die: Graph of the distribution function.

Discrete versus Continuous Random Variables

Discrete Random Variable	**Continuous Random Variable**
Probability **Mass** Function (PMF) $p(x_i) = P(X = x_i)$ 1. $p(x_i) \ge 0$, for all i 2. $\sum_{i=1}^{\infty} p(x_i) = 1$	Probability **Density** Function (PDF) $f(x)$ 1. $f(x) \ge 0$, for all x in R_X 2. $\int_{R_X} f(x)dx = 1$ 3. $f(x) = 0$, if x is not in R_X

Cumulative Distribution Function (CDF) $p(X \le x)$

$p(X \le x) = \sum_{x_i \le x} p(x_i)$	$p(X \le x) = \int_{-\infty}^{x} f(t)dt = 0$ $p(a \le X \le b) = \int_a^b f(x)dx$

Mean or Expected Value

Expectation of discrete random variable X

$$\mu_X = E(X) = \sum_{i=1}^{n} x_i \cdot p(x_i)$$

Expectation of continuous random variable X

$$\mu_X = E(X) = \int_{-\infty}^{+\infty} x \cdot f(x)dx$$

Example: Mean and variance

- When a die is thrown, each of the possible faces 1, 2, 3, 4, 5, 6 (the xi's) has a probability of 1/6 (the p(xi)'s) of showing. The expected value of the face showing is therefore:

$\mu = E(X) = (1 \times 1/6) + (2 \times 1/6) + (3 \times 1/6) + (4 \times 1/6) + (5 \times 1/6) + (6 \times 1/6) = 3.5$

- Notice that, in this case, E(X) is 3.5, which is not a possible value of X.

Variance

- The **variance** is a measure of the 'spread' of a distribution about its average value.
- Variance is symbolized by $V(X)$ or $Var(X)$ or σ^2.
  - The **mean** is a way to describe the location of a distribution,
  - the **variance** is a way to capture its scale or degree of being spread out. The unit of variance is the square of the unit of the original variable.

Variance

- The Variance of the random variable X is defined as:

$$V(X) = \sigma_X^2 = E(X - E(X))^2 = E(X^2) - E(X)^2$$

  - where E(X) is the expected value of the random variable X.
- The **standard deviation** is defined as the square root of the variance, i.e.:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{V(X)} = s$$

Coefficient of Variation

- The Coefficient of Variance of the random variable X is defined as:

$$CV(X) = \frac{V(X)}{E(X)} = \frac{\sigma_X}{\mu_X}$$

- Gives useful information about the distribution. Ex. cv=1 for any exponential distribution regardless of λ. Therefore if we found cv close to 1 in some distribution, we may suggest that it is an exp. distribution

Mean and Variance

E(X) the expected value

Discrete: $E(x) = \sum_i x_i p(x_i)$

Continuous: $E(x) = \int_0^{\infty} xf(x)$

Var(x) the variance

Discrete $Var(X) = \sum_{i=0}^{n} (x_i - \mu_i)^2 . p(x_i) = \sum_{i=0}^{n} x_i^2 . p(x_i) - \left(\sum_{i=0}^{n} x_i . p(x_i) \right)^2$

Continuous $Var(X) = \int_{-\infty}^{\infty} (x - \mu_x)^2 . f(x) dx = \left(\int_{-\infty}^{\infty} x^2 . f(x) dx \right)^2 - \mu_x^2$

Discrete Probability Distribution

- Bernoulli Trials
- Binomial Distribution
- Geometric Distribution
- Poisson Distribution
- Poisson Process

Bernoulli Trials

Any simple trial with two possible outcomes. p and q

EX: Tossing a coin, repeat, with counting # of success ***p***
"the number of heads"

Then # of failure ***q=***(1-***p)***, ***"the number of tails"***

P(HHT) =p.p.q

P(TTT)=q.q.q

If we have k as the number of successes and n-k failures

Then the probability is $p^k q^{n-k}$

Binomial Random Variable

If we have $X : S \rightarrow \{0,1,2,3\}$

Where X is the number of successes

$$X(sss) = 3$$

$$X(sfs) = X(ssf) = 2 \quad \text{….etc}$$

X now is a random variable.

X is named Binomial random variable resulted from n Bernoulli trials denoted: $b(n, p)$

Modeling of Random Events with Two-States

Binomial Random Variable

Now the probability that $X = k$ $\quad 0 \leq k \leq n$

that is all strings with k success and $n - k$ fails, there are $\binom{n}{k}$ different ways

$$P(X = k) = \binom{n}{k} p^k q^{n-k}$$

Remark: $\sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} = (p+q)^n = 1$

Geometric Random Variable

- Consider independent Bernoulli trials are performed until success $\quad s, fs, ffs, fffs, \ldots$

$$P(X = n) = P(fff \ldots fs) = q^{n-1} p = pq^{n-1}$$

- Remark: $\sum_{n=1}^{\infty} pq^{n-1} = p(1 + q + q^2 + \ldots) = p \frac{1}{1-q} = 1$
- Exr: For a geometric variable x compute $P(X > k)$

Geometric Random Variable

PMF: $p(X = k) = \begin{cases} q^{k-1}p, & k = 0,1,2,...,n \\ 0, & \text{otherwise} \end{cases}$

CDF: $F(X) = p(X \le k) = 1 - (1-p)^k$

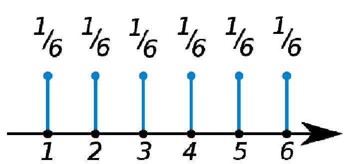
Expected Value: $E[X] = \frac{1}{p}$

Variance: $V[X] = \sigma^2 = \frac{q}{p^2} = \frac{1-p}{p^2}$

Uniform Random Variable

- An R.V. Takes values $1,2,3,...n$ with equal probabilities

$$P(X = k) = \frac{1}{n}$$



Poisson

- For $X = b(n, p)$ with large ***n*** and small ***p***, it is useless to compute the exact $P(X = k)$, as it involves huge calculations of factorial ***n***

$$P(X = k) = \binom{n}{k} p^k q^{n-k} = \frac{n(n-1)....(n-k+1)}{fact(k)} p^k q^{n-k}$$

- For large n, n-k+1 is approximated to n

$$P(X = k) \approx \frac{n^k}{k!} p^k (1-p)^n = \frac{(np)^k}{k!} \left[(1-p)^{\frac{1}{p}} \right]^{np}$$

$$\lim (1-x)^{\frac{1}{x}} = e^{-1}, small x$$

$$P(X = k) = \frac{(np)^k}{k!} e^{-np}$$

$$P(X = k) = \frac{(\lambda)^k}{k!} e^{-\lambda}, \lambda = np$$

Poisson

- Ex. A production line with .4 percent of its items are defective , n=500 items are taken for a quality control. What is the probability that 0, 1, 3 items of them are defective
- That is X=b(500,.004) aprox. To Poisson

$$P(X = k) = \frac{(\lambda)^k}{k!} e^{-\lambda}$$

$$\lambda = 500 * .004 = 2$$

$$P(x = 0) = e^{-2}$$

$$P(x = 1) = 2e^{-2}$$

$$P(x = 3) = \frac{4}{3} e^{-2}$$

Example: Poisson Distribution

- The number of cars that enter the parking follows a Poisson distribution with a mean rate equal to λ = 20 **cars**/**hour**
  - The probability of having **exactly** **15 cars** entering the parking in **one hour**:

$$p(15) = P(X = 15) = \frac{20^{15}}{15!} \cdot \exp(-20) = 0.051649$$

Applications of Poisson

- **Context:** number of events occurring in a fixed period of time
  - Events occur with a known average rate and are independent
- Possion distribution is characterized by the **average rate** λ
  - The average number of arrival in the fixed time period.

- **Examples**
  - The number of cars passing a fixed point in a 5 minute interval. Average rate: λ = 3 cars/5 minutes
  - The number of calls received by a switchboard during a given period of time. Average rate: λ =3 call/minutes
  - The number of message coming to a router per second
  - The number of travelers arriving to the airport for flight registration

Poisson Distribution

- The Poisson distribution with the average rate parameter λ

PMF: $p(k) = P(X = k) = \begin{cases} \frac{\lambda^k}{k!}\exp(-\lambda) & \text{for } k = 0,1,2,\ \ldots. \\ 0, & \text{otherwise} \end{cases}$

CDF: $F(k) = p(X \le k) = \sum_{i=0}^{k} \frac{\lambda^i}{i!} \cdot \exp(-\lambda)$

Expected value: $E[X] = \lambda$

Variance: $V[X] = \lambda$

Continuous Probability Distribution

- uniform Distribution
- exponential Distribution
- Normal Distribution
- Standard Normal Process

Continuous Uniform Distribution

- The **continuous uniform distribution** is a family of probability distributions such that for each member of the family, all intervals of the same length on the distribution's support are **equally probable**
- A random variable X is **uniformly distributed** on the interval $[a,b]$, ***U(a,b)***, if its PDF and CDF are:

PDF: $f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

CDF: $F(x) = \begin{cases} 0, & x \prec a \\ \frac{x-a}{b-a}, & a \le x \prec b \\ 1, & x \ge b \end{cases}$

Expected value: $E[X] = \frac{a+b}{2}$

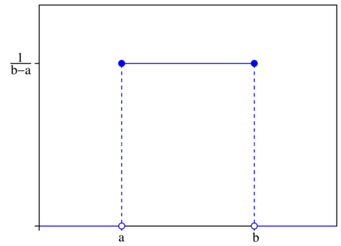
Variance: $V[X] = \frac{(a+b)^2}{12}$

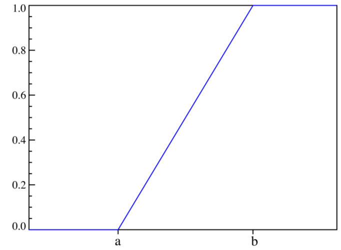
Uniform Distribution U(a,b)

- The PDF is $f(x) = const = \frac{1}{b-a}$
- **Properties**
  - $p(x1 \le X \le x2)$ is proportional to the length of the interval

$$F(X_2) - F(X_1) = \frac{X_2 - X_1}{b-a}$$

- **Special case:** a **standard uniform** distribution U(0,1).
  - Very useful for random number generators in simulators



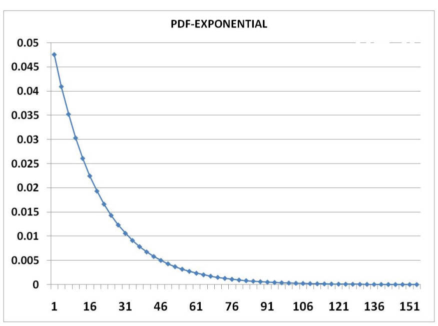


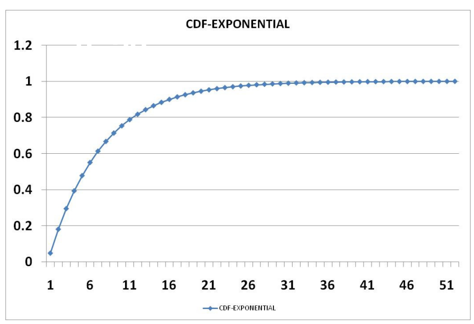
Exponential Distribution

$$\text{PDF:} \quad f(x) = \begin{cases} \lambda \cdot \exp(-\lambda \cdot x), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{CDF:} \quad F(x) = \begin{cases} 0, & x < 0 \\ \int_0^x \lambda e^{-\lambda t} dt = 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

Exponential Distribution





$$f(x) = \begin{cases} \frac{1}{20} \cdot \exp\left(-\frac{x}{20}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - \exp\left(-\frac{x}{20}\right), & x \geq 0 \end{cases}$$

Exponential Distribution

- (Special interest)The **exponential distribution** describes the times between events in a Poisson process, in which events occur continuously and independently at a constant average rate.
- A random variable X is **exponentially distributed** with parameter $\mu=1/\lambda > 0$ if its PDF and CDF are:

PDF: $f(x)=\begin{cases}\lambda\cdot\exp(-\lambda\cdot x), & x\ge 0\\ 0, & \text{otherwise}\end{cases}$ $\Rightarrow$ $f(x)=\begin{cases}\frac{1}{\mu}\cdot\exp\left(-\frac{x}{\mu}\right), & x\ge 0\\ 0, & \text{otherwise}\end{cases}$

CDF: $F(x)=\begin{cases}0, & x<0\\ \int_0^x \lambda e^{-\lambda t}dt = 1-e^{-\lambda x}, & x\ge 0\end{cases}$ $\Rightarrow$ $F(x)=\begin{cases}0, & x<0\\ 1-\exp\left(-\frac{x}{\mu}\right), & x\ge 0\end{cases}$

Expected value: $E[X]=\frac{1}{\lambda}=\mu$

Variance: $V[X]=\frac{1}{\lambda^2}=\mu^2$

Example: Continuous Random Variables

Ex.: modeling the lifetime of a device

- Time is a **continuous random variable**
- Random Time is typically modeled as **exponential distribution**
- We assume that with **average** lifetime of a device is 2 years

$$f(x)=\begin{cases}\frac{1}{2}e^{-x/2}, & \text{x} \ge 0\\ 0, & \text{otherwise}\end{cases}$$

- Probability that the device's life is between 2 and 3 years is:

$$P(2\le x\le 3)=\frac{1}{2}\int_2^3 e^{-x/2}dx = 0.14$$

The life time Ex.

- **Cumulative Distribution Function**: A device has the CDF:

$$F(x) = \frac{1}{2}\int_0^x e^{-t/2}dt = 1 - e^{-x/2}$$

  - The probability that the device lasts for less than 2 years:

$$P(0 \le X \le 2) = F(2) - F(0) = F(2) = 1 - e^{-1} = 0.632$$

  - The probability that it lasts between 2 and 3 years:

$$P(2 \le X \le 3) = F(3) - F(2) = (1 - e^{-(3/2)}) - (1 - e^{-1}) = 0.145$$

The life time Ex.

Expected Value and Variance

- Example: The mean of life of the previous device is:

$$E(X) = \frac{1}{2}\int_0^\infty xe^{-x/2}dx = -xe^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2}dx = 2$$

- To compute variance of *X*, we first compute *E*(X^2):

$$E(X^2) = \frac{1}{2}\int_0^\infty x^2e^{-x/2}dx = -x^2e^{-x/2}\Big|_0^\infty + \int_0^\infty e^{-x/2}dx = 8$$

- Hence, the variance and standard deviation of the device's life are:

$$V(X) = 8 - 2^2 = 4$$

$$\sigma = \sqrt{V(X)} = 2$$

Exponential Distribution

- **The memoryless property:** In probability theory, **memoryless** is a property of certain probability distributions: the **exponential distributions** and the **geometric distributions**, wherein any derived probability from a set of random samples is distinct and has no information (i.e. "memory") of earlier samples.
- Formally, the memoryless property is:

  For all s and t greater or equal to 0:

$$p(X > s + t \mid X > s) = p(X > t)$$

- This means that the future event do not depend on the past event, but only on the present event

Normal Distribution

- The **Normal distribution**, also called the **Gaussian distribution**, is an important family of continuous probability distributions, applicable in many fields.
- Each member of the family may be defined by two parameters, **location** and **scale**: **the mean** ("average", μ) and **variance** (standard deviation squared, σ2) respectively.
- The importance of the normal distribution as a model of quantitative phenomena in the **natural** and **behavioral** sciences is due in part to the **Central Limit Theorem**.
- It is usually used to model system error (e.g. channel error), the distribution of natural phenomena, height, weight, etc.

Normal or Gaussian Distribution

- A **continuous random variable** X, taking all real values in the range (-∞,+∞) is said to follow a **Normal distribution** with parameters **μ** and **σ** if it has the following PDF and CDF:

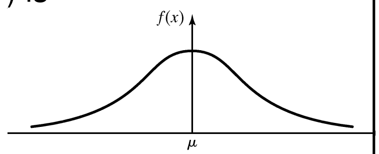
$$\text{PDF: } f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right]$$

$$\text{CDF: } F(x) = \frac{1}{2} \cdot \left(1 + erf\left(\frac{x-\mu}{\sigma \cdot \sqrt{2}} \right) \right)$$

where

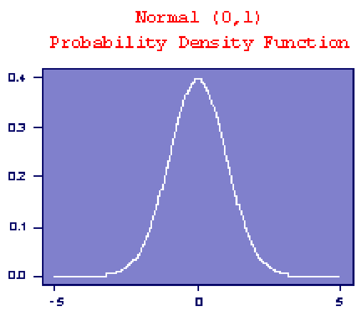
$$\text{Error Function: } erf(x) = \frac{2}{\sqrt{\pi}} \cdot \int_0^x \exp\left(-t^2\right)$$

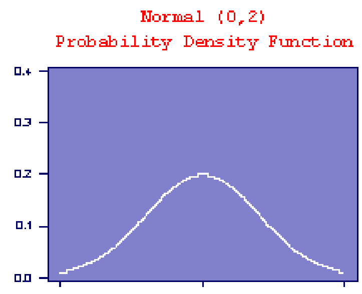
- The Normal distribution is denoted as $X \sim N\left(\mu, \sigma^2\right)$
- This probability density function (PDF) is
  - a symmetrical, bell-shaped curve,
  - centered at its expected value μ.
  - The variance is σ^2.



Normal distribution

- Example
- The simplest case of the normal distribution, known as the **Standard Normal Distribution**, has expected value zero and variance one. This is written as $N(0,1)$.





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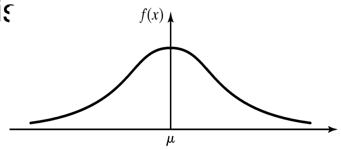


Standard Normal Distribution

Independent of μ and σ, using the **standard normal distribution**:

- Transformation of variables: let

$$Z \sim N(0,1)$$

$$Z = \frac{X-\mu}{\sigma}$$

$$F(x) = P(X \le x) = P\left(Z \le \frac{x-\mu}{\sigma}\right)$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= \int_{-\infty}^{(x-\mu)/\sigma} \phi(z) dz = \Phi\left(\tfrac{x-\mu}{\sigma}\right)$$, where $$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

Standard Normal Distribution

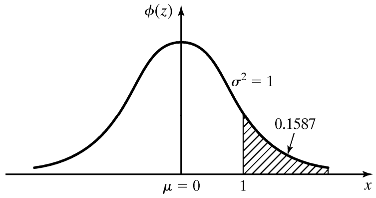
- Note that $f_Z(x)$ ***is positive for all*** $-\infty < x < \infty$ ***, hence Z takes on all real*** values, its range is the entire real line. Also note that $f_Z(x)$ ***is an even function***
- The graph of $f_Z(x)$ ***is a bell-shaped curve, symmetric about the y-axis.***
- This curve is called a gaussian curve. Its maximum is attained at ***x = 0,*** then it decreases on both sides of its top point. Actually, it decreases very fast.

Normal Distribution

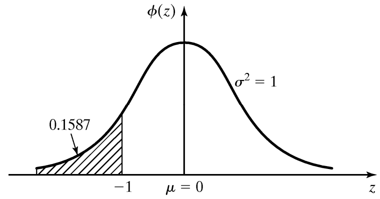
- Example: The time required to load a transporting truck, ***X***, is distributed as ***N(12,4)***
  - The probability that the truck is loaded in less than 10 hours:

$$F(10) = \Phi\left(\frac{10-12}{2}\right) = \Phi(-1) = 0.1587$$

  - Using the symmetry property, $\Phi(1)$ is the complement of Φ (-1)



(a)



(b)

Empirical Distributions

- An Empirical Distribution is a distribution whose parameters are the observed values in a sample of data.
  - May be used when it is impossible or unnecessary to establish that a random variable has any particular parametric distribution.
  - **Advantage**: no assumption beyond the observed values in the sample.
  - **Disadvantage**: sample might not cover the entire range of possible values.

Empirical Distributions

- In statistics, an **empirical distribution function** is a **cumulative probability distribution function** that concentrates probability 1/n at each of the n numbers in a sample.
- Let $x_1, \ldots, x_n$ be iid random variables in with the CDF equal to $F(x)$.
- The empirical distribution function $F_n(x)$ based on sample $x_1, \ldots, x_n$ is a step function defined by

$$F_n(x) = \frac{\text{number of element in the sample } \leq x}{n} = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$$

where $I(A)$ is the **indicator of event** A. $I(X_i \leq x) = \begin{cases} 1 & \text{if } (X_i \leq x) \\ 0 & \text{otherwise} \end{cases}$

- For a fixed value x, $I(X_i \leq x)$ is a **Bernoulli (Trial)** random variable with parameter $p=F(x)$, hence $nF_n(x)$ is a **binomial** random variable with mean $nF(x)$ and variance $nF(x)(1-F(x))$.