Chapter 8
Generating Random Variate

- If a Pseudo random number Y is used to generate another variable X, then X is called random variate.
- Note that the sequence X may or may not be uniform.
- Generally, all random variates require the use of a U(0,1) distribution.

Random Variate

- The exact technique used to generate an observation varies with the type of distribution used. However, in general we're looking for techniques that are:
  - Exact
  - Efficient in terms of time, storage, and setup
  - Low in complexity
  - Robust.

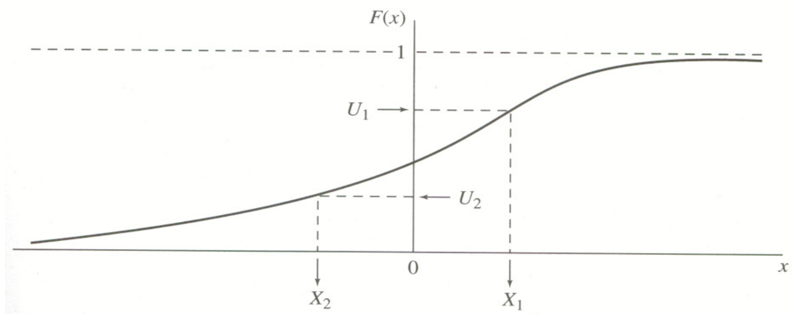
Generating Random Variates

- Say we have fitted an exponential distribution to interarrival times of customers
- Every time we anticipate a new customer arrival (place an arrival even on the events list), we need to generate a realization of the arrival times
- Already know how to generate unit uniform
- Can we use this to generate exponential? (And other distributions)

Two Types of Approaches

- Direct
  - Obtain an analytical expression
  - Inverse transform
    - Requires inverse of the distribution function
  - Composition & Convolution
    - For special forms of distribution functions
- Indirect
  - Acceptance-rejection

Inverse-Transform Method



- Place r_i on the y-axis and find the corresponding x_i

$$r = F(x) \Rightarrow x = F^{-1}(r)$$

Formulation

- Algorithm

$$1.\ \text{Generate}\ U \sim U(0,1)$$

$$2.\ \text{Return}\ X = F^{-1}(U)$$

- Proof Since *F*(*x*) is a monotone function, then

$$\begin{aligned} P(X \le x) &= P\left(F^{-1}(U) \le x\right) \\ &= P(U \le F(x)) \\ &= F(x) \end{aligned}$$

Example

- Let $$f(x)=\begin{cases}4x^3 & 0<x\le 1\\ 0 & O.W.\end{cases}$$

$$F(x)=\int_0^x 4x^3dx = x^4$$

- i.e. $$r = x^4$$

$$x_i = F^{-1}(r_i) = \sqrt[4]{r_i} \qquad 0<r_i\le 1$$

- That is use a pseudo random generator to find r_i then find x_i for the new random variate

Ex. Uniform random variate

- Let $$f(x)=\begin{cases}\dfrac{1}{b-a} & a<x\le b\\ 0 & O.W.\end{cases}$$

- i.e. $$F(x)=\frac{x-a}{b-a}=r \qquad 0<r\ \le 1$$

$$x = a + r(b-a)$$

- Use a pseudo random generator to find r_i then find x_i for the new random variate uniformly distributed on (a,b].

Exponential R.V.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \lambda > 0, x \geq 0 \\ 0 & O.W. \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$r = 1 - e^{-\lambda x} \qquad 0 < r \leq 1$$

$$e^{-\lambda x} = 1 - r$$

$$x = \frac{-1}{\lambda} \ln(1 - r)$$

Exponential R.V.

- Algorithm:

$$1.\ \text{Generate } U \sim U(0,1)$$

$$2.\ \text{Return } X = \frac{-1}{\lambda} \ln(1 - U)$$

Exponential R.V.

- Algorithm:

$$1.\ \text{Generate}\ U \sim U(0,1)$$

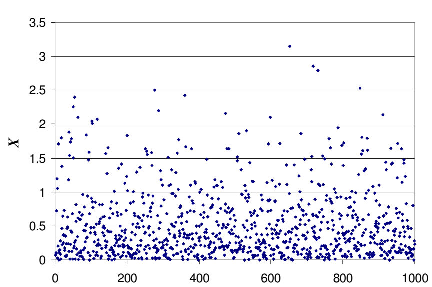
$$2.\ \text{Return}\ X_U = \frac{-1}{\lambda}\ln(1-U)$$

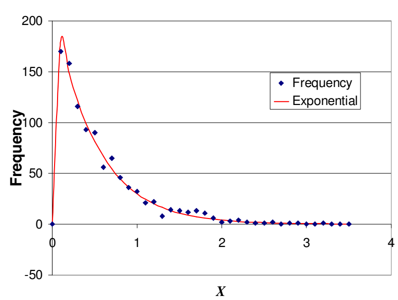
- Remark: Since $U \sim U(0,1)$ then also $1-U \sim U(0,1)$

And hence for sake of computation we may replace *1-U* by *U* in the above algorithm.

Exponential Distribution

1000 Exponential Random Numbers ($\lambda = 2$)





Exponential R.V.

- Example:

- For a sample of 5 numbers and $\lambda = 3.8$
- Use a R.N.G. to find $r_i{}'s$

r_i	0.135	0.639	0.424	0.01	0.843
ln r_i	-2.002	-0.448	-0.858	-4.61	-171
x_i	0.526	0.118	0.226	1.213	0.045

Exponential distribution

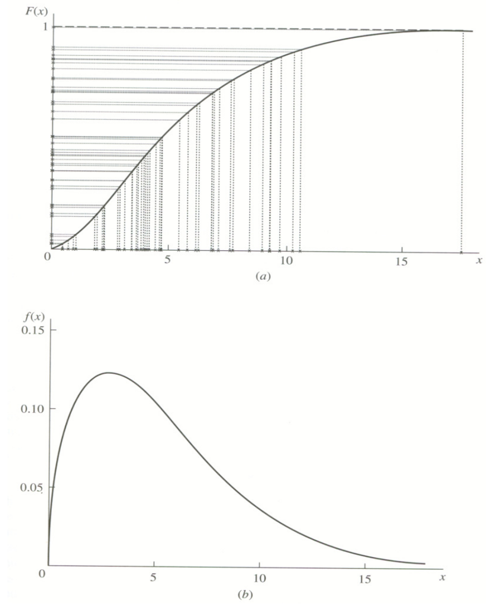
Needed for many applications

- Probability of waiting time on phone calls
- Radio active decay materials
- Time arrival of customers to some server
- Time of failure of machines/ components

Example: Weibull

- The CDF of a Weibull is

$$F(X) = 1 - e^{\left(\frac{x}{\alpha}\right)^{\beta}}$$



Inverse Transformation-Weibull

$$r = 1 - e^{-\left(\frac{X}{\alpha}\right)^{\beta}}$$

$$(1-r) = e^{-\left(\frac{X}{\alpha}\right)^{\beta}}$$

$$\ln(1-r) = -\left(\frac{X}{\alpha}\right)^{\beta}$$

$$\left[-\ln(1-r)\right]^{\frac{1}{\beta}} = \frac{X}{\alpha}$$

$$\alpha\left[-\ln(1-r)\right]^{\frac{1}{\beta}} = X$$

- Exr. Write the algorithm for generating a Weibull random variate

Advantages of Inverse Transform Method

- Straightforward method, easy to use
- Variance reduction techniques has an advantage if inverse transform method is used

Disadvantages of the inverse method

- Must evaluate the inverse of the distribution function
  - May not exist in closed form
  - Could still use numerical methods
- May not be the fastest way

Discrete R.V.: Bernoulli

- Mass function

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & \text{otherwise} \end{cases}$$

- Algorithm

1. Generate $U \sim U(0,1)$
2. If $U \le p$ return $X = 1$. Otherwise return $X = 0$

Geometric Distribution

$p(x) = p(1-p)^x$, where x = 0, 1, 2, ...

$$F(x) = \sum_{j=0}^{x} p(1-p)^x = 1-(1-p)^{x+1}$$

$$R = 1-(1-p)^{x+1}$$

$$\Leftrightarrow (1-p)^{x+1} = 1-R$$

$$\Leftrightarrow (x+1)\ln(1-P) = \ln(1-R)$$

$$\Leftrightarrow x = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F^{-1}(R) = \frac{\ln(1-R)}{\ln(1-p)} - 1$$

$$F(x-1) < R \le F(x)$$

$$X = \lfloor F^{-1}(R) \rfloor = \left\lfloor \frac{\ln(1-R)}{\ln(1-p)} - 1 \right\rfloor$$

Geometric Distribution

- Algorithm to generate random variate for

$$p(x) = p(1-p)^x, \text{ where x} = 0, 1, 2, \ldots\ldots$$

  - Generate `U~U(0,1)` uniform random number
  - Return $\left\lfloor \frac{\ln(1-U)}{\ln(1-p)} - 1 \right\rfloor$

Geometric

- Mass function $p(x) = \begin{cases} p(1-p)^x & x \in \{0,1,\ldots,t\} \\ 0 & \text{otherwise} \end{cases}$
- Use inverse-transform

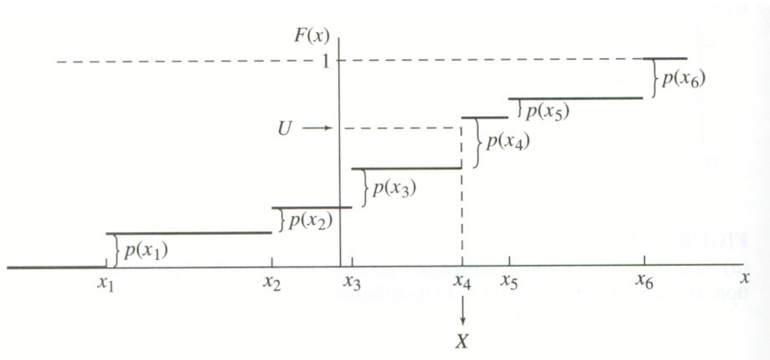
  1. Generate $U \sim U(0,1)$

  2. Return $X = \left\lfloor \frac{\ln(1-U)}{\ln(1-p)} \right\rfloor$

- Remark

For simplicity you may return $X = \left\lfloor \frac{\ln U}{\ln(1-p)} \right\rfloor$

Discrete Distributions



Formulation

- Algorithm

  X can take values $x_1, x_2, \ldots$

  1. Generate $U \sim U(0,1)$
  2. Return $X = \min\{I : U \le F(x_I)\}$

- Prove: Need to show

$$P(X = x_i) = p(x_i) \forall i$$

Proof of inverse transform method for Discrete case

- Kelton 8.2.1
- We need to show that $P(X = x_i) = p(x_i)$ for all i

for $i = 1$, $X = x_1$ iff $U \leq F(x_1) = p(x_1)$

(since the x_i are in increasing order)

since, $U \sim U(0,1)$ $P(X = x_1) = p(x_1)$

For, $i \geq 2$, $X = x_i$ iff $F(x_{i-1}) < U \leq F(x_i)$,

since, the algorithm chooses i such that $U \leq F(x_i)$

$0 \leq F(x_{i-1}) < F(x_i) \leq 1$

$P(X = x_i) = P[F(x_{i-1}) < U \leq F(x_i)] = F(x_i) - F(x_{i-1}) = p(x_i)$