Ch 3: Fuzzy Relations

Classical Relation

Definition 3.1. *(Classical relation). A subset $R \subseteq X \times Y$ where X and Y are classical sets, is a classical relation.*

A classical relation can be characterized by a function $R : X \times Y \to \{0, 1\}$,

$$R(x, y) = \begin{cases} 1 & \text{if} \quad (x, y) \in R \\ 0 & \quad\text{otherwise} \end{cases} .$$

Fuzzy Relation

Definition 3.2. *(Fuzzy Relation, Sanchez [129], Di Nola-Sessa-Pedrycz-Sanchez [48], De Baets [42]) Let X, Y be two classical sets. A mapping $R : X \times Y \to [0, 1]$ is called a* ***fuzzy relation****. The number $R(x, y) \in [0, 1]$ can be interpreted as the degree of relationship between x and y.*

Remark 3.3. *A fuzzy relation can be seen as a fuzzy subset of the set $X \times Y$. We denote by $\mathcal{F}(X \times Y)$ the family of all fuzzy relations between elements of X and Y.*

Example

Example 3.4. $R =$ "*much greater than*"

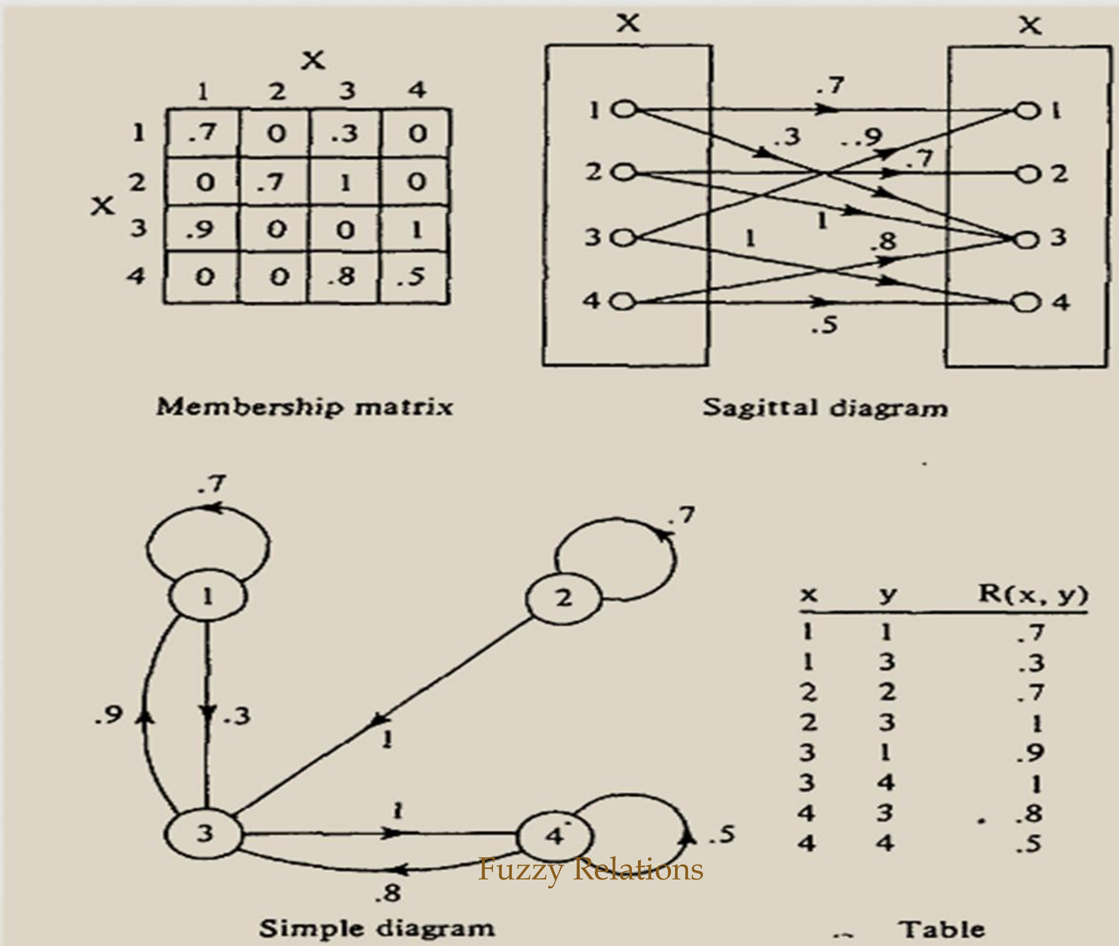
$$R(x,y) = \begin{cases} \frac{1}{1+\frac{100}{(x-y)^2}} & if \quad x > y \\ 0 & otherwise \end{cases} .$$

Representation

A fuzzy relation between elements in two finite sets $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_n\}$ can be represented as a matrix

$$R = \begin{pmatrix} R(x_1, y_1) & R(x_1, y_2) & ... & R(x_1, y_n) \\ R(x_2, y_1) & R(x_2, y_2) & ... & R(x_2, y_n) \\ ... & ... & ... & ... \\ R(x_m, y_1) & R(x_m, y_2) & ... & R(x_m, y_n) \end{pmatrix}.$$

Different Representations



Example

Let R be a fuzzy relation between the two sets X = {New York City, Paris} and Y = {Beijing, New York City, London}, which represents the relational concept "very far." This relation can be written in list notation as

$$R(X, Y) = 1/\text{NYC, Beijing} + 0/\text{NYC, NYC} + .6/\text{NYC, London} + .9/\text{Paris, Beijing} + .7/\text{Paris, NYC} + .3/\text{Paris, London}.$$

This relation can also be represented by the following two-dimensional membership array (matrix):

	NYC	*Paris*
Beijing	1	.9
NYC	0	.7
London	.6	.3

Fuzzy Set Operations

$$N(R(x,y)) = \bar{R}(x,y) = 1 - R(x,y),$$

$$(R \vee S)(x,y) = R(x,y) \vee S(x,y),$$

$$(R \wedge S)(x,y) = R(x,y) \wedge S(x,y).$$

T-Norm, T-Conorm, and The Inverse (Transpose)

$$R^{-1}(x, y) = R(y, x),$$

$$T(R, P)(x, y) = T(R(x, y), P(x, y)),$$

$$S(R, P)(x, y) = S(R(x, y), P(x, y)),$$

where T, S are a t-norm and a t-conorm respectively.

Max-Min Composition

Definition 3.5. *Let $R \in \mathcal{F}(X \times Y)$ and $S \in \mathcal{F}(Y \times Z)$ be fuzzy relations. Then $R \circ S \in \mathcal{F}(X \times Z)$, defined as*

$$R \circ S(x, z) = \bigvee_{y \in Y} R(x, y) \wedge S(y, z),$$

is called the ***max-min composition*** *of the fuzzy relations R and S.*

If $R \in \mathcal{F}(X \times X)$ then we can define $R^2 = R \circ R$, and generally $R^n = R \circ R^{n-1}$, $n \geq 2$.

Max-Min Composition: Discrete Case

Let $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_m\}$, and $Z = \{z_1, ..., z_p\}$ be finite sets. If $R = (r_{ij})_{i=1,...,n, j=1,...,m} \in \mathcal{F}(X \times Y)$, and $S = (s_{jk})_{j=1,...,m, k=1,...,p} \in \mathcal{F}(Y \times Z)$ are discrete fuzzy relations then the composition $T = (t_{ik})_{i=1,...,n, k=1,...,p} = R \circ S \in \mathcal{F}(X \times Z)$ is given by

$$t_{ik} = \bigvee_{j=1}^{m} r_{ij} \wedge s_{jk},$$

$i = 1, ..., n$, $k = 1, ..., p$.

Example

Example 3.7. *If* $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ *then* $R \circ S = \begin{pmatrix} 0.3 & 0.3 \\ 0.8 & 0.6 \end{pmatrix}$.

Example

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & .5 & .7 \\ .5 & .4 & .5 & .6 \end{bmatrix}.$$

For example,

$$\begin{aligned} .8(= r_{11}) &= \max[\min(.3, .9), \min(.5, .3), \min(.8, 1)] \\ &= \max[\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})], \\ .4(= r_{32}) &= \max[\min(.4, .5), \min(.6, .2), \min(.5, 0)] \\ &= \max[\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})]. \end{aligned}$$

Properties

Proposition 3.8. *(i) The max-min composition is associative, i.e.,*

$$(R \circ S) \circ Q = R \circ (S \circ Q),$$

where $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ *and* $Q \in \mathcal{F}(Z \times U)$.

(ii) Let $R_1, R_2 \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$. *If* $R_1 \leq R_2$ *then*

$$R_1 \circ Q \leq R_2 \circ Q.$$

Properties

Proposition 3.9. *For any* $R, S \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$ *we have*

(i) $(R \vee S) \circ Q = (R \circ Q) \vee (S \circ Q)$

(ii) $(R \wedge S) \circ Q \leq (R \circ Q) \wedge (S \circ Q)$.

Remark 3.10. *Equality in (ii) does not hold. Indeed, if we consider* $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *then* $(R \wedge S) \circ Q = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ *while* $(R \circ Q) \wedge (S \circ Q) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Min-Max Composition

Definition 3.11. *(e.g. Nobuhara-Bede-Hirota [118]) Let* $R \in \mathcal{F}(X \times Y)$ *and* $S \in \mathcal{F}(Y \times Z)$. *Then* $R \bullet S \in \mathcal{F}(X \times Z)$, *defined as*

$$R \bullet S(x,z) = \bigwedge_{y \in Y} R(x,y) \vee S(y,z)$$

is called the ***min-max composition*** *of the fuzzy relations R and S.*

Example

Example 3.12. *If* $R = \begin{pmatrix} 0.3 & 0.7 & 0.2 \\ 1 & 0 & 0.9 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0.8 & 0.3 \\ 0.1 & 0 \\ 0.5 & 0.6 \end{pmatrix}$ *then*
$R \bullet S = \begin{pmatrix} 0.5 & 0.3 \\ 0.1 & 0 \end{pmatrix}$.

Properties

Proposition 3.13. *(i) The min-max composition is associative, i.e., for any* $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ *and* $T \in \mathcal{F}(Z \times U)$ *we have*

$$(R \bullet S) \bullet T = R \bullet (S \bullet T).$$

(ii) Consider $R_1, R_2 \in \mathcal{F}(X \times Y)$, $Q \in \mathcal{F}(Y \times Z)$. *If* $R_1 \leq R_2$ *then*

$$R_1 \bullet Q \leq R_2 \bullet Q.$$

Properties

Proposition 3.14. *For any* $R, S \in \mathcal{F}(X \times Y)$ *and* $T \in \mathcal{F}(Y \times Z)$ *we have*

(i) $(R \wedge S) \bullet T = (R \bullet T) \wedge (S \bullet T)$.

(ii) $(R \vee S) \bullet T \geq (R \bullet T) \vee (S \bullet T)$.

Remark 3.15. *Equality in (ii) does not hold. Indeed,* $R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ *then* $(R \vee S) \bullet T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *while* $(R \bullet T) \vee (S \bullet T) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

Properties

Proposition 3.16. *If we consider the standard negation we have* $\overline{R \circ S} = \bar{R} \bullet \bar{S}$ *and* $\overline{R \bullet S} = \bar{R} \circ \bar{S}$.

Generalization

Remark 3.17. *The min-max composition can be naturally generalized to min-t-conorm compositions*

$$R \bullet_S P(x,z) = \bigwedge_{y \in Y} R(x,y) S P(y,z),$$

where S is an arbitrary t-conorm.

Min → Composition

Let → be the standard **Gödel implication** defined as

$$x \to y = \sup\{z \in [0,1] | x \wedge z \leq y\} = \begin{cases} 1 & \text{if} \quad x \leq y \\ y & \quad \text{otherwise} \end{cases}.$$

Min → Composition

Proposition 3.18. *For any* $x, y, z \in [0,1]$ *we have*

(i) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

(ii) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$.

(iii) $x \to (y \vee z) = (x \to y) \vee (x \to z)$.

(iv) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

(v) $x \wedge (x \to y) \le y$.

(vi) $x \to (x \wedge y) \ge y$.

(vii) $(x \to y) \to y \ge x$.

Min → Composition

Definition 3.19. *The* ***min→ composition*** *can be defined as:*

$$R \triangleleft S(x,z) = \bigwedge_{y \in Y} R(x,y) \to S(y,z).$$

Often in the literature (see e.g. De Baets [42]) it is called subcomposition and a dual operation is considered as

$$R \triangleright S(x,z) = \bigwedge_{y \in Y} S(y,z) \to R(x,y)$$

called the supercomposition. The relation between the two is given by the next proposition.

Relation between subcomposition and supercomposition

Proposition 3.20. *For* $R_1 \in \mathcal{F}(X \times Y)$ *and* $R_2 \in \mathcal{F}(Y \times Z)$ *we have*

(i) $R_1 \triangleleft R_2 = (R_2^{-1} \triangleright R_1^{-1})^{-1}$;

(ii) $R_1 \triangleright R_2 = (R_2^{-1} \triangleleft R_1^{-1})^{-1}$.

Properties

Proposition 3.21. *If* $R, S \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$ *are such that* $R \leq S$ *then* $R \triangleleft Q \geq S \triangleleft Q$.

Proposition 3.22. *For any* $R, S \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$ *we have*

(i) $(R \vee S) \triangleleft Q = (R \triangleleft Q) \wedge (S \triangleleft Q)$.

(ii) $(R \wedge S) \triangleleft Q \geq (R \triangleleft Q) \vee (S \triangleleft Q)$.

Fuzzy Relational Equations with Max-Min and Min $\longrightarrow$ Compositions

We consider the following two fuzzy relational equations

$$R \circ P = Q$$

and

$$R \triangleleft P = Q$$

with $R \in \mathcal{F}(X \times Y)$, $P \in \mathcal{F}(Y \times Z)$ and $Q \in \mathcal{F}(X \times Z)$.

Fuzzy Relational Equations with Max-Min and Min → Compositions

Theorem 3.23. *The following inequalities hold true:*

(i) $P \leq R^{-1} \triangleleft (R \circ P)$;

(ii) $R \circ (R^{-1} \triangleleft Q) \leq Q$;

(iii) $R \leq (P \triangleleft (R \circ P)^{-1})^{-1}$;

(iv) $(P \triangleleft Q^{-1})^{-1} \circ P \leq Q$.

Fuzzy Relational Equations with Max-Min and Min → Compositions

Theorem 3.24. *(Sanchez [129]) (i) Consider the equation $R \circ P = Q$ with unknown P. The equation has solutions if and only if $R^{-1} \triangleleft Q$ is a solution and in this case it is the greatest solution of this equation.*

(ii) Consider the equation $R \circ P = Q$ with unknown R. The equation has solutions if and only if $(P \triangleleft Q^{-1})^{-1}$ is a solution and in this case it is the greatest solution of this equation.

Example

Example 3.25. *Let us consider the fuzzy relational equation*

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.6 \end{pmatrix} \circ P = \begin{pmatrix} 0.3 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.5 \\ 0.3 & 0.5 & 0.6 \end{pmatrix}.$$

Then

$$R^{-1} \triangleleft Q = \begin{pmatrix} 0.3 & 1 & 1 \\ 0.3 & 1 & 1 \\ 0.3 & 0.5 & 1 \end{pmatrix},$$

Example (Cont.)

and since

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.6 \end{pmatrix} \circ \begin{pmatrix} 0.3 & 1 & 1 \\ 0.3 & 1 & 1 \\ 0.3 & 0.5 & 1 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 & 0.4 \\ 0.3 & 0.5 & 0.5 \\ 0.3 & 0.5 & 0.6 \end{pmatrix},$$

then $R^{-1} \triangleleft Q$ *is a solution of the equation. From the previous theorem it follows that it is the greatest solution of the given equation.*

Fuzzy Relational Equations with Max-Min and Min → Compositions

Theorem 3.26. *The following inequalities hold true:*

(i) $(Q \triangleleft P^{-1}) \triangleleft P \geq Q$;

(ii) $(R \triangleleft P) \triangleleft P^{-1} \geq R$;

(iii) $R^{-1} \circ (R \triangleleft P) \leq P$;

(iv) $R \triangleleft (R^{-1} \circ Q) \geq Q$.

Fuzzy Relational Equations with Max-Min and Min $\rightarrow$ Compositions

Theorem 3.27. *(Miyakoshi-Shimbo [112]) (i) Consider the equation* $R \triangleleft P = Q$ *with unknown* R. *The equation has solutions if and only if* $Q \triangleleft P^{-1}$ *is a solution and in this case it is the greatest solution of this equation.*

(ii) Consider the equation $R \triangleleft P = Q$ *with unknown* P. *The equation has solutions if and only if* $R^{-1} \circ Q$ *is a solution and in this case it is the least solution of this equation.*

Example

Example 3.28. *Let us consider the fuzzy relational equation*

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.6 \end{pmatrix} \triangleleft P = \begin{pmatrix} 0.3 & 0.4 & 0.4 \\ 1 & 0.6 & 0.5 \\ 0.5 & 0.5 & 0.6 \end{pmatrix}.$$

Then

$$R^{-1} \circ Q = \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.6 \end{pmatrix}.$$

Example (Cont.)

Since

$$\begin{pmatrix} 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.5 \\ 0.5 & 0.4 & 0.6 \end{pmatrix} \triangleleft \begin{pmatrix} 0.5 & 0.5 & 0.5 \\ 0.4 & 0.4 & 0.4 \\ 0.5 & 0.5 & 0.6 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0.5 & 0.5 & 1 \end{pmatrix}.$$

is not a solution of the equation, by the preceding theorem it follows that the equation has no solutions.

Max-t-Norm Composition

Definition 3.29. *The max-min composition can be naturally generalized to max-t-norm compositions*

$$R \circ_T P(x,z) = \bigvee_{y \in Y} R(x,y)\ T\ P(y,z),$$

where T is an arbitrary t-norm.

Properties

Proposition 3.30. *(i) The max-t-norm composition is associative, i.e.,*

$$(R \circ_T S) \circ_T Q = R \circ_T (S \circ_T Q),$$

for any $R \in \mathcal{F}(X \times Y)$, $S \in \mathcal{F}(Y \times Z)$ *and* $Q \in \mathcal{F}(Z \times U)$.

(ii) If $R_1 \leq R_2$ *then*

$$R_1 \circ_T Q \leq R_2 \circ_T Q$$

for any $R_1, R_2 \in \mathcal{F}(X \times Y)$ *and* $Q \in \mathcal{F}(Y \times Z)$.

Properties

Proposition 3.31. *For any* $R, S \in \mathcal{F}(X \times Y)$, $Q \in \mathcal{F}(Y \times Z)$ *and any t-norm* T *we have*

(i) $(R \vee S) \circ_T Q = (R \circ_T Q) \vee (S \circ_T Q)$

(ii) $(R \wedge S) \circ_T Q \leq (R \circ_T Q) \wedge (S \circ_T Q)$.

Min $\rightarrow_T$ Composition

Let T be an arbitrary continuous t-norm and $\rightarrow_T$ be the R-implication defined as

$$x \rightarrow_T y = \sup\{z | x \; T \; z \leq y\}.$$

Min $\rightarrow_T$ Composition

Proposition 3.32. *For any $x, y, z \in [0,1]$ and for any t-norm T the residual implication $\rightarrow_T$ has the following properties:*

(i) $xT(x \rightarrow_T y) \leq y$.

(ii) $x \rightarrow_T (xTy) \geq y$.

(iii) $(x \rightarrow_T y) \rightarrow_T y \geq x$.

Min $\rightarrow_T$ Composition

Definition 3.33. *The* ***min***$\rightarrow_T$ ***composition*** *can be defined in a similar way as the min $\rightarrow$ composition:*

$$R \triangleleft_T S(x,z) = \bigwedge_{y \in Y} R(x,y) \rightarrow_T S(y,z)$$

Min $\rightarrow_T$ Composition

Proposition 3.34. *If* $R, S \in \mathcal{F}(X \times Y)$ *are such that* $R \leq S$ *and if* $Q \in \mathcal{F}(Y \times Z)$ *then* $R \lhd_T Q \geq S \lhd Q$.

Fuzzy Relational Equations with Max-t-Norm and Min $\rightarrow_T$ Compositions

We consider the following two fuzzy relational equations with max-t-norm and min $\rightarrow_T$ compositions

$$R \circ_T P = Q$$

and

$$R \triangleleft_T P = Q$$

with $R \in \mathcal{F}(X \times Y)$, $P \in \mathcal{F}(Y \times Z)$ and $Q \in \mathcal{F}(X \times Z)$.

Fuzzy Relational Equations with Max-t-Norm and Min $\rightarrow_T$ Compositions

Theorem 3.35. *The following inequalities hold true:*

(i) $P \leq R^{-1} \lhd_T (R \circ_T P)$;

(ii) $R \circ_T (R^{-1} \lhd_T Q) \leq Q$;

(iii) $R \leq (P \lhd_T (R \circ_T P)^{-1})^{-1}$;

(iv) $(P \lhd_T Q^{-1})^{-1} \circ_T P \leq Q$.

Fuzzy Relational Equations with Max-t-Norm and Min $\rightarrow_T$ Compositions

Theorem 3.36. *(Sanchez [129], Miyakoshi-Shimbo [112]) (i) Consider the equation* $R \circ_T P = Q$ *with unknown* P*. The equation has solutions if and only if* $R^{-1} \triangleleft_T Q$ *is a solution and in this case it is the greatest solution of this equation.*

(ii) Consider the equation $R \circ_T P = Q$ *with unknown* R*. The equation has solutions if and only if* $(P \triangleleft Q^{-1})^{-1}$ *is a solution and in this case it is the greatest solution of this equation.*

Fuzzy Relational Equations with Max-t-Norm and Min $\rightarrow_T$ Compositions

Theorem 3.37. *The following inequalities hold true:*

(i) $(Q \lhd_T P^{-1}) \lhd_T P \geq Q$;
(ii) $(R \lhd_T P) \lhd_T P^{-1} \geq R$;
(iii) $R^{-1} \circ_T (R \lhd_T P) \leq P$;
(iv) $R \lhd_T (R^{-1} \circ_T Q) \geq Q$.

Fuzzy Relational Equations with Max-t-Norm and Min $\rightarrow_T$ Compositions

Theorem 3.38. *(Miyakoshi-Shimbo [112]) (i) Consider the equation* $R \triangleleft_T P = Q$ *with unknown* R*. The equation has solutions if and only if* $Q \triangleleft_T P^{-1}$ *is a solution and in this case it is the greatest solution of this equation.*

(ii) Consider the equation $R \triangleleft_T P = Q$ *with unknown* P*. The equation has solutions if and only if* $R^{-1} \circ_T Q$ *is a solution and in this case it is the least solution of this equation.*

The Use of Neural Networks

- Aim: To illustrate the way in which fuzzy relation equations can be represented by neural networks.
- Our discussion is restricted to the form:

$$\mathbf{P} \circ \mathbf{Q} = \mathbf{R},$$

where $\circ$ is the max-product composition. Let $\mathbf{P} = [p_{ij}]$, $\mathbf{Q} = [q_{jk}]$, $\mathbf{R} = [r_{ik}]$, where $i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$, $k \in \mathbb{N}_s$. We assume that relations $\mathbf{Q}$ and $\mathbf{R}$ are given, and we want to determine $\mathbf{P}$. The above equation represents the set of equations

$$\max_{j \in \mathbb{N}_m} p_{ij} q_{jk} = r_{ik}$$

for all $i \in \mathbb{N}_n$, $k \in \mathbb{N}_s$.

The Use of Neural Networks

- To solve for pij, we can use a feedforward neural network with m inputs and only one layer with n neurons.
- The activation function employed by the neurons is not the sigmoid function, but the so-called linear activation function f defined by:

$$f(a) = \begin{cases} 0 & \text{if } a < 0 \\ a & \text{if } a \in [0, 1] \\ 1 & \text{if } a > 1. \end{cases}$$

The Use of Neural Networks

- The output y_i of neuron i is defined by

$$y_i = f(\max_{j \in \mathbb{N}_m} W_{ij} x_j) \quad (i \in \mathbb{N}_n).$$

The training set consists of columns q_k of matrix Q as inputs ($X_j = q_{jk}$ for each j in N_m, k in N_s) and columns r_k of matrix R as expected outputs ($y_i = r_{ik}$ for each i in N_n, k in N_s).

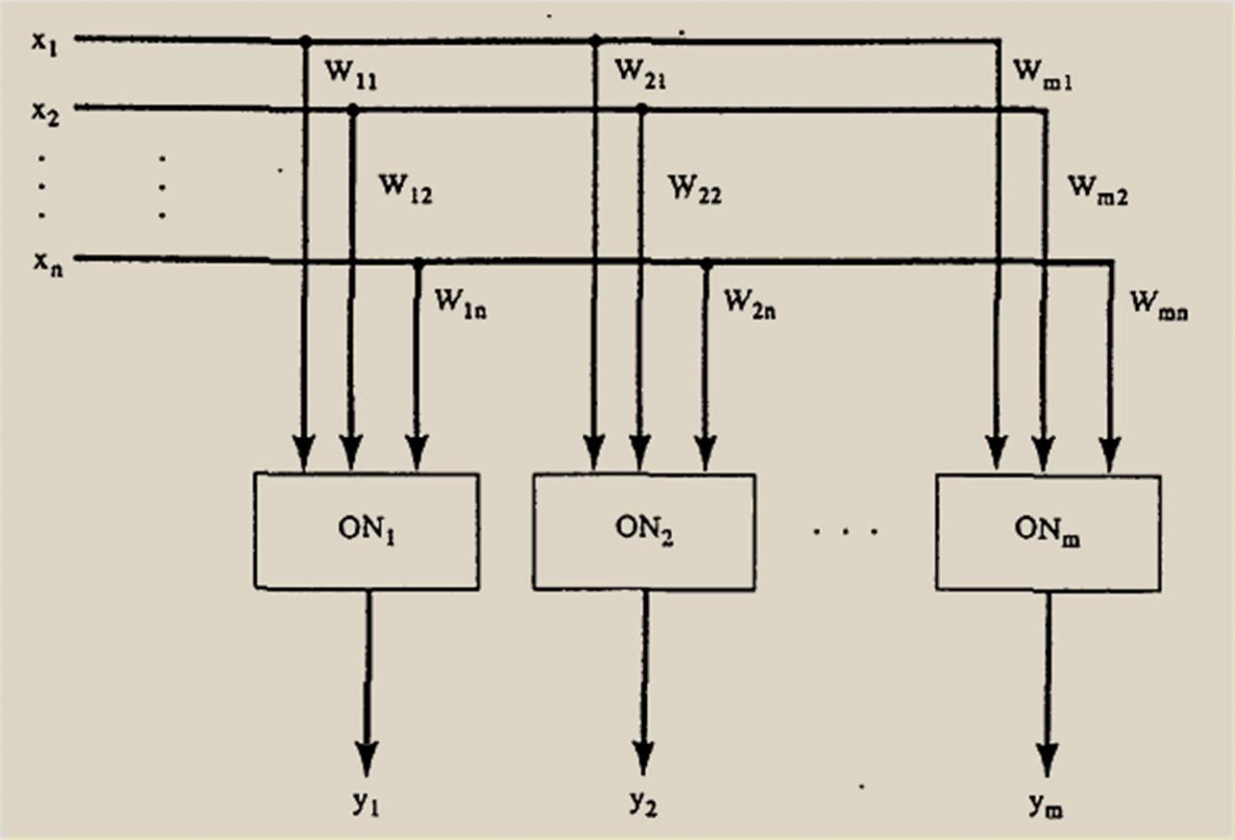
The Use of Neural Networks

- The solution is then expressed by the weights W_{ij} as

$$p_{ij} = W_{ij}$$

for all $i \in \mathbb{N}_n$, $j \in \mathbb{N}_m$.

The Use of Neural Networks



The Use of Neural Networks: Example

$$\mathbf{P} \circ \begin{bmatrix} .1 \\ .2 \\ .3 \end{bmatrix} = \begin{bmatrix} .12 \\ .18 \\ .27 \end{bmatrix}.$$

We form a neural network with three inputs and three neurons in the output layer. The training set consists of only one input $\langle .1, .2, .3 \rangle$ and one expected output $\langle .12, .18, .27 \rangle$. This training pair is applied to the learning algorithm repeatedly until the error function reaches zero. The speed of convergence depends on the choice of initial values of the weights and on the chosen learning rate. In our experiment, the cost function reached zero after 109 cycles. The final weights are shown in Next Fig. Hence, the solution is

$$\mathbf{P} = \begin{bmatrix} .1324 & .2613 & .4 \\ .2647 & .404 & .6 \\ .2925 & .5636 & .9 \end{bmatrix}.$$

The Use of Neural Networks: Example